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Arens-irregularity of tensor product of Banach algebras

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Dedicated to the Memory of Charalambos J. Papaioannou

(Communicated by M. Eshaghi Gordji)

Abstract

We introduce Banach algebras arising from tensor norms. By these Banach algebras we make Arens regular Banach algebras such that α tensor product becomes irregular, where α is tensor norm. We illustrate injective tensor product, does not preserve bounded approximate identity and it is not algebra norm.

Keywords: Arens products, Arens regularity, compact operators, approximable operators, nuclear operators, tensor norm, approximate identity, approximation property.
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1. Introduction and preliminaries

1.1. Arens product

Let X and Y be Banach spaces. We denote all bounded linear operators from X to Y by $\mathcal{B}$(X,Y), compact operators from X to Y by $\mathcal{K}$(X,Y) and $\mathcal{F}$(X,Y) be finite rank operators. Let $\mathcal{A}$(E), the approximable operators, be the closure of $\mathcal{F}$(E) in $\mathcal{K}$(E). We denote the dual and bidual space of X by X^* and X^{**}. We use the dual pair notation $\langle\cdot,\cdot\rangle$, so we write $\langle\mu, x\rangle = \mu(x)$ for $\mu \in X^*$ and $x \in X$. Recall that there is a canonical isometry $K_X : X \longrightarrow X^{**}$ defined by $\langle K_X(x), \mu\rangle = \mu(x)$ for $\mu \in X^*$ and $x \in X$. We say that X is reflexive if K_X is an isomorphism. Let $\mathcal{A}$ be a Banach algebra. We define norm-decreasing bilinear maps $\mathcal{A}^* \times \mathcal{A} \longrightarrow \mathcal{A}^*$ and $\mathcal{A} \times \mathcal{A}^* \longrightarrow \mathcal{A}^*$ by

$$\langle\mu \cdot a, b\rangle = \langle\mu, ab\rangle, \quad \langle a \cdot \mu, b\rangle = \langle\mu, ba\rangle \quad (\mu \in \mathcal{A}^*,\ a, b \in \mathcal{A}),$$

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and we define norm-decreasing bilinear maps $\mathcal{A}^{**} \times \mathcal{A}^{*} \longrightarrow \mathcal{A}^{*}$ and $\mathcal{A}^{*} \times \mathcal{A}^{**} \longrightarrow \mathcal{A}^{*}$ by

$$\langle \Phi \cdot \mu, a\rangle = \langle \Phi, \mu \cdot a\rangle, \quad \langle \mu \cdot \Phi, a\rangle = \langle \Phi, a \cdot \mu\rangle \quad (\Phi \in \mathcal{A}^{**},\ \mu \in \mathcal{A}^{*},\ a \in \mathcal{A}).$$

Then we define norm-decreasing bilinear maps $\Box, \Diamond : \mathcal{A}^{**} \times \mathcal{A}^{**} \longrightarrow \mathcal{A}^{**}$ by

$$\langle \Phi \Box \Psi, \mu\rangle = \langle \Phi, \Psi \cdot \mu\rangle, \quad \langle \Phi \Diamond \Psi, \mu\rangle = \langle \Psi, \mu \cdot \Phi\rangle \quad (\Phi, \Psi \in \mathcal{A}^{**},\ \mu \in \mathcal{A}^{*}).$$

$\Box$ and $\Diamond$ are termed the first and second Arens products, respectively, and when $\Box$ and $\Diamond$ coincide, we say that $\mathcal{A}$ is Arens regular. We say that $\mathcal{A}$ is Arens irregular if $\Box \neq \Diamond$. We have that $a \cdot \Phi = K_{\mathcal{A}}(a) \Box \Phi = K_{\mathcal{A}}(a) \Diamond \Phi$ for $\Phi \in \mathcal{A}^{**}$ and $a \in \mathcal{A}$. So $K_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}^{**}$ becomes a homomorphism when $\mathcal{A}^{**}$ is given either Arens product. $\Box$ and $\Diamond$ are extend the product on $\mathcal{A}$. A good survey of results about the Arens product is [6], see also [[2],section 2.6]. If $\mathcal{A}$ is a reflexive as a Banach space, then $\mathcal{A}$ is trivially Arens regular. Consider the space $\ell^1(\mathbb{Z})$. If we define a product pointwise, then $\ell^1(\mathbb{Z})$ is Arens regular. Conversely, if we use the convolution product on $\ell^1(\mathbb{Z})$ coming from the additive group $\mathbb{Z}$, then $\ell^1(\mathbb{Z})$ is not Arens regular: in fact, the two Arens products agree only on the original algebra (see [13] or [11]). Every C^* algebra is Arens regular [2].

