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Contractive maps in Mustafa-Sims metric spaces

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Dedicated to the Memory of Charalambos J. Papaioannou

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Abstract

The fixed point result in Mustafa-Sims metrical structures obtained by Karapinar and Agarwal [Fixed Point Th. Appl., 2013, 2013:154] is deductible from a corresponding one stated in terms of anticipative contractions over the associated (standard) metric space.

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1. Introduction

Let X be a nonempty set; and $d : X \times X \to R_+ := [0, \infty[$ be a *metric* over it; the couple (X, d) is called a *metric space*. Call the subset Y of X, *almost singleton* (in short: *asingleton*), provided $[y_1, y_2 \in Y$ implies $y_1 = y_2]$; and *singleton*, if, in addition, Y is nonempty; note that, in this case, $Y = \{y\}$, for some $y \in X$. Further, let $T \in \mathcal{F}(X)$ be a selfmap of X. [Here, given $A, B \neq \emptyset$, $\mathcal{F}(A, B)$ stands for the class of all functions $f : A \to B$; if $A = B$, we write $\mathcal{F}(A)$ in place of $\mathcal{F}(A, A)$]. Denote $\text{Fix}(T) = \{x \in X; x = Tx\}$; each point of this set is referred to as *fixed* under T. The determination of such elements is to be performed in the context below, comparable with the one in Rus [32, Ch 2, Sect 2.2]:

1a) We say that T is a *Picard operator* (modulo d) if, for each $x \in X$, the iterative sequence $(T^n x; n \geq 0)$ is d-convergent

1b) We say that T is a *strong Picard operator* (modulo d) if, for each $x \in X$, $(T^n x; n \geq 0)$ is d-convergent, and $\lim_n(T^n x)$ belongs to $\text{Fix}(T)$

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1c) We say that T is a *globally strong Picard operator* (modulo d), if it is a strong Picard operator (modulo d), and (in addition), Fix(T) is an asingleton (or, equivalently: singleton).

The sufficient (regularity) conditions for such properties are being founded on *orbital* concepts (in short: o-concepts). Namely, call the sequence $(z_n; n \geq 0)$ in X, *orbital (modulo T)*, when it is a subsequence of $(T^n x; n \geq 0)$, for some $x \in X$.

1d) Call (X, d), *o-complete*, provided (for each o-sequence) d-Cauchy $\Longrightarrow$ d-convergent

1e) We say that T is (o, d)-*continuous*, if $[(z_n)$=o-sequence and $z_n \xrightarrow{d} z]$ imply $Tz_n \xrightarrow{d} Tz$.

When the orbital properties are ignored, these conventions may be written in the usual way; we do not give details.

Concerning the existence results for such points, a basic one was obtained in 1974 by Ćirić [8]. Call the selfmap T, $(d; \alpha)$-*contractive* (where $\alpha \geq 0$), provided

(a01) $(\forall x, y \in X)$: $d(Tx, Ty) \leq$
$\alpha \max\{d(x, Tx), d(x, y), d(x, Ty), d(Tx, y), d(y, Ty)\}$.

Note that, by the definition of "max" operator, this property gives its "implicit" version

(a02) $(\forall x, y \in X)$: $d(Tx, Ty) \leq \alpha A(x, y)$;
where $A(x, y) = \text{diam}[T(x; 1) \cup T(y; 1)]$.

Here, $\text{diam}(U) = \sup\{d(x, y); x, y \in U\}$ is the *diameter* of the subset $U \subseteq X$; and

$T(x; n) := \{T^i x; 0 \leq i \leq n\}$, $x \in X$, $n \geq 0$;

referred to as: the *orbital n-segment* generated by x. The reciprocal inclusion [(a02) $\Longrightarrow$ (a01)] is also true, when $0 \leq \alpha < 1$, as it can be directly seen.

Theorem 1.1. *Suppose that T is $(d; \alpha)$-contractive, for some $\alpha \in [0, 1[$. In addition, let (X, d) be o-complete. Then, T is globally strong Picard (modulo d).*

This result extends the ones in Banach [5], Kannan [19], and Zamfirescu [39]; see also Hardy and Rogers [16]. Since all quoted statements have a multitude of applications to operator equations, Theorem 1.1 was the subject of many extensions. The most natural one is to pass from the "linear" type contraction above to (implicit) "functional "contractive conditions like

(a03) $F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0$,
for all $x, y \in X$;

where $F : R_+^6 \to R$ is a function. For a basic extension of this type, we refer to Daneš [9]; further choices of F may be found in Rhoades [31] and the references therein. Note that, all such conditions are *non-anticipative*; i.e., the right member of (a03) does not contain terms like $d(T^i u, T^j v)$, $u, v \in \{x, y\}$, where $i + j \geq 3$; so, the question arises of to what extent it is possible to have *anticipative* contractions (in the above sense). A positive answer to this was recently obtained, in the "linear" setting of Theorem 1.1, by Dung [12]. It is our aim in the present exposition to give a further extension of this last result, within the functional context we just quoted. As an argument for its usefulness, a fixed point theorem in Mustafa-Sims metric spaces due to Karapinar and Agarwal [20] is derived. This, among others, shows that a reduction of their statement to standard metrical ones is possible, along the lines described by Jleli and Samet [18]; in contradiction with authors' claim. Further aspects will be delineated elsewhere.

2. Functional anticipative contractions

Let (X, d) be a metric space; and T be a selfmap of X. In the following, we are interested to solve the problem of introductory part with the aid of (implicit) contractive conditions like

(b01) $(\forall x, y \in X)$: $d(Tx, Ty) \le \Phi(d(x, Tx), d(x, T^2x), d(x, y), d(x, Ty);$
$d(Tx, T^2x), d(Tx, y), d(Tx, Ty); d(T^2x, y), d(T^2x, Ty); d(y, Ty));$

where $\Phi : R_+^{10} \to R_+$ is a certain function. As precise, these conditions are anticipative counterparts of the (non-anticipative) condition (a03). To describe them, some conventions are needed. Given $\varphi \in \mathcal{F}(R_+)$, we say that T is *anticipative* $(d; \varphi)$*-contractive*, provided

(b02) $(\forall x, y \in X)$: $d(Tx, Ty) \le \varphi(B(x, y))$;
where $B(x, y) = \mathrm{diam}[T(x; 2) \cup T(y; 1)]$.

The functions φ to be considered here are introduced as follows. Call $\varphi \in \mathcal{F}(R_+)$, *increasing*, provided [$t_1 \le t_2$ implies $\varphi(t_1) \le \varphi(t_2)$]; denote the class of all these as $\mathcal{F}(in)(R_+)$. For an easy reference, we list the basic properties for such functions to be used further:

i) Given $\varphi \in \mathcal{F}(in)(R_+)$, we say that it is *regressive*, in case

$\varphi(t) < t$, for all $t > 0$; hence, $\varphi(0) = 0$.

Note that this property holds in case of φ being *super regressive*:

$\varphi(s + 0) < s$, for all $s > 0$; hence, $\varphi(0) = 0$.

Here, as usually, $\varphi(s + 0) = \lim_{t \to s+} \varphi(t)$ is the *right limit* of φ at $s > 0$.

ii) Call $\varphi \in \mathcal{F}(in)(R_+)$, *Matkowski admissible* [22], provided

(b03) $\varphi^n(t) \to 0$ as $n \to \infty$, for all $t > 0$;

here, for each $n \ge 0$, φ^n stands for the n-th *iterate* of φ. Note that, any such function is regressive; cf. Matkowski [23]. On the other hand, each super regressive $\varphi \in \mathcal{F}(in)(R_+)$ is Matkowski admissible; an implicit proof of this may be found in Boyd and Wong [6].

iii) For the last property, we need a convention. Let $\varphi \in \mathcal{F}(in)(R_+)$ be regressive. Denote $\psi(t) = t - \varphi(t)$, $t \in R_+$; it is an element of $\mathcal{F}(R_+)$; referred to as the *complement* of φ. Remember that, the *coercive* property for the complement function $\psi(.)$ means:

(b04) $\lim_{t\to\infty}(\psi(t)) = \infty$: i.e.: $\forall \alpha > 0$, $\exists \beta > \alpha$: $[t > \beta \Longrightarrow \psi(t) > \alpha]$.

By definition, it will be referred to as: φ is *complement coercive*; note that, passing to the negation operator, this property may be written as:

(b05) $\forall \alpha > 0$, $\exists \beta > \alpha$: $[t \le \alpha + \varphi(t) \Longrightarrow t \le \beta]$.

As a consequence, the function

(b06) $\chi(r) = \sup\{t \in R_+; t \le r + \varphi(t)\}$, $r \in R_+$

is well defined, as an element of $\mathcal{F}(in)(R_+)$; moreover (as $\varphi \in \mathcal{F}(in)(R_+)$ is regressive)

$$\chi(0) = 0;\ \chi(r) \ge r, \text{ for all } r > 0. \tag{2.1}$$

We are now in position to state our basic result of this section.

Theorem 2.1. *Suppose that T is anticipative $(d;\varphi)$-contractive, for some regressive, Matkowski admissible, and complement-coercive function $\varphi \in \mathcal{F}(in)(R_+)$. In addition, let (X,d) be o-complete; and one of the extra conditions below holds*

2a) *T is o-continuous [(x_n)=o-sequence and $x_n \xrightarrow{d} x$ imply $Tx_n \xrightarrow{d} Tx$]*

2b) *φ is super regressive [$\varphi(s+0) < s$, $\forall s > 0$].*

Then, T is globally strong Picard (modulo d); i.e.,

j) $\mathrm{Fix}(T) = \{z\}$, *for some $z \in X$*

jj) $T^n x \xrightarrow{d} z$ *as $n \to \infty$, for each $x \in X$.*

Proof . We firstly check the asingleton property of $\mathrm{Fix}(T)$. Let $z_1, z_1 \in \mathrm{Fix}(T)$; and suppose by contradiction that $z_1 \neq z_2$; hence, $d(z_1,z_2) > 0$. Clearly,

$$T(z_1;2) = \{z_1\},\ T(z_2;1) = \{z_2\};\ \text{whence, } B(z_1,z_2) = d(z_1,z_2);$$

so that, by the contractive condition (and φ=regressive)

$$d(z_1,z_2) = d(Tz_1,Tz_2) \leq \varphi(d(z_1,z_2)) < d(z_1,z_2);$$

contradiction. Hence, necessarily $z_1 = z_2$; and the asingleton property follows. It remains now to establish the strong Picard property (modulo d) for T. Fix some $x_0 \in X$; and put $(x_n = T^n x_0; n \geq 0)$; clearly, this is an orbital sequence. If $x_n = x_{n+1}$ for some $n \geq 0$, we are done; so, without loss, one may assume that

(b07) $x_n \neq x_{n+1}$ (hence, $\rho_n := d(x_n,x_{n+1}) > 0$), $\forall n$.

Remember that, for each $x \in X$ and each $n \geq 0$, $T(x;n) = \{T^i x; 0 \leq i \leq n\}$ stands for the *orbital n-segment* generated by x. Put also

(b08) $T(x;\infty) = \{T^i x; i \geq 0\} = \cup\{T(x;n); n \geq 0\}$;

and call it: the *orbital set* generated by x. By the introduced notations, we have, for each $k \geq 0$,

$$T(x_k;n) = \{x_h; k \leq h \leq k+n\},\ n \geq 0;\ T(x_k;\infty) = \{x_h; h \geq k\}. \quad (2.2)$$

Moreover, by the working hypothesis above,

$$\mathrm{diam}T(x_k;n) \geq \rho_k := d(x_k,x_{k+1}) > 0, \text{ for all } k \geq 0,\ n \geq 1. \quad (2.3)$$

There are several steps to be passed.

I) We start with the following useful evaluation

Lemma 2.2. *Under the introduced notations,*

$$d(x_i,x_j) \leq \varphi(\mathrm{diam}T(x_{i-1};j-i+1)), \text{ whenever } 1 \leq i \leq j. \quad (2.4)$$

Proof .(Lemma 2.2) The case of $i = j$ is clear; so, without loss, one may assume $i < j$; hence, $i+1 \leq j$. By definition (and (2.2) above)

$$B(x_{i-1},x_{j-1}) = \mathrm{diam}[T(x_{i-1};2) \cup T(x_{j-1};1)] = \mathrm{diam}\{x_{i-1},x_i,x_{i+1},x_{j-1},x_j\}$$
$$\leq \mathrm{diam}\{x_n; i-1 \leq n \leq j\} = \mathrm{diam}T(x_{i-1};j-i+1);$$

wherefrom, combining with the contractive condition,

$$d(x_i,x_j) \leq \varphi(B(x_{i-1},x_{j-1})) \leq \varphi(\mathrm{diam}T(x_{i-1};j-i+1)).$$

This ends the argument. □

II) The following consequence of this fact is to be noted.

Lemma 2.3. *Denote* $\alpha = \rho_0 := d(x_0, x_1)(> 0)$*, and* $\beta = \chi(\alpha)$ *(hence,* $\beta \geq \alpha$*; see above). Then,*

$$\text{diam}T(x_0; n) \leq \beta, \quad \textit{for all } n \geq 1; \tag{2.5}$$

so, necessarily, $\text{diam}T(x_0; \infty) \leq \beta$*.*

Proof .(Lemma 2.3) The case $n = 1$ is clear, via $\beta \geq \alpha$; so, we may assume that $n \geq 2$. For each (i, j) with $1 \leq i \leq j \leq n$, we have, by Lemma 2.2,

$$d(x_i, x_j) \leq \varphi(\text{diam}T(x_{i-1}; j - i + 1)) \leq \varphi(\text{diam}T(x_0; n)) < \text{diam}T(x_0; n);$$

so that, necessarily,

$$\text{diam}T(x_0; n) = d(x_0, x_k), \text{ for some } k \in \{1, ..., n\}.$$

On the other hand, the same auxiliary statement gives

$$d(x_1, x_k) \leq \varphi(\text{diam}T(x_0; k)) \leq \varphi(\text{diam}T(x_0; n)).$$

Putting these together yields, by the triangle inequality,

$$\text{diam}T(x_0; n) = d(x_0, x_k) \leq d(x_0, x_1) + d(x_1, x_k) \leq$$
$$d(x_0, x_1) + \varphi(\text{diam}T(x_0; n)) = \alpha + \varphi(\text{diam}T(x_0; n));$$

wherefrom, $\text{diam}T(x_0; n) \leq \chi(\alpha) = \beta$; as claimed. □

III) The following d-Cauchy property of our iterative sequence is now available.

Lemma 2.4. *With the same notations as before, one has*

$$\text{diam}T(x_n; \infty) \leq \varphi^n(\beta), \ \textit{for all } n \geq 0; \tag{2.6}$$

hence, $(x_n; n \geq 0)$ *is a* d*-Cauchy o-sequence.*

Proof .(Lemma 2.4) The case $n = 0$ is established in Lemma 2.3; so, we may assume that $n \geq 1$. By Lemma 2.2 one has, for each (i, j) with $n \leq i < j$,

$$d(x_i, x_j) \leq \varphi(\text{diam}T(x_{i-1}, j - i + 1)) \leq \varphi(\text{diam}T(x_{n-1}; \infty)).$$

Passing to supremum over such (i, j), yields $\text{diam}T(x_n; \infty) \leq \varphi(\text{diam}T(x_{n-1}; \infty))$. After n steps, one thus gets

$$\text{diam}T(x_n; \infty) \leq \varphi^n(\text{diam}T(x_0; \infty)) \leq \varphi^n(\beta);$$

and conclusion follows. □

IV) As (X, d) is o-complete, $x_n \xrightarrow{d} z$, for some (uniquely determined) $z \in X$. There are two alternatives to be discussed.

Case IV-1. Suppose that T is o-continuous. Then, $(y_n := Tx_n = x_{n+1}; n \geq 0)$, d-converges to Tz. On the other hand, $(y_n; n \geq 0)$ is a subsequence of $(x_n; n \geq 0)$; so that, $y_n \xrightarrow{d} z$. As d is sufficient, this yields $z = Tz$.

Case IV-2. Suppose that φ is super regressive. To get the desired fact, we use a *reductio ad absurdum* argument. Namely, assume that $z \neq Tz$; i.e., $b := d(z, Tz) > 0$. From the contractive property, we have

$$d(x_{n+1}, Tz) \leq \varphi(B(x_n, z)), \text{for each } n \geq 0; \tag{2.7}$$

where (cf. the previous notations),

$$B(x_n, z) = \mathrm{diam}[T(x_n, 2) \cup T(z; 1)] = \mathrm{diam}\{x_n, x_{n+1}, x_{n+2}, z, Tz\},\ n \geq 0.$$

Note that, by the d-continuity of the map $(x, y) \mapsto d(x, y)$, the sequence $(\lambda_n := d(x_{n+1}, Tz); n \geq 0)$ fulfills $\lambda_n \to b$ as $n \to \infty$. On the other hand, by the very definition above, the sequence $(\mu_n := B(x_n, z); n \geq 0)$ fulfills

$$\mu_n \geq b,\ \forall n;\ \mu_n \to b,\ \text{as } n \to \infty.$$

There are two sub-cases to discuss.

Sub-case IV-2-1. Suppose that

(b09) for each $h \geq 0$, there exists $k > h$, such that $\mu_k = b$.

As a consequence, there exists a sequence of ranks $(i(n); n \geq 0)$ with $i(n) \to \infty$ as $n \to \infty$, such that $\mu_{i(n)} = b$, $\forall n$. Passing to limit as $n \to \infty$, over this subsequence, in the contractive property (2.7), yields $b \leq \varphi(b)$; contradiction.

Sub-case IV-2-2. Assume that the opposite alternative is true: there exists a certain rank $h \geq 0$, such that

(b10) $n > h \Longrightarrow \mu_n > b$; hence $\mu_n \to b+$ as $n \to \infty$.

Passing to limit in the same contractive property (2.7), gives $b \leq \varphi(b+0) < b$; again a contradiction. Summing up, the working hypothesis about $z \in X$ cannot be accepted; so, we necessarily have $z = Tz$. The proof is thereby complete. □

In particular, assume that the function φ is linear; i.e.: $\varphi(t) = \alpha t$, $t \in R_+$, for some $\alpha \in [0, 1[$. Then, φ is increasing, super regressive, Matkowski admissible and complement-coercive. By Theorem 2.1 we get the fixed point statement in Dung [12]. Given $\alpha \geq 0$, call T, *anticipative* $(d; \alpha)$*-contractive*, provided

(b11) $(\forall x, y \in X)$: $d(Tx, Ty) \leq \alpha B(x, y)$;
where (see above) $B(x, y) = \mathrm{diam}\{x, Tx, T^2x, y, Ty\}$.

Theorem 2.5. *Suppose that T is anticipative $(d; \alpha)$-contractive, for some $\alpha \in [0, 1[$. In addition, let (X, d) be o-complete. Then, T is globally strong Picard (modulo d).*

(C) For the applications to be considered, the following particular case of this theorem will be useful. Denote, for $x, y \in X$,

(b12) $P(x, y) := \max\{d(x, Tx) + d(Tx, y), d(T^2x, y) + d(T^2x, Ty),$
$d(Tx, T^2x) + d(Tx, y), d(Tx, y) + d(Tx, Ty), d(x, y), d(x, Ty), d(y, Ty)\}$,
(b13) $Q(x, y) := \max\{d(x, Tx) + d(Tx, T^2x), d(x, Tx) + d(Tx, y),$
$d(T^2x, Ty) + d(y, Ty), d(Tx, T^2x) + d(T^2x, Ty), d(x, y), d(x, Ty)\}$.

Further, given some $\gamma \geq 0$, we say that T is $(d, P, Q; \gamma)$*-contractive*, provided

(b14) $(\forall x, y \in X)$: $d(Tx, Ty) \leq \gamma \max\{P(x, y), Q(x, y)\}$.

Note that, by the convention above, this contractive condition is anticipative.

The following fixed point result is available.

Theorem 2.6. *Suppose that T is $(d, P, Q; \gamma)$-contractive, for some $\gamma \in [0, 1/2[$. In addition, let (X, d) be complete. Then, T is globally strong Picard (modulo d).*

Proof . By the very conventions above, one has

$$P(x,y), Q(x,y) \leq 2B(x,y),\ \forall x,y \in X.$$

So, by the accepted contractive conditions, it follows that

$$(\forall x,y \in X):\ d(Tx,Ty) \leq 2\gamma B(x,y).$$

Hence, the preceding result applies, with $\alpha = 2\gamma$. This ends the argument. □

As a consequence, Theorem 2.6 is indeed reducible to the developments above. However, for simplicity reasons, it would be useful having a separate proof of it.

Proof .(Theorem 2.6) [**Alternate**] First, we establish the asingleton property of Fix(T). Let r,s be two points in Fix(T). By definition, $P(r,s) = 2d(r,s)$, $Q(r,s) = d(r,s)$; so that, from the contractive condition,

$$d(r,s) = d(Tr,Ts) \leq 2\gamma d(r,s).$$

This, along with $0 \leq 2\gamma < 1$, yields $d(r,s) = 0$; whence, $r = s$. It remains now to establish the strong Picard (modulo d) property of T. To this end, we start from

$$P(x,Tx), Q(x,Tx) \leq d(x,Tx) + d(Tx,T^2x),\ \forall x \in X. \tag{2.8}$$

By the contractive condition, we therefore get

$$d(Tx,T^2x) \leq \beta d(x,Tx),\ \forall x \in X; \tag{2.9}$$

where $0 \leq \beta := \gamma/(1-\gamma) < 1$. Fix some $x_0 \in X$; and put $(x_n = T^n x_0; n \geq 0)$. By the above evaluation,

$$d(x_n, x_{n+1}) \leq \beta^n d(x_0,x_1),\ \forall n.$$

This tells us that $(x_n; n \geq 0)$ is a d-Cauchy sequence. As (X,d) is complete, there must be some (uniquely determined) $r \in X$ such that $x_n \xrightarrow{d} r$. We claim that $r = Tr$; and this completes the argument. By the contractive condition,

$$d(x_{n+1},Tr) \leq \gamma \max\{P(x_n,r), Q(x_n,r)\},\ \forall n. \tag{2.10}$$

But, from the very definitions above, one has, for all $n \geq 0$,

$$\begin{aligned}P(x_n,r) = \max\{&d(x_n,x_{n+1}) + d(x_{n+1},r), d(x_{n+2},r) + d(x_{n+2},Tr),\\ &d(x_{n+1},x_{n+2}) + d(x_{n+1},r), d(x_{n+1},r) + d(x_{n+1},Tr),\\ &d(x_n,r), d(x_n,Tr), d(r,Tr)\},\end{aligned}$$

$$\begin{aligned}Q(x_n,r) = \max\{&d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}), d(x_n,x_{n+1}) + d(x_{n+1},r),\\ &d(x_{n+2},Tr) + d(r,Tr), d(x_{n+1},x_{n+2}) + d(x_{n+2},Tr), d(x_n,r), d(x_n,Tr)\}.\end{aligned}$$

This yields

$$\lim_n P(x_n,r) = d(r,Tr), \lim_n Q(x_n,r) = 2d(r,Tr);$$

whence, passing to limit in the relation (2.10), one gets $d(r,Tr) \leq 2\gamma d(r,Tr)$. As $0 \leq 2\gamma < 1$, this yields $d(r,Tr) = 0$; so that, $r = Tr$. The proof is complete. □

Note that, further extensions of the obtained facts are possible, in the class of *dislocated metric spaces* defined under the model of Hitzler [17, Ch 1, Sect 1.4]; see also Amini-Harandi [2]. Some other aspects may be found in Yeh [38]; see also Popa [30].

3. Dhage metrics

As already precise in the introductory part, there are many generalizations of the Banach's fixed point theorem. Here, we shall be interested in the *structural* way of extension, consisting of the "dimensional" parameters attached to the ambient metric being increased. For example, this is the case when the initial metric $d : X \times X \to R_+$ is to be substituted by a *generalized metric* $\Lambda : X \times X \times X \to R_+$ which fulfills – at this level – the conditions imposed to the standard case. An early construction of this type was proposed in 1963 by Gaehler [14]; the resulting map $B : X \times X \times X \to R_+$ was referred to as a *2-metric* on X. Short after, this structure was intensively used in many fixed point theorems, under the model due to Namdeo et al [28], Negoescu [29] and others; see also Ashraf [3, Ch 3], for a consistent references list. However, it must be noted that this 2-metric is not a true generalization of an ordinary metric; for – as shown in Ha et al [15] – the associated real function $B(.,.,.)$ is not B-continuous in its arguments. This, among others, led Dhage [10] to construct – via different geometric reasons – a new such object.

(A) Let X be some nonempty set. By a *Dhage metric* (in short: *D-metric*) over X, we shall mean any map $D : X \times X \times X \to R_+$, with the properties

(c01) $D(x, y, z) = D(x, z, y) = D(y, x, z) = D(y, z, x) =$
$D(z, x, y) = D(z, y, x)$, $\forall x, y, z \in X$ (symmetric)

(c02) $(x = y = z) \Longleftrightarrow D(x, y, z) = 0$ (reflexive sufficient)

(c03) $D(x, y, z) \le D(x, y, u) + D(x, u, z) + D(u, y, z)$,
for all $x, y, z \in X$ and all $u \in X$ (tetrahedral).

In this case, the couple (X, D) will be termed a *D-metric* space.

Define a sequential D-convergence ($\xrightarrow{D}$) on (X, D) according to: $x_n \xrightarrow{D} x$ iff $D(x_m, x_n, x) \to 0$ as $m, n \to \infty$; i.e.,

(c04) $\forall \varepsilon > 0, \exists i(\varepsilon)$: $m, n \ge i(\varepsilon) \Longrightarrow D(x_m, x_n, x) \le \varepsilon$.

Note that this concept obeys the general rules in Kasahara [21]. By definition, $x_n \xrightarrow{D} x$ will be referred to as: x is the *D-limit* of (x_n). The set of all these will be denoted D-$\lim_n(x_n)$; if it is nonempty, then (x_n) is called *D-convergent*; the class of all D-convergent sequences will be denoted Conv(X, D). Further, let the D-Cauchy structure on (X, D) be introduced as: call the sequence (x_n) in X, *D-Cauchy*, provided $D(x_m, x_n, x_p) \to 0$ as $m, n, p \to \infty$; i.e.:

(c05) $\forall \varepsilon > 0, \exists j(\varepsilon)$: $m, n, p \ge j(\varepsilon) \Longrightarrow D(x_m, x_n, x_p) \le \varepsilon$.

The class of all these will be indicated as Cauchy(X, D); it fulfills the general requirements in Turinici [36].

By definition, the pair (Conv(X, D), Cauchy(X, D)) will be called the *conv-Cauchy structure* attached to (X, D). Note that, by the properties of D, each D-convergent sequence is D-Cauchy too; referred to as: (X, D) is *regular*. The converse is not in general true; when it holds, we say that (X, D) is *complete*.

(B) According to Dhage's topological results in the area, this new metric corrects the "bad" properties of a 2-metric. As a consequence, his construction was interesting enough so as to be used in the deduction of many fixed point results; see, for instance, Dhage [11] and the references therein. The setting of all these is to be described as below. Let (X, D) be a D-metric space; and $T \in \mathcal{F}(X)$ be a selfmap of X. The determination of the points in Fix(T) is to be performed under the lines of Section 1, adapted to our context:

3a) We say that T is a *Picard operator* (modulo D) if, for each $x \in X$, the iterative sequence $(T^n x; n \geq 0)$ is D-convergent

3b) We say that T is a *strong Picard operator* (modulo D) if, for each $x \in X$, $(T^n x; n \geq 0)$ is D-convergent, and $D - \lim_n(T^n x)$ belongs to $\mathrm{Fix}(T)$

3c) We say that T is a *globally strong Picard operator* (modulo D), if it is a strong Picard operator (modulo D) and (in addition), $\mathrm{Fix}(T)$ is an asingleton (or, equivalently: singleton).

Sufficient conditions guaranteeing these properties are of D-metrical type. The simplest one is the following. Call T, $(D;\alpha)$*-contractive* (for some $\alpha \geq 0$) if

(c06) $D(Tx, Ty, Tz) \leq \alpha D(x, y, z), \forall x, y, z \in X.$

The following fixed point statement in Dhage [10] is a cornerstone for all further developments in the area.

Theorem 3.1. *Let (X, D) be complete bounded; and $T : X \to X$ be $(D;\alpha)$-contractive, for some $\alpha \in [0, 1[$. Then, T is a globally strong Picard operator (modulo D).*

In the last part of his reasoning, Dhage tacitly used the D-continuity of the application $(x, y, z) \mapsto D(x, y, z)$, expressed as

$$[x_n \xrightarrow{D} x,\ y_n \xrightarrow{D} y,\ z_n \xrightarrow{D} z] \text{ imply } D(x_n, y_n, z_n) \to D(x, y, z).$$

But, as proved in Naidu, Rao and Rao [26], the described property is not in general valid. This must be related with the developments in Mustafa and Sims [24]; according to which, an appropriate construction of topological and/or uniform structures over (X, D) is not in general possible; we do not give details. Returning to the above discussion, note that – technically speaking – it would be possible that the conclusion in Dhage's fixed point theorem be retainable, with a different proof. However, as results from an illuminating example provided by Naidu, Rao and Rao [27], this last hope fails as well; so that, ultimately, Dhage's fixed point result is not true.

For the sake of completeness, we shall present this example, with certain small modifications.

Example 3.2. Fix in the following some $\gamma \in]0, 1[$; note that the sequence $(a_n := \gamma^n; n \in N)$ is strictly descending in $R_+^0 :=]0, \infty[$ with $a_n \to 0$. Put $X = \{a_n; n \in N\}$; and let us introduce a mapping $D : X \times X \times X \to R_+$ as

$D(x, y, z) = 0$, if $\mathrm{card}\{x, y, z\} = 1$,
$D(x, y, z) = \min\{\max\{x, y\}, \max\{y, z\}, \max\{z, x\}\}$, otherwise.

I) We firstly show that D is a Dhage metric on X. In fact, the symmetry of D is clear; as well as (via $X \subset R_+^0$), the reflexive sufficiency of the same. It remains to establish that D is tetrahedral. Let $x, y, z \in X$ be arbitrary fixed. By the symmetry of $D(.,.,.)$, we may assume that $x \leq y \leq z$; whence, $D(x, y, z) = y$. Let $u \in X$ be arbitrary fixed. If $u \leq y$, we have $D(u, y, z) = y = D(x, y, z)$; and if $u > y$, one derives $D(x, y, u) = y = D(x, y, z)$; so that, we are done.

II) We now assert that (X, D) is complete. Let (x_n) be a D-Cauchy sequence in X. There are two cases to consider.

Case 1. Assume that, for some $k \in N$, $\beta > 0$,

(c07) $\{n > k; x_n < \beta\}$ is empty; i.e.: $x_n \in X(\beta, \leq), \forall n > k$.

Here, $X(\beta, \leq) = \{x \in X; \beta \leq x\}$, $\beta > 0$. As $X(\beta, \leq)$ is finite, there exists a strictly ascending sequence $(i(n); n \in N)$ (hence $i(n) \to \infty$ as $n \to \infty$) such that $(y_n := x_{i(n)}; n \in N)$ is constant: $y_n = y_0, \forall n \in N$. Then, evidently, $x_n \xrightarrow{D} y_0$.

Case 2. Assume that the opposite to (c07) alternative holds:

(c08) $\forall k \in N$, $\forall \beta > 0$, $\exists h = h(k, \beta) > k$: $x_h < \beta$.

We claim that $x_n \to 0$, in the usual metric of R. Assume not; i.e., for some $\varepsilon > 0$,

(c09) $\forall p \in N$, $\exists q > p$: $x_q \geq \varepsilon$.

Given this $\varepsilon > 0$, there exists, by hypothesis, $i = i(\varepsilon) \in N$ such that

$$D(x_m, x_n, x_p) < \varepsilon, \ \forall m, n, p \geq i. \tag{3.1}$$

For the obtained i, there exist, via (c09), a couple of ranks $m, n \in N$ with $i < m < n$, $x_m, x_m \geq \varepsilon$. On the other hand, from (c08) (with $k = n$, $\beta = \varepsilon$) there exists $p > n$ with $x_p < \varepsilon$; hence, $x_p < \min\{x_m, x_n\}$. This by definition, gives

$$(m, n, p \geq i \text{ and}) \ D(x_m, x_n, x_p) = \min\{x_m, x_n\} \geq \varepsilon,$$

in contradiction with (3.1); and our claim follows.

Let $v \in X$ be arbitrary fixed. By the previous fact, it follows that, for each ε in $]0, v[$ there must be some rank $j = j(\varepsilon) \in N$ such that $x_n < \varepsilon < v$ (hence $\max\{v, x_n\} = v$), $\forall n \geq j$. This, by definition, yields

$$D(v, x_m, x_n) = \max\{x_m, x_n\} < \varepsilon, \ \forall m, n \geq j;$$

whence, $x_n \xrightarrow{D} v$ as $n \to \infty$; so that (as $v \in X$ was arbitrarily chosen) $D - \lim_n(x_n) = X$.

III) In addition to this, note that $D(x, y, z) \leq 1$, $\forall x, y, z \in X$; wherefrom, (X, D) is bounded.

IV) Let $T : X \to X$ be introduced as: $T(a_n) = a_{n+1}$, $n \in N$. Clearly,

$$D(Tx, Ty, Tz) \leq \gamma D(x, y, z), \ \forall x, y, z \in X; \tag{3.2}$$

i.e., T is $(D; \gamma)$-contractive. Summing up, conditions of Theorem 3.1 are holding for these data. However, its conclusion is not valid; because $\text{Fix}(T)$ is empty.

A conv-Cauchy motivation of this negative conclusion comes from the fact that the convergence structure $\text{Conv}(X, D)$ attached to our D-metric space is "too large"; i.e.: for many sequences (x_n) in X, $D - \lim_n(x_n)$ is the whole of X. A method of correcting this property was already proposed in the above quoted papers; however, we must say that – until now, at least – it was not followed by consistent applications. Hence, summing up: under the admitted conditions upon the underlying structure, a genuine fixed point theory in D-metric spaces is not (yet) available.

4. Mustafa-Sims metrics

The drawbacks of Dhage metrical structures we just exposed, determined Mustafa and Sims [25] to look for a different perspective upon this matter. Some basic aspects of it will be described further.

(A) Let X be a nonempty set. By a *Mustafa-Sims metric* (in short: *MS-metric*) on X, we mean any map $G : X \times X \times X \to R_+$, with

(d01) $G(.,.,.)$ is symmetric and reflexive (see above)
(d02) $G(x, x, y) = 0$ implies $x = y$ (plane sufficient)
(d03) $G(x, x, y) \leq G(x, y, z)$, $\forall x, y, z \in$X, $y \neq z$ (MS-property)
(d04) $G(x, y, z) \leq G(x, u, u) + G(u, y, z)$, $\forall x, y, z, u \in X$ (MS-triangular).

In this case, the couple (X, G) will be referred to as a *MS-metric space.*

The following direct consequences of these axioms are valid.

Proposition 4.1. *We have, for each $x, y, z, u \in X$,*

$$G(x,y,z) \leq G(x,x,y) + G(x,x,z) \tag{4.1}$$

$$G(x,y,y) \leq 2G(x,x,y),\ G(x,x,y) \leq 2G(x,y,y) \tag{4.2}$$

$$G(x,y,z) \leq G(x,u,z) + G(u,y,z) \tag{4.3}$$

$$G(x,y,z) \leq (2/3)[G(x,u,y) + G(y,u,z) + G(z,u,x)] \tag{4.4}$$

$$G(x,y,z) \leq G(x,u,u) + G(y,u,u) + G(z,u,u). \tag{4.5}$$

Proof .i) From (d04) and (d01) we have (taking $u = y$), $G(x,y,z) \leq G(x,y,y) + G(z,y,y)$; this, again via (d01), gives (4.1), by replacing (x, y) with (y, x).

ii) The first half of (4.2) follows at once from (4.1) by taking $z = y$; and the second part is obtainable by replacing (x, y) with (y, x).

iii) By (d01), it results that (d03) may be written as

(d05) $G(x,y,y) \leq G(x,y,z)$, $\forall x,y,z \in$X, $x \neq z$.

Combining with (d04), we get (for $x \neq z$)

$$G(x,y,z) \leq G(x,u,u) + G(u,y,z) \leq G(x,u,z) + G(u,y,z);$$

i.e.: (4.3) holds, when $x \neq z$. It remains to establish the case $x = z$ of this relation:

$$G(y,x,x) \leq G(x,u,x) + G(y,x,u). \tag{4.6}$$

Clearly, the alternative $u \neq y$ is obtainable from (d05). On the other hand, the alternative $u = y$ means

$$G(y,x,x) \leq G(x,y,x) + G(y,x,y);$$

evident. Hence (4.3) is true.

iv) By a repeated application of (4.3),

$$\begin{aligned} G(x,y,z) &\leq G(x,u,y) + G(x,u,z), \\ G(x,y,z) &\leq G(y,u,z) + G(y,u,x), \\ G(x,y,z) &\leq G(z,u,x) + G(z,u,y). \end{aligned}$$

Adding these, relation (4.4) follows.

v) By (4.1), we have

$$G(y,u,z) \leq G(y,u,u) + G(z,u,u).$$

Replacing in (d04) gives (4.5). The proof is complete. □

Remark 4.2. In particular, (4.4) tells us that the MS-metric $G(.,.,.)$ is tetrahedral. Moreover, $G(.,.,.)$ is sufficient. In fact, assume that $G(x,y,z) = 0$; but, e.g., $y \neq z$. From (d03), we then get $G(x,x,y) = 0$; wherefrom, by (d02), $x = y$. In this case, the working hypothesis becomes $G(y,y,z) = 0$; so, again via (d02), $y = z$; contradiction. Hence, summing up, $G(.,.,.)$ is a D-metric on X.

(B) By an *almost metric* on X, we mean any map $g : X \times X \to R_+$ with

(d06) $g(x, y) \leq g(x, z) + g(z, y), \forall x, y, z \in X$ (*triangular*)
(d07) $x = y \Longleftrightarrow g(x, y) = 0$ (*reflexive sufficient*);

see also Turinici [37]. Some basic examples of such objects are to be obtained, in the MS-metric space (X, G), as follows. Define a quadruple of maps $b, c, d, e : X \times X \to R_+$ according to: for each $x, y \in X$,

(d08) $b(x, y) = G(x, y, y)$, $c(x, y) = G(x, x, y) = b(y, x)$
(d09) $d(x, y) = \max\{b(x, y), c(x, y)\}$, $e(x, y) = b(x, y) + c(x, y)$.

Proposition 4.3. *Under the above conventions,*

j) *The mappings $b(.,.)$ and $c(.,.)$ are triangular and reflexive sufficient; hence, these are almost metrics on X*

jj) *The mappings $d(.,.)$ and $e(.,.)$ are triangular, reflexive sufficient and symmetric; hence, these are (standard) metrics on X*

jjj) *In addition, the following relations are valid*

$$b \leq 2c \leq 2d \leq 4b, c \leq 2b \leq 2d \leq 4c,\ \ d \leq e \leq 2d. \tag{4.7}$$

Proof .j) It will suffice establishing the assertions concerning the map $b(.,.)$. The reflexive sufficient property is a direct consequence of (d01) and (d02). On the other hand, the triangular property is a direct consequence of (d04). In fact, by this condition, we have (taking $y = z$)

$$G(x, y, y) \leq G(x, u, u) + G(u, y, y);$$

and, from this we are done.

jj) Evident, by the involved definition.

jjj) The first and second part are immediate, by Proposition 4.1. The third part is evident. Hence the conclusion. □

Remark 4.4. A formal verification of **j)** is to be found in Jleli and Samet [18]. On the other hand, **jj)** (modulo e) was explicitly asserted in Mustafa and Sims [25]. This determines us to conclude that **j)** is also clarified by the quoted authors.

(C) Having these precise, we may now pass to the conv-Cauchy structure of a MS-metric space (X, G).

Define a sequential G-convergence ($\xrightarrow{G}$) on (X, G) according to: $x_n \xrightarrow{G} x$ iff $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$; i.e.,

(d10) $\forall \varepsilon > 0, \exists i(\varepsilon)$: $m, n \geq i(\varepsilon) \Longrightarrow G(x_m, x_n, x) \leq \varepsilon$.

As before, this concept obeys the general rules in Kasahara [21]. By definition, $x_n \xrightarrow{G} x$ will be referred to as: x is the *G-limit* of (x_n). The set of all these will be denoted G-lim$_n(x_n)$; if it is nonempty, then (x_n) is called *G-convergent*; the class of all G-convergent sequences will be denoted Conv(X, G). Call the convergence ($\xrightarrow{G}$), *separated* when G-lim$_n(x_n)$ is an asingleton, for each sequence (x_n) of X. Further, let the G-Cauchy structure on (X, G) be introduced as: call (x_n), *G-Cauchy*, provided $G(x_m, x_n, x_p) \to 0$ as $m, n, p \to \infty$; i.e.:

(d11) $\forall \varepsilon > 0, \exists j(\varepsilon)$: $m, n, p \geq j(\varepsilon) \Longrightarrow G(x_m, x_n, x_p) \leq \varepsilon$.

The class of all these will be indicated as Cauchy(X, G); it fulfills the general requirements in Turinici [36]. By definition, the pair (Conv(X, G), Cauchy(X, G)) will be called the *conv-Cauchy structure* attached to (X, G). Call (X, G), *regular* when each G-convergent sequence is G-Cauchy too; and *complete*, if the converse holds: each G-Cauchy sequence is G-convergent.

In parallel to this, we may introduce a conv-Cauchy structure attached to any $g \in \{b, c, d, e\}$. This, essentially, consists in the following. Define a sequential g-convergence $(\xrightarrow{g})$ on (X, g) according to: $x_n \xrightarrow{g} x$ iff $g(x_n, x) \to 0$. This will be referred to as: x is the *g-limit* of (x_n). The set of all these will be denoted g-lim$_n(x_n)$; if it is nonempty, then (x_n) is called *g-convergent*; the class of all g-convergent sequences will be denoted Conv(X, g). Call the convergence $(\xrightarrow{g})$, *separated* when g-lim$_n(x_n)$ is an asingleton, for each sequence (x_n) of X. Further, let the g-Cauchy structure on (X, g) be introduced as: call the sequence (x_n) in X, *g-Cauchy*, provided $g(x_m, x_n) \to 0$ as $m, n \to \infty$; the class of all these will be indicated as Cauchy(X, g). By definition, (Conv(X, g), Cauchy(X, g)) will be called the *conv-Cauchy structure* attached to (X, g). Call (X, g), *regular*, when each g-convergent sequence is g-Cauchy; and *complete*, when the converse holds: each g-Cauchy sequence is g-convergent.

Proposition 4.5. *Under the above conventions,*

i) *$(\forall (x_n) \subseteq X, \forall x \in X)$: $x_n \xrightarrow{G} x$ is equivalent with*

$$x_n \xrightarrow{g} x \text{ for some/all } g \in \{b, c, d, e\} \tag{4.8}$$

ii) *the convergence structures $(\xrightarrow{G})$ and $(\xrightarrow{g})$ (for $g \in \{b, c, d, e\}$) are separated.*

Proof .i) Assume that $x_n \xrightarrow{G} x$; i.e.: $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$. This yields $G(x_n, x_n, x) \to 0$ as $n \to \infty$; i.e.: $x_n \xrightarrow{c} x$; wherefrom, combining with Proposition 4.3, (4.8) is clear. Conversely, assume that (4.8) holds. Taking (4.7) into account, gives $x_n \xrightarrow{b} x$; wherefrom, by means of (4.1), $x_n \xrightarrow{G} x$.

ii) Clearly, $(\xrightarrow{g})$ is separated, for $g \in \{d, e\}$. This, by the preceding step, gives the desired fact. Hence the conclusion. □

Likewise, the following characterization of the Cauchy property is available.

Proposition 4.6. *The following are valid:*

j) *$(\forall (x_n) \subseteq X)$: (x_n) is G-Cauchy is equivalent with*

$$(x_n) \text{ is } g\text{-Cauchy, for some/all } g \in \{b, c, d, e\} \tag{4.9}$$

jj) *(X, G) and (X, g) (for $g \in \{b, c, d, e\}$) are regular*

jjj) *(X, G) is complete iff (X, g) is complete, for some/all $g \in \{b, c, d, e\}$.*

Proof .j) Assume that (x_n) is G-Cauchy; i.e.: $G(x_m, x_n, x_p) \to 0$ as $m, n, p \to \infty$. This, in particular, yields $G(x_m, x_n, x_n) \to 0$ as $m, n \to \infty$; i.e.: (x_n) is b-Cauchy; so that, combining with Proposition 4.3, (4.9) follows. Conversely, assume that (4.9) holds. Taking (4.7) into account, one gets that (x_n) is b-Cauchy; wherefrom, by means of (4.1), we are done.

jj) The assertion is clear for (X, G), by Proposition 4.1; as well as for (X, g) (where $g \in \{d, e\}$), by its metric properties. The remaining situations ($g \in \{b, c\}$) follow from Proposition 4.5 and **j)** above.

jjj) Evident, by the previously obtained facts. □

(D) Let (X, G) be a MS-metric space. Given the function $\Lambda : X \times X \times X \to R$, call it *sequentially G-continuous*, provided

$x_n \xrightarrow{G} x$, $y_n \xrightarrow{G} y$, $z_n \xrightarrow{G} z$ imply $\Lambda(x_n, y_n, z_n) \to \Lambda(x, y, z)$.

A basic example of this type is just the one of $G(.,.,.)$. To verify this, the following auxiliary fact is to be used (cf. Mustafa and Sims [25]):

Proposition 4.7. *The map $G(.,.,.)$ is d-Lipschitz:*

$$|G(x, y, z) - G(u, v, w)| \leq d(x, u) + d(y, v) + d(z, w),\ \forall x, y, z, u, v, w \in X; \tag{4.10}$$

hence, in particular, $G(.,.,.)$ is d-continuous.

Proof . By the MS-triangular property of G,

$$G(u, v, w) \leq G(v, y, y) + G(y, u, w),$$
$$G(u, w, y) \leq G(u, x, x) + G(x, y, w),$$
$$G(w, x, y) \leq G(w, z, z) + G(z, x, y);$$

so that (by the adopted notations)

$$G(u, v, w) - G(x, y, z) \leq d(u, x) + d(v, y) + d(w, z).$$

In a similar way, one gets (by replacing (x, y, z) with (u, v, w))

$$G(x, y, z) - G(u, v, w) \leq d(x, u) + d(y, v) + d(z, w).$$

These, by the symmetry of $d(.,.)$, give the written conclusion. □

As a direct consequence of this, we have (taking Proposition 4.5 into account)

Proposition 4.8. *The map $G(.,.,.)$ is sequentially G-continuous in its variables.*

This property allows us to get a partial answer to a useful global question. Call the MS-metric $G(.,.,.)$, *symmetric* if

$G(x, y, y) = G(x, x, y)$, $\forall x, y \in X$.

Note that, under the conventions above, this may be expressed as: $b = c$; wherefrom: $d = b = c$, $e = 2b = 2c$. The class of symmetric MS-metrics is nonempty. For example, given the metric $g(.,.)$ on X, its associated MS-metric

$G(x, y, z) = \max\{g(x, y), g(y, z), g(z, x)\}$, $x, y, z \in X$

is symmetric, as it can be directly seen. On the other hand, the class of all non-symmetric MS-metrics is also nonempty; see Mustafa and Sims [25] for an appropriate example. Hence, the question of a certain MS-metric on X being or not symmetric is not trivial. An appropriate answer to this may be given as follows. Call the MS-metric space (X, G), *perfect* provided

for each $x \in X$ there exists a sequence (x_n) in $X \setminus \{x\}$ with $x_n \xrightarrow{G} x$.

Proposition 4.9. *Suppose that (X, G) is perfect. Then, $G(.,.,.)$ is symmetric.*

Proof . Let $x, y \in X$ be arbitrary fixed. Further, let (y_n) be a sequence in $X \setminus \{y\}$ with $y_n \xrightarrow{G} y$. From the MS-property of $G(.,.,.)$,

$$G(x, x, y) \leq G(x, y, y_n), \text{ for all } n.$$

Passing to limit as $n \to \infty$ yields, via Proposition 4.8, $G(x, x, y) \leq G(x, y, y)$. As $x, y \in X$ were arbitrary, one gets (under our notations) $c(x, y) \leq b(x, y)$, $\forall x, y \in X$. This gives (by a substitution of (x, y) with (y, x)), $b(x, y) \leq c(c, y)$; wherefrom $b = c$. The proof is complete. □

It follows from this that the class of all non-symmetric MS-metrics over X is not very large. Further aspects will be delineated elsewhere.

5. Contractions over MS-metric spaces

From the developments above, it follows that the metrical construction proposed by Mustafa and Sims [25] corrects certain errors of the preceding one, due to Dhage [10]. As a consequence, this structure was intensively used in many fixed point theorems; see, for instance, Aage and Salunke [1], Aydi, Shatanawi and Vetro [4], Choudhury and Maity [7], Saadati et al [33], Shatanawi [35], to quote only a few. But, as recently proved by Jleli and Samet [18], most fixed point results on MS-metric spaces are directly reducible to their corresponding statements on almost/standard metric spaces. Under this perspective, a return of certain writers in the area to 2-metric spaces must be not surprising; see, e.g., Dung et al [13]. Clearly, it is possible that *not* all fixed point results over MS-metric spaces be obtainable in this way; to substantiate the claim, an example was proposed by Karapinar and Agarwal [20]. Some conventions are in order. Let (X, G) be a MS-metric space; with, in addition,

(e01) (X, G) is complete; hence, so is (X, d).

Here, d is the associated to G standard metric we just introduced; namely,

$$d(x, y) = \max\{G(x, y, y), G(x, x, y)\},\ x, y \in X. \tag{5.1}$$

Remember that, by the MS-triangular inequality,

$$G(x, y, z) \le G(x, y, y) + G(y, y, z),\ \forall x, y, z \in X;$$

and this gives the so-called *strong triangle inequality*:

$$G(x, y, z) \le d(x, y) + d(y, z),\ \forall x, y, z \in X. \tag{5.2}$$

Further, let T be a selfmap of X. The question of determining its fixed points is to be treated with the aid of Picard concepts in Section 3 (modulo G). Sufficient conditions for these properties are G-counterparts of the ones in Section 1.

(A) Define, for $x, y, z \in X$,

(e02) $M(x, y, z) = \max\{G(x, Tx, y), G(y, T^2x, Ty), G(Tx, T^2x, Ty),$
$G(y, Tx, Ty), G(x, Tx, z), G(z, T^2x, Tz),$
$G(Tx, T^2x, Tz), G(z, Tx, Ty), G(x, y, z),$
$G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz),$
$G(z, Tx, Tx), G(x, Ty, Ty), G(y, Tz, Tz)\}.$

Given $\gamma \ge 0$, let us say that T is $(G, M; \gamma)$-*contractive*, provided

(e03) $G(Tx, Ty, Tz) \le \gamma M(x, y, z)$, $\forall x, y, z \in X$.

We are now in position to state the announced result (in our notations):

Theorem 5.1. *Suppose that T is $(G, M; \gamma)$-contractive, for some $\gamma \in [0, 1/2[$. Then,*
i) *T is a globally strong Picard operator (modulo d)*
ii) *T is a globally strong Picard operator (modulo G).*

According to the authors, this fixed point statement is an illustration of the following assertion: there are *many* fixed point results over MS-metric structures, to which the reduction techniques in Jleli and Samet (see above) are not applicable. It is our aim in the following to show that, actually, the above stated fixed point theorem cannot be viewed as such an exception [i.e.: as an illustration of this (hypothetical for the moment) alternative]. Precisely, we shall establish that Theorem 5.1 is reducible to the anticipative fixed point result over standard metric spaces given in a preceding place. This will follow from the proposed

Proof .(Theorem 5.1) By the accepted condition,

$$G(Tx,Ty,Ty) \le \gamma M(x,y,y),\ G(Tx,Tx,Ty) \le \gamma M(x,x,y),\ \forall x,y \in X; \tag{5.3}$$

and this, from the definition of d (see above), gives

$$d(Tx,Ty) \le \gamma \max\{M(x,y,y), M(x,x,y)\},\ \forall x,y \in X. \tag{5.4}$$

Now, let us deduce from such a "mixed" contractive relation in terms of (d,G), some contractive relation in terms of d only. To this end, we have

$$\begin{aligned}&M(x,y,y) = \max\{G(x,Tx,y), G(y,T^2x,Ty), G(Tx,T^2x,Ty),\\ &G(y,Tx,Ty), G(x,Tx,y), G(y,T^2x,Ty),\\ &G(Tx,T^2x,Ty), G(y,Tx,Ty), G(x,y,y),\\ &G(x,Tx,Tx), G(y,Ty,Ty), G(y,Ty,Ty),\\ &G(y,Tx,Tx), G(x,Ty,Ty), G(y,Ty,Ty)\};\end{aligned}$$

or equivalently (eliminating the identical terms)

$$\begin{aligned}&M(x,y,y) = \max\{G(x,Tx,y), G(y,T^2x,Ty), G(Tx,T^2x,Ty),\\ &G(y,Tx,Ty), G(x,y,y), G(x,Tx,Tx),\\ &G(y,Ty,Ty), G(y,Tx,Tx), G(x,Ty,Ty)\}.\end{aligned}$$

By the strong triangle inequality,

$$\begin{aligned}&G(x,Tx,y) \le d(x,Tx) + d(Tx,y),\\ &G(y,T^2x,Ty) \le d(T^2x,y) + d(T^2x,Ty),\\ &G(Tx,T^2x,Ty) \le d(Tx,T^2x) + d(T^2x,Ty),\\ &G(y,Tx,Ty) \le d(Tx,y) + d(Tx,Ty);\end{aligned}$$

and this yields (by avoiding the smaller terms)

$$\begin{aligned}&M(x,y,y) \le P(x,y) := \max\{d(x,Tx) + d(Tx,y),\\ &d(T^2x,y) + d(T^2x,Ty), d(Tx,T^2x) + d(Tx,y),\\ &d(Tx,y) + d(Tx,Ty), d(x,y), d(x,Ty), d(y,Ty)\}.\end{aligned} \tag{5.5}$$

Similarly, we have

$$\begin{aligned}&M(x,x,y) = \max\{G(x,Tx,x), G(x,T^2x,Tx), G(Tx,T^2x,Tx),\\ &G(x,Tx,Tx), G(x,Tx,y), G(y,T^2x,Ty),\\ &G(Tx,T^2x,Ty), G(y,Tx,Tx), G(x,x,y),\\ &G(x,Tx,Tx), G(x,Tx,Tx), G(y,Ty,Ty),\\ &G(y,Tx,Tx), G(x,Tx,Tx), G(x,Ty,Ty)\};\end{aligned}$$

or equivalently (eliminating the identical terms)

$$\begin{aligned}&M(x,x,y) = \max\{G(x,x,Tx), G(x,Tx,T^2x), G(Tx,Tx,T^2x),\\ &G(x,Tx,Tx), G(x,Tx,y), G(y,T^2x,Ty),\\ &G(Tx,T^2x,Ty), G(y,Tx,Tx), G(x,x,y),\\ &G(y,Ty,Ty), G(x,Ty,Ty)\}.\end{aligned}$$

By the strong triangle inequality

$$\begin{aligned}&G(x,Tx,T^2x)\le d(x,Tx)+d(Tx,T^2x),\\&G(x,Tx,y)\le d(x,Tx)+d(Tx,y),\\&G(y,T^2x,Ty)\le d(T^2x,Ty)+d(y,Ty),\\&G(Tx,T^2x,Ty)\le d(Tx,T^2x)+d(T^2x,Ty);\end{aligned}$$

and this yields (by avoiