Int. J. Nonlinear Anal. Appl. **2** (2011) No.2, 103–108
ISSN: 2008-6822 (electronic)
http://www.ijnaa.com

STRONGLY $[V_2, \lambda_2, M, P]$ −SUMMABLE DOUBLE SEQUENCE SPACES DEFINED BY ORLICZ FUNCTION

AYHAN ESI

Abstract. In this paper we introduce strongly $[V_2, \lambda_2, M, p]$ −summable double vsequence spaces via Orlicz function and examine some properties of the resulting these spaces. Also we give natural relationship between these spaces and S_{λ_2}−statistical convergence.

1. Introduction

Before we enter the motivation for this paper and the presentation of the main results we give some preliminaries.

By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > n$, [1]. We shall write more briefly as "$P-$*convergent*".

The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l. Let l_∞^n the space of all bounded double such that

$$\|x_{k,l}\|_{(\infty,2)} = \sup_{k,l} |x_{k,l}| < \infty.$$

Recall in [8] that an *Orlicz function* M is continuous, convex, nondecreasing function define for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called the *modulus function* and characterized by Ruckle [10]. An Orlicz function M is said to satisfy Δ_2-*condition* for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$.

Let $\lambda = (\lambda_r)$ be a nondecreasing sequence of positive numbers tending to infinity and $\lambda_{r+1} \leq \lambda_r + 1, \lambda_1 = 1$. The generalized *de la Vallee-Poussin mean* is defined by

$$t_r(x) = \frac{1}{\lambda_r} \sum_{k \in I_r} x_k, \;\; I_r = [r - \lambda_r + 1, r].$$

A single sequence $x = (x_k)$ is said to be $(V, \lambda)-$*summable* to a number L if $t_r(x) \to L$ as $r \to \infty$, [4]. If $\lambda_r = r$, then the $(V, \lambda)-$*summability* is reduced to $(C, 1)-$*summability*, [5, 9].

Date: Received: September 2010; Revised: June 2011.
2000 *Mathematics Subject Classification.* Primary 40A99; Secondary 40A05.
Key words and phrases. P-convergent, double statistical convergence, Orlicz function.

Using these notations we now present the following new definitions:

2. Definitions and Results

Definition 2.1. The double sequence $\lambda_2 = (\lambda_{m,n})$ of positive real numbers tending to infinity such that

$$\lambda_{m+1,n} \leq \lambda_{m,n} + 1,\ \ \lambda_{m,n+1} \leq \lambda_{m,n} + 1,$$

$$\lambda_{m,n} - \lambda_{m+1,n} \leq \lambda_{m,n+1} - \lambda_{m+1,n+1}, \lambda_{1,1} = 1,$$

and

$$I_{m,n} = \left\{(k,l):\ m - \lambda_{m,n} + 1 \leq k \leq m,\ n - \lambda_{m,n} + 1 \leq l \leq n\right\}.$$

The generalized double de Vallee-Poussin mean is defined by

$$t_{m,n} = t_{m,n}\left(x_{k,l}\right) = \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} x_{k,l}.$$

A double number sequence $x = (x_{k,l})$ is said to be $(V_2, \lambda_2)-$*summable* to a number L if $P - \lim_{m,n} t_{m,n} = L$. If $\lambda_{m,n} = mn$, then the $(V_2, \lambda_2)-$*summability* is reduced to $(C,1,1)-$*summability*, [2]. We write

$$[V_2, \lambda_2] = \left\{ x = (x_{k,l}) : P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} |x_{k,l} - L| = 0, \text{ for some } L \right\}$$

for sets of double sequences $x = (x_{k,l})$. We say that $x = (x_{k,l})$ is strongly $[V_2, \lambda_2]-$*summable* to L, that is $x = (x_{k,l}) \to L\left([V_2, \lambda_2]\right)$.

Definition 2.2. A double number sequence $x = (x_{k,l})$ is $S_{\lambda_2}-P-$*convergent* to L if provided that for every $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \left|\left\{(k,l) \in I_{m,n} :\ \ |x_{k,l} - L| \geq \varepsilon\right\}\right| = 0.$$

We will denote the set of all double $S_{\lambda_2} - P -$ *convergent* sequences by S_{λ_2}.

Let M be an Orlicz function and $p = (p_{k,l})$ be any factorable double sequence of strictly positive real numbers, we define the following sequence spaces:

$$[V_2, \lambda_2, M, p]_o = \left\{ x = (x_{k,l}) : P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} \left[M\left(\frac{|x_{k,l}|}{\rho} \right) \right]^{p_{k,l}} = 0, \right.$$
$$\left. \text{for some } \rho > 0 \right\},$$

$$[V_2, \lambda_2, M, p] = \left\{ x = (x_{k,l}) : P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} \left[M\left(\frac{|x_{k,l} - L|}{\rho} \right) \right]^{p_{k,l}} = 0, \right.$$
$$\left. \text{for some } \rho > 0 \text{ and } L \right\},$$

and

$$[V_2, \lambda_2, M, p]_\infty = \left\{ x = (x_{k,l}) : \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} \left[M\left(\frac{|x_{k,l}|}{\rho} \right) \right]^{p_{k,l}} < \infty, \right.$$
$$\left. \text{for some } \rho > 0 \right\}.$$

We shall denote $[V_2, \lambda_2, M, p]_o$, $[V_2, \lambda_2, M, p]$ and $[V_2, \lambda_2, M, p]_\infty$ as $[V_2, \lambda_2, M]_o$, $[V_2, \lambda_2, M]$ and $[V_2, \lambda_2, M]_\infty$, respectively when $p_{k,l} = 1$ for all k and l. Also note that if $M(x) = x$ and $p_{k,l} = 1$ for all k and l, then $[V_2, \lambda_2, M, p]_o = [V_2, \lambda_2]_o$, $[V_2, \lambda_2, M, p] = [V_2, \lambda_2]$ and $[V_2, \lambda_2, M, p]_\infty = [V_2, \lambda_2]_\infty$ and $M(x) = x$ then $[V_2, \lambda_2, M, p]_o = [V_2, \lambda_2, p]_o$, $[V_2, \lambda_2, M, p] = [V_2, \lambda_2, p]$ and $[V_2, \lambda_2, M, p]_\infty = [V_2, \lambda_2, p]_\infty$.
The proof of the first theorem is standard thus we omitted.

Theorem 2.3. *For any Orlicz function M a bounded factorable positive double number sequence $p = (p_{k,l})$, the spaces $[V_2, \lambda_2, M, p]_o$, $[V_2, \lambda_2, M, p]$ and $[V_2, \lambda_2, M, p]_\infty$ are linear spaces.*

Before the proof of below theorem we need the following lemma.

Lemma 2.4. *Let M be an Orlicz function which satisfies Δ_2- condition and let $0 < \delta < 1$. Then for each $x \geq \delta$, we have $M(x) < K\delta^{-1}M(2)$ for some constant $K > 0$.*

Theorem 2.5. *For any Orlicz function M which satisfies Δ_2- condition we have $[V_2, \lambda_2, p] \subset [V_2, \lambda_2, M, p]$.*

Proof. Let $x = (x_{k,l}) \in [V_2, \lambda_2, p]$, then

$$A_{m,n} = P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} |x_{k,l} - L|^{p_{k,l}} \quad \text{for some } L. \tag{2.1}$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \leq t \leq \delta$. Write $y_{k,l} = |x_{k,l} - L|$ and consider

$$\sum_{(k,l)\in I_{m,n}} [M(y_{k,l})]^{p_{k,l}} = \sum_{(k,l)\in I_{m,n}: y_{k,l}\leq\delta} [M(y_{k,l})]^{p_{k,l}} + \sum_{(k,l)\in I_{m,n}: y_{k,l}>\delta} [M(y_{k,l})]^{p_{k,l}}.$$

Since M is continuous

$$\sum_{(k,l)\in I_{m,n}: y_{k,l}\leq\delta} [M(y_{k,l})]^{p_{k,l}} < \varepsilon$$

and for $y_{k,l} > \delta$, we use the fact that

$$y_{k,l} < \frac{y_{k,l}}{\delta} < 1 + \frac{y_{k,l}}{\delta}.$$

Since M is nondecreasing and convex, it follows that

$$M(y_{k,l}) < M\left(1 + \frac{y_{k,l}}{\delta}\right) < \frac{1}{2}M(2) + \frac{1}{2}M\left(\frac{2y_{k,l}}{\delta}\right).$$

Since M satisfies Δ_2- *condition*, therefore

$$M(y_{k,l}) < \frac{1}{2}K\frac{y_{k,l}}{\delta}M(2) + \frac{1}{2}K\frac{y_{k,l}}{\delta}M(2) = K\frac{y_{k,l}}{\delta}M(2).$$

Hence

$$\sum_{(k,l)\in I_{m,n}:y_{k,l}>\delta}[M(y_{k,l})]^{p_{k,l}}<\max\left(1,K\delta^{-1}M(2)\right)^{H}A_{m,n}$$

where $H=\sup_{k,l}p_{k,l}$. This and from (2.1), we obtain $[V_2,\lambda_2,p]\subset[V_2,\lambda_2,M,p]$. □

3. λ_2-Statistical Convergence

The notion of statistical convergence for single sequences was introduced by Fast [3] and studied by various authors. Mursaleen [6] introduced the concept of $\lambda-$statistical convergence as follows: A sequence $x=(x_k)$ is said to be $\lambda-$*statistically convergent* or $S_\lambda-$*convergent* to L if for every $\varepsilon>0$,

$$\lim_r\frac{1}{\lambda_r}\left|\{k\in I_r:\ \ |x_k-L|\geq\varepsilon\}\right|=0,\ I_r=[r-\lambda_r+1,r],$$

where the vertical bars indicate the number of elements in the enclosed set.

Now we extend this definition for double sequences.

Definition 3.1. The double number sequence $x=(x_{k,l})$ is called $S_{\lambda_2}-P-$*convergent* to the number L provided that for every $\varepsilon>0$

$$P-\lim_{m,n}\frac{1}{\lambda_{m,n}}\left|\{(k,l)\in I_{m,n}:\ \ |x_{k,l}-L|\geq\varepsilon\}\right|=0.$$

In this case we write $S_{\lambda_2}-\lim x=L$ and we say that the double sequence $x=(x_{k,l})$ is λ_2- *statistically convergent* to L. If $\lambda_{m,n}=mn$ for all m and n, we obtain all $P-$statistical convergent double sequence space st_2 which was defined by Mursaleen and Edely [7].

Theorem 3.2. *Let M be an Orlicz function. For double λ_2 sequence $[V_2,\lambda_2,M]\subset S_{\lambda_2}$ and the inclusion is strict.*

Proof. Suppose that $x=(x_{k,l})\in[V_2,\lambda_2,M]$ and $\varepsilon>0$. Then we obtain the following for every m and n,

$$\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}}M\left(\frac{|x_{k,l}-L|}{\rho}\right)\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}:\left|x_{k,l}-L\right|\geq\varepsilon}M\left(\frac{|x_{k,l}-L|}{\rho}\right)$$

$$\geq\frac{M\left(\frac{\varepsilon}{\rho}\right)}{\lambda_{m,n}}\left|\{(k,l)\in I_{m,n}:\ \ |x_{k,l}-L|\geq\varepsilon\}\right|.$$

Hence $x = (x_{k,l}) \in S_{\lambda_2}$. To show this inclusion is strict, we can establish an example as follows: Let $M(x) = x$ and

$$x_{k,l} = \begin{pmatrix} 1 & 2 & 3 & \ldots & \left[\sqrt[3]{\lambda_{m,n}}\right] & 0 & 0 & \ldots \\ 2 & 2 & 3 & \ldots & \left[\sqrt[3]{\lambda_{m,n}}\right] & 0 & 0 & \ldots \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 2 & \left[\sqrt[3]{\lambda_{m,n}}\right] & \left[\sqrt[3]{\lambda_{m,n}}\right] & \ldots & \left[\sqrt[3]{\lambda_{m,n}}\right] & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}$$

and

$$P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \left|\{(k,l) \in I_{m,n} : \ |x_{k,l} - L| \geq \varepsilon\}\right| = P - \lim_{m,n} \frac{\left[\sqrt[3]{\lambda_{m,n}}\right]}{\lambda_{m,n}} = 0.$$

Therefore $S_{\lambda_2} - \lim x = 0$ and $x = (x_{k,l}) \in S_{\lambda_2}$. But

$$P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} |x_{k,l}| = P - \lim_{m,n} \frac{\left[\sqrt[3]{\lambda_{m,n}}\right] \left(\left[\sqrt[3]{\lambda_{m,n}}\right] \left(\left[\sqrt[3]{\lambda_{m,n}}\right] + 1\right)\right)}{2\lambda_{m,n}} = \frac{1}{2}.$$

Therefore $x = (x_{k,l}) \notin [V_2, \lambda_2, M]$. This completes the proof. □

Theorem 3.3. *$[V_2, \lambda_2, M] = S_{\lambda_2}$ if and only if the Orlicz function M is bounded.*

Proof. Suppose that M is bounded and $x = (x_{k,l}) \in S_{\lambda_2}$. Since M is bounded then there exists an integer K such that $M(x) \leq K$ for all $x \geq 0$. Then for each m and n, we have

$$\begin{aligned} \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} \left[M\left(\frac{|x_{k,l} - L|}{\rho}\right)\right] &= \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n} : \left|x_{k,l} - L\right| \geq \varepsilon} \left[M\left(\frac{|x_{k,l} - L|}{\rho}\right)\right] \\ &+ \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n} : \left|x_{k,l} - L\right| < \varepsilon} \left[M\left(\frac{|x_{k,l} - L|}{\rho}\right)\right] \\ &\leq \frac{K}{\lambda_{m,n}} \left|\{(k,l) \in I_{m,n} : \ |x_{k,l} - L| \geq \varepsilon\}\right| + M(\varepsilon) \end{aligned}$$

and thus the Pringsheim limit on m and n grant us the result.

Conversely, suppose that M is unbounded so that there is a positive double sequence (z_{mn}) with $M(z_{mn}) = (\lambda_{m,n})^2$ for $m, n = 1, 2, \ldots$. Now the sequence $x = (x_{k,l})$ defined by $x_{k,l} = z_{mn}$ if $k, l = (\lambda_{m,n})^2$ for $m, n = 1, 2, \ldots$ and $x_{k,l} = 0$, otherwise. Then we have

$$\frac{1}{\lambda_{m,n}} \left|\{(k,l) \in I_{m,n} : \ |x_{k,l} - L| \geq \varepsilon\}\right| \leq \frac{\sqrt{\lambda_{m,n}}}{\lambda_{m,n}} \to 0, \text{ as } m, n \to \infty.$$

Hence $x_{k,l} \to L = 0\,(S_{\lambda_2})$. But $x = (x_{k,l}) \notin [V_2, \lambda_2, M]$, contradicting $[V_2, \lambda_2, M] = S_{\lambda_2}$. This completes the proof. □

In the next theorem we prove the following relation.

Theorem 3.4. *$x=(x_{k,l})\in st_2$ implies $x=(x_{k,l})\in S_{\lambda_2}$ if*

$$\liminf_{m,n}\frac{1}{\lambda_{m,n}}>0. \tag{3.1}$$

Proof. For given $\varepsilon>0$, we have

$$\{(k,l)\in I_{m,n}:k\leq m \text{ and } l\leq n,\ \ |x_{k,l}-L|\geq\varepsilon\}\supset\{(k,l)\in I_{m,n}:\ \ |x_{k,l}-L|\geq\varepsilon\}.$$

Therefore

$$\frac{1}{mn}\left|\{(k,l)\in I_{m,n}:k\leq m \text{ and } l\leq n,\ \ |x_{k,l}-L|\geq\varepsilon\}\right|$$

$$\geq\frac{1}{mn}\left|\{(k,l)\in I_{m,n}:\ \ |x_{k,l}-L|\geq\varepsilon\}\right|=\frac{\lambda_{m,n}}{mn}.\frac{1}{\lambda_{m,n}}\left|\{(k,l)\in I_{m,n}:\ \ |x_{k,l}-L|\geq\varepsilon\}\right|.$$

Taking the Pringsheim limit on m and n and using (3.1), we get desired result. This completes the proof. □

References

1. A. Pringsheim, Zur Theori der zweifach unendlichen Zahlenfolgen, Math. Ann. **53**(1900), 289–321.
2. G. M. Robinson, Divergent double sequences and series, Trans. Am. Math. Soc. **28**(1926), 50–73.
3. H. Fast, Sur la convergence statistique, Colloq. Math., **2**(1951), 241–244.
4. L. Leindler, Über die la Vallee-Pousinche summierbarkeit allgemeiner orthogonalreihen, Acta Math. Hung., **16**(1965), 375–378.
5. L. L. Silverman, On the definition of the sum of a divergent series, Ph.D. Thesis, University of Missouri Studies, Mathematics Series, 1913.
6. Mursaleen, $\lambda-$statistical convergence, Math. Slovaca, **50**(2000), 111–115.
7. M. Mursaleen and O. H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl. **288(1)**(2003), 223–231.
8. M. A. Krasnosel"ski and Y. B. Rutickii, Convex function and Orlicz spaces, Groningen, Nederland, 1961.
9. O. Toeplitz, Über allgenmeine linear Mittelbrildungen, Prace Mat.-Fiz. (Warzaw) **22**(1913), 113–119 (Polish).
10. W. H. Ruckle, FK-spaces in which the sequence of coordinate vectors is bounded, Canad. J. Math., **25**(1973), 973–978.

Adiyaman University, Science and Art Faculty, Department of Mathematics, 02040, Adiyaman, Turkey.

E-mail address: aesi23@hotmail.com