1.2. Tensor norms

We will now draw the definitions and results about tensor norms which we will need. Let X and Y be Banach spaces, and norm the tensor product $X \otimes Y$ by

$$\pi(u) = \inf\{\sum_{i=1}^{n} \|x_i\|\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i\} \quad (u \in X \otimes Y).$$

Then $\pi(\cdot)$ is the projective tensor norm, and the completion of $(X \otimes Y, \pi)$ is $X \widehat{\otimes} Y$ or $X \widehat{\otimes}_\pi Y$, the projective tensor product of X and Y. We define the injective tensor product ε of X and Y by

$$\varepsilon(u, X\widehat{\otimes}Y) = \sup\{|\sum_{i=1}^{n}\langle \mu, x_i\rangle\langle \lambda, y_i\rangle| : \mu \in X^*, \quad , \lambda \in Y^*, \quad \|\mu\| = \|\lambda\| = 1\},$$

where $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$. We denote the completion of $X \otimes Y$ respect to ε by $X \widehat{\otimes}_\varepsilon Y$ or $X \check{\otimes} Y$. In fact we can consider every tensor elements as a operator in $\mathcal{B}(X^*, Y)$ and injective norm is operator norm $\|.\|_o$. We can check that $\varepsilon(\cdot) \le \pi(\cdot)$, so $X \widehat{\otimes}_\pi Y \subseteq X \widehat{\otimes}_\varepsilon Y$.

Definition 1. *Let X and Y be normed vectors spaces and α be a norm on $X \otimes Y$. Then α is a reasonable crossnorm if $\varepsilon(\cdot) \le \alpha(\cdot) \le \pi(\cdot)$ for each $u \in X \otimes Y$. A uniform crossnorm is an assignment to each pair, (X, Y), of Banach spaces, of a reasonable crossnorm α on $X \otimes Y$ such that we have the following. Let E, X, F and Y be Banach spaces, and let $T \in B(E, X)$ and $S \in B(F, Y)$. Then we form the bilinear map*

$$T \otimes S : E \times F \longrightarrow X \otimes Y;\ (x, y) \longmapsto T(x) \otimes S(y) \quad (x \in E, y \in F),$$

which extends to $E \otimes F$ by the tensorial property. Then we insist that $\|T \otimes S\| \le \|T\|\|S\|$ with respect to the norm α on $E \otimes F$ and on $X \otimes Y$. For $u \in X \otimes Y$, we often write $\alpha(u, X \otimes Y)$, instead of just $\alpha(u)$, to avoid confusion. Let E be a closed subspace of X, let F be a closed subspace of Y, and $u \in E \otimes F$. By considering the inclusion maps $E \longrightarrow X$ and $F \longrightarrow Y$, we identify u with its image in $X \otimes Y$, and hence for a uniform crossnorm α, we see that

$$\alpha(u, X \otimes Y) \le \alpha(u, E \otimes F) \qquad (u \in E \otimes F).$$

Definition 2. *Let α be a uniform crossnorm. Then α is finitely generated if, for each pair of Banach spaces X and Y, and each $u \in X \otimes Y$, we have*

$$\alpha(u, X \otimes Y) = \inf\{\alpha(u, E \otimes F) : dimE < \infty, dimF < \infty, u \in E \otimes F\}.$$

We call a finitely generated uniform crossnorm a tensor norm. We denote the completion of the normed space $(X \otimes Y, \alpha)$ by $X\widehat{\otimes}_\alpha Y$.

Both the injective and projective tensor norms are tensor norms. We can check $(X \otimes Y)^* = X^* \otimes Y^*$, when X or Y is finite dimensional spaces.

Definition 3. *Let α be a tensor norm. Then the dual tensor norm to α is $\alpha^{'}$, and is given by setting*

$$(X\widehat{\otimes}_\alpha Y)^* = X^*\widehat{\otimes}_{\alpha'} Y^*$$

when X and Y are finite dimensional spaces and extending $\alpha^{'}$ to all Banach spaces by finite-generation. We can show that $\alpha^{'}$ is a tensor norm. Then we have that $\alpha^{''} = \alpha$, $\varepsilon^{'} = \pi$ and $\pi^{'} = \varepsilon$. So when X and Y are finite dimensional spaces, we have $(X\widehat{\otimes}_\pi Y)^* = X^*\widehat{\otimes}_\varepsilon Y^*$ and $(X\widehat{\otimes}_\varepsilon Y)^* = X^*\widehat{\otimes}_\pi Y^*$. For a tensor norm α define α^s by the embedding $X\widehat{\otimes}_{\alpha^s} Y \longrightarrow (X^*\widehat{\otimes}_\alpha Y^*)^*$ for any Banach spaces X and Y. Thus $\alpha^s = \alpha^{'}$ on finite dimensional spaces, but not, in general, on infinite-dimensional spaces.

Definition 4. *Let α be a tensor norm such that $(\alpha^{'})^s = \alpha^{''} = \alpha$ on $X \otimes Y$ whenever at least one of X and Y are in finite dimensional spaces. Then α is said to be accessible. Suppose further that we always have $(\alpha^{'})^s = \alpha$. Then α is totally accessible. We can show that ε is totally accessible, and that π is accessible [15].*

We refer the reader to [15] or [2] for more details on the tensor norms.

2. Nuclear and integral operators; the approximation property

We recommend [5] or [4] for better characteristics on the topic of this section.

Definition 5. *Let X and Y be Banach spaces. There is a natural, norm decreasing map $J : X^* \otimes Y \longrightarrow \mathcal{B}(X, Y)$ given by*

$$J(u) = J(\sum_{i=1}^{n} \mu_i \otimes y_i)(x) = \sum_{i=1}^{n} \langle \mu_i, x\rangle y_i \quad (u = \sum_{i=1}^{n} \mu_i \otimes y_i \in X^* \otimes Y).$$

We may check that J is onto $\mathcal{F}(X, Y)$ and also

$$\varepsilon(u) = \sup\{\sum_{i=1}^{n} \langle \mu_i, x\rangle y_i : \|x\| \leq 1\} = \|J(u)\|_o.$$

So J on $X^*\widehat{\otimes}_\varepsilon Y$ is isometry and we have $J(X^*\widehat{\otimes}_\varepsilon Y) = \overline{\mathcal{F}(X, Y)} = \mathcal{A}(X, Y)$. We will write from now $X^*\widehat{\otimes}_\varepsilon Y = \overline{\mathcal{F}(X, Y)} = \mathcal{A}(X, Y)$. We may index J by J_α on $X^*\widehat{\otimes}_\alpha Y$. We may check that $\|J_\alpha\|_o \leq \alpha(u)$ for $u \in X^* \otimes Y$, and so J_α is norm-decreasing, so $\widetilde{J_\alpha} : \frac{X^*\widehat{\otimes}_\alpha Y}{\ker J_\alpha} \longrightarrow J_\alpha(X^*\widehat{\otimes}_\alpha Y)$, defined by $\widetilde{J_\alpha}(u + \ker J_\alpha) = J_\alpha(u)$ for $u \in X^* \otimes Y$, is one to one and onto. The image of J_α equipped with the quotient norm, is the set of α-nuclear operators, denoted $\mathcal{N}_\alpha(X, Y)$, with norm $\|.\|_{N_\alpha}$. The nuclear operators, $\mathcal{N}(X, Y)$, are the π-nuclear operators and $\|.\|_{N_\pi} = \|.\|_N$.

We can check that the α-nuclear operators form an operator ideal, and so $\mathcal{N}(X)$ is Banach algebra (see [3]).

As shown in ([15], Chapter 2), for projective tensor product, we have that $(X\widehat{\otimes}_\pi Y)^* = \mathcal{B}(X, Y^*)$. As the swap map $X\widehat{\otimes}_\pi Y \longrightarrow Y\widehat{\otimes}_\pi X$ is an isometry, we can naturally identify $(X\widehat{\otimes}_\pi Y)^*$ with $\mathcal{B}(Y, X^*)$ as well as with $\mathcal{B}(X, Y^*)$.

Definition 6. *Let α be a tensor norm. As $\alpha \leq \pi$ for each pair of Banach spaces X and Y, the formal identity map $I_\alpha : X\widehat{\otimes}_\pi Y \longrightarrow X\widehat{\otimes}_\alpha Y$ is norm decreasing. For $\mu \in (X\widehat{\otimes}_\alpha Y)^*$, we have*

$$T := I_\alpha^*(\mu) \in (X\widehat{\otimes}_\pi Y)^* = \mathcal{B}(X, Y^*).$$

It is easy to see that

$$\langle \mu, \sum_{i=1}^{n} x_i \otimes y_i \rangle = \sum_{i=1}^{n} \langle T(x_i), y_i \rangle \quad (\sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y),$$

so that we can identify $(X\widehat{\otimes}_\pi Y)^*$ with a subspace of $\mathcal{B}(X, Y^*)$, denoted by $\mathcal{B}_{\alpha'}(X, Y^*)$, the α'-integral operators, and give it the norm $\|.\|_{\alpha'}$ induced by the identification of $\mathcal{B}_{\alpha'}(X, Y^*)$ with $(X\widehat{\otimes}_\pi Y)^*$. The ε-integral operators are just the bounded operators. We call the π-integral operators just the integral operators and denote them by $\mathcal{I}(X, Y^*) = \mathcal{B}_\pi(X, Y^*)$.

We can check that the α-integral operators form an operator ideal, so $\mathcal{I}(X^*)$ is a Banach algebra (see [3]).

Proposition 7. *Let X and Y be Banach spaces. Then the map $J_\alpha : X^*\widehat{\otimes}_\alpha Y \longrightarrow \mathcal{B}(X, Y)$ maps into $\mathcal{B}_\alpha(X, Y)$, and the arising inclusion $\mathcal{B}_\alpha(X, Y) \longrightarrow \mathcal{N}_\alpha(X, Y)$ is norm-decreasing; that is, $\|T\|_{N_\alpha} \geq \|T\|_\alpha$ for each $T \in \mathcal{N}_\alpha(X, Y)$.*

Proof . See ([15], Section 8.1). $\square$

We can see by ([4],Corollary 3.15) that $\mathcal{I}(X) \nsubseteq \mathcal{B}(X)$, when X is infinite-dimensional.

Proposition 8. *Let X and Y be Banach spaces, such that X or Y be a reflexive space, then $\mathcal{N}(X, Y^*) = \mathcal{I}(X, Y^*)$ with the same norm.*

Proof . It follows from ([4],Corollary 3.19). $\square$

Definition 9. *Let X be a Banach space. Then X has the approximation property if, for each compact set $K \subseteq X$ and each $\epsilon > 0$, there is $T \in \mathcal{F}(X)$ such that $\|T(x) - x\| < \epsilon$ for each $x \in K$. If there is $M > 0$, such that each T satisfying in above, be bounded to M, then we say X has the bounded approximation property. When $M = 1$, we say X has the metric approximation property.*

There are Banach spaces with the approximation property, but without the bounded approximation property (see [8]). We can show that for $1 \leq p \leq \infty$ and any measure μ, $L^p(\mu), \ell^p, C(X)$ for each compact space X, along with their duals have the metric approximation property. There are spaces without the approximation property (the first was constructed in [7]). In fact, for $p \neq 2$, ℓ^p contains subspaces without the approximation property (see [16]), and $\mathcal{B}(\ell^2)$ does not have the approximation property (see [17]).

Theorem 10. *Let X be a Banach space. Then X has the approximation property if and only if the map $J_\pi : X^*\widehat{\otimes}_\pi X \longrightarrow \mathcal{N}(X)$ is injective.*

Proof . See ([15], Proposition 4.6]). □

Theorem 11. *Let X and Y be Banach spaces and α be tensor norm. If either X^* or Y has the approximation property, then $\mathcal{N}_\alpha(X,Y) = X^*\widehat{\otimes}_\alpha Y$.*

Proof . See ([15], Proposition 8.7). □

Proposition 12. *If either X^* or Y has the approximation property, then $\mathcal{K}(X,Y) = X^*\widehat{\otimes}_\varepsilon Y$.*

Proof . See ([15], Corollary 4.13). □

We may check that, if X and Y are Banach spaces with the bounded(metric) approximation property, then both $X\widehat{\otimes}_\pi Y$ and $X\widehat{\otimes}_\varepsilon Y$ have the bounded(metric) approximation property(see [15], chapter 4).

Proposition 13. *Let X be a Banach space such that X^* has the (bounded) approximation property. Then X has the (bounded) approximation property.*

Proof . See ([5], Section 16.3). □

The converse of this result is false; there exist Banach spaces, which have the approximation property whose duals do not have this property. For example, the aforementional example of [7], $\mathcal{B}(\ell^2)$, is dual of $\ell^2\widehat{\otimes}_\pi\ell^2$, which has the approximation property.

3. Arens-irregularity of tensor products

We may launch this section by the following theorem, which is beneficial for next result.

Theorem 14. *Let X be a infinite-dimensional reflexive Banach space. If X has the approximation property, then $\mathcal{K}(X)$ is not reflexive.*

Proof . Since X is reflexive and has the approximation property, so do X^*, and by Proposition 12 and 8, we have

$$\mathcal{K}(X)^* = \mathcal{A}(X)^* = (X^*\widehat{\otimes}_\varepsilon X)^* = \mathcal{I}(X^*) = \mathcal{N}(X^*) = X^{**}\widehat{\otimes}_\varepsilon X^*,$$

so that

$$\mathcal{K}(X)^{**} = (X^{**}\widehat{\otimes}_\varepsilon X^*)^* = \mathcal{B}(X^{**}) = \mathcal{B}(X).$$

On the other hand X is infinite-dimensional space, so $Id_X \notin \mathcal{K}(X)$, and $\mathcal{K}(X) \nsubseteq \mathcal{K}(X)^{**}$. Therefore $\mathcal{K}(X)$ is not reflexive. □

ℓ^2 is an example for the exitance of X in the Theorem 14.

Theorem 15. *Let X be a reflexive Banach space, let α be a tensor norm. Suppose that either α is accessible, or X has the approximation property. Then $\mathcal{N}_\alpha(X)$ is Arens regular.*

Proof . See ([4], Theorem 5.39) □

Both π and ε are accessible, so when X is a reflexive Banach space, $\mathcal{N}(X)$ and $\mathcal{A}(X)$ are Arens regular. For $\mathcal{A}(X)$ the converse is true (See [18]). In fact $\mathcal{A}(X)$ is Arens regular if and only if X is a reflexive Banach space.

There are many Arens regular Banach algebras which projective tensor product of them is Arens regular. For example, Hilbert space is C^*-algebra, so is Arens regular Banach algebra, and projective tensor product of two Hilbert spaces is again Hilbert space, so is Arens regular algebra. Is it true always? Is it true for the other tensor products? We may make Arens regular Banach algebras, that projective tensor product of them is not Arens regular Banach algebras. Also we may make some special Arens regular Banach algebras, that α tensor product of them is Banach algebras, but is not Arens regular.

Example 16. *Let X be a reflexive Banach space and be infinite-dimensional space and X has the approximation property. We have $\mathcal{K}(X)^* = \mathcal{A}(X)^* = \mathcal{N}(X^*)$. $\mathcal{A}(X)$ and $\mathcal{N}(X^*)$ are Arens regular Banach algebras. Let $\mathcal{A} = \mathcal{K}(X)$ and $\mathcal{B} = \mathcal{K}(X)^*$. Both $\mathcal{A}$ and $\mathcal{B}$ are Arens regular Banach algebras, and by Theorem 14, $\mathcal{K}(X)$ is not reflexive. But we have $\mathcal{B}\widehat{\otimes}_\varepsilon\mathcal{A} = \mathcal{A}(\mathcal{K}(X))$, so by Theorem 15, $\mathcal{B}\widehat{\otimes}_\varepsilon\mathcal{A}$ is Arens irregular.*

Corollary 17. *Injective tensor product does not preserve Arens regularity. It means if $\mathcal{A}$ and $\mathcal{B}$ are Arens regular Banach algebras, and $\mathcal{A}\widehat{\otimes}_\varepsilon\mathcal{B}$ be Banach algebra, then $\mathcal{A}\widehat{\otimes}_\varepsilon\mathcal{B}$ is not necessarily Arens regular Banach algebra.*

Now we will introduce examples like above example for more tensor norms. First consider following theorem.

Theorem 18. *Let X be a Banach space which is not reflexive, let α be an accessible tensor norm. Then $\mathcal{N}_\alpha(X)$ is not Arens regular Banach algebra.*

Proof . See ([4], Theorem 5.14). □

Projective norm is an accessible tensor norm, so when X is not reflexive, $\mathcal{N}(X)$ is not Arens regular.

Example 19. *Let α be an accessible tensor norm. ℓ^2 has all conditions of Theorem 14. So $\mathcal{K}(\ell^2)$ is not reflexive. Let $\mathcal{A} = \mathcal{K}(\ell^2)$ and $\mathcal{B} = \mathcal{K}(\ell^2)^* = \mathcal{N}(\ell^{2*})$. Again both $\mathcal{A}$ and $\mathcal{B}$ are Arens regular Banach algebras. ℓ^2 has the (metric) approximation property, so $\ell^2\widehat{\otimes}_\varepsilon\ell^2$ does. ℓ^2 is a Hilbert space, so we have*

$$\mathcal{B} = \mathcal{N}(\ell^{2*}) = \ell^{2**}\widehat{\otimes}_\varepsilon\ell^{2*} = \ell^2\widehat{\otimes}_\varepsilon\ell^2.$$

So $\mathcal{B}$ has the approximation property. By Proposition 11, $\mathcal{B}\widehat{\otimes}_\alpha\mathcal{A} = \mathcal{N}_\alpha(\mathcal{K}(\ell^2))$. $\mathcal{K}(\ell^2)^*$ has the approximation property, so $\mathcal{K}(\ell^2)$ has the approximation property as a Banach space. Also $\mathcal{K}(\ell^2)$ is not reflexive, So by Theorem 18, $\mathcal{N}_\alpha(\mathcal{K}(\ell^2))$ is Arens irregular, so that $\mathcal{B}\widehat{\otimes}_\alpha\mathcal{A}$ is Arens irregular. Specially, $\mathcal{N}(\mathcal{K}(\ell^2))$ is Arens irregular.

Corollary 20. *Let α be an accessible tensor norm, then α tensor product does not preserve Arens regularity. It means if $\mathcal{A}$ and $\mathcal{B}$ are Arens regular Banach algebras, and $\mathcal{A}\widehat{\otimes}_\alpha\mathcal{B}$ be Banach algebra, then $\mathcal{A}\widehat{\otimes}_\alpha\mathcal{B}$ is not necessarily Arens regular Banach algebra. Furthermore, projective tensor product does not preserve Arens regularity.*

4. approximate identity and tensor products

Definition 21. *Let $\mathcal{A}$ be a Banach algebra. A bounded net $\{e_\gamma\}$ in $\mathcal{A}$ is a bounded left (right) approximate identity when $e_\gamma a \longrightarrow a(ae_\gamma \longrightarrow a)$ for each $a \in \mathcal{A}$. A bounded approximate identity is a bounded left and right approximate identity.*

Every C^*-algebra has an approximate identity [2]. For $p > 1$ and locally compact group G, $L^p(G)$ don't have bounded left approximate identity ([10], Theorem 34.40).

Projective tensor norm is algebra norm [2]. It means if $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras, then $\mathcal{A}\widehat{\otimes}_\pi\mathcal{B}$ is Banach algebra. But injective tensor norm is not algebra norm. By the following processes we illustrate another reason to show injective tensor norm, is not algebra norm.

We use the terminology of [12] for the following definition.

Definition 22. *Let X and Y be Banach spaces. A seminorm ρ on $X \otimes Y$ is called admissible if there are positive constant m and M such that $m\varepsilon \leq \rho \leq M\pi$.*

Every tensor norm is admissible norm.

Theorem 23. *Let ρ be an admissible algebra norm on $\mathcal{A} \otimes \mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ each possess a bounded left (right) approximate identity so does $\mathcal{A}\widehat{\otimes}_\rho\mathcal{B}$. Conversely, if $\mathcal{A}\widehat{\otimes}_\rho\mathcal{B}$ has a bounded left (right) approximate identity then so do $\mathcal{A}$ and $\mathcal{B}$.*

Proof . See ([12] Theorem 2). □

Theorem 24. *Let X be Banach space. X^* has the bounded approximation property if and only if $\mathcal{A}(X)$ has a bounded approximate identity.*

Proof . See ([9], Theorem 3.3). □

Example 25. *Assume G be locally compact group. $L^2(G)$ dose not have bounded approximate identity. By [2], $L^2(G)$ is a Hilbert space. So $L^2(G)^* = L^2(G)$. Let $\mathcal{A} = L^2(G)$ and $\mathcal{B} = L^2(G)^*$. We have $\mathcal{B}\widehat{\otimes}_\varepsilon\mathcal{A} = \mathcal{K}(L^2(G))$. $\mathcal{B}$ has the bounded approximation property, so by Theorem 24, $\mathcal{B}\widehat{\otimes}_\varepsilon\mathcal{A}$ has bounded approximate identity, while both $\mathcal{A}$ and $\mathcal{B}$ do not have bounded approximate identity. So by Theorem 23, we have following corollary.*

Corollary 26. *Injective tensor norm is not norm algebra.*

5. Conclusion

We have demonstrated that for accessible tensor norms α Arens regularity of tensor product of Banach algebras is not preserved. Is it true for non-accessible tensor norms? To answer this question, we need to find Banach spaces without having the bounded approximation property to establish the result in [4].

By this fact that having the bounded approximation property for dual Banach space X, X^*, is equivalent to have bounded approximate identity for the approximate operator, $\mathcal{A}(X)$, we could conclude that injective tensor norm is not algebra norm. To find which one of tensor norms is not algebra norm, we want to know if there any relationship between having the bounded approximation property for Banach space X or its dual, X^*, and having bounded approximate identity for $\alpha-$Nuclear operators on X.

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References

[1] P. Civin and B. Yood, The second conjugate space of a Banach algebra as an algebra, Pacific J. Math. 11 (1961) 847-870.

[2] H. G. Dales, Banach Algebras and Automatic Continuity, London Mathematical Society Monographs, New Series, 24. Oxford University Press, New York, 2000.

[3] M. Daws, Arens regularity of the algebra of operators on a Banach space, Bull. London Math. Soc. 36 (2004) 493-503.

[4] M. Daws, Arens-regularity of algebras arising from tensor norms, New York J. Math. 13 (2007) 215-270.

[5] A. Defant and K. Floret, Tensor norms and operator ideals, North- Holland Mathematics Studies, 176. North-Holland Publishing Co., Amsterdam, 1993.
[6] J. Duncan and S. A. R. Hosseiniun, The second dual of a Banach algebra, Proc. Roy. Soc. Edinburgh Sect. A 84 (1979) 309-325.
[7] P. Enflo, A counterexample to the approximation problem in Banach spaces, Acta Math. 130 (1973) 309-317.
[8] T. Figiel and W. B. Johnson, The approximation property does not imply the bounded approximation property, Proc. Amer. Math. Soc. 41 (1973) 197-200.
[9] N. Gronbaek and George A. Willis, Approximate identities in Banach algebras of compact operators, Canad. Math. Bull. 36 (1993) 45-53.
[10] E. Hewitt and K. A. Ross, Abstract Harmonic Analysis, I , II , Springer, Berlin 1963, 1970.
[11] A. Lau and V. Losert, On the second conjugate algebra of $L^1(G)$ of a locally compact group, J. London Math. Soc. 37 (1988) 464-470.
[12] R. J. Loy, Identities in tensor products of Banach algebras, Bull. Austral. Math. Soc. Vol. 2 (1970), 253-260.
[13] M. Neufang, A unified approach to the topological centre problem for certain Banach algebras arising in abstract harmonic analysis, Arch. Math. 82(2004) 164-171.
[14] Theodore W. Palmer, Banach Algebras and the General Theory of *-Algebras, Vol. 1. Algebras and Banach algebras, Encyclopedia of Mathematics and its Applications, 49. Cambridge University Press, Cambridge, 1994.
[15] Raymond A. Ryan, Introduction to Tensor Products of Banach Spaces, Springer Monographs in Mathematics, Springer-Verlag, London, 2002.
[16] A. Szankowski, Subspaces without the approximation property, Israel J. Math. 30 (1978) 123-129.
[17] A. Szankowski, B(H) does not have the approximation property, Acta Math. 147 (1981) 89-108.
[18] N. J. Young, Periodicity of functionals and representations of normed algebras on reflexive spaces, Proc. Edinburgh Math. Soc. (2) 20 (1976/77) 99-120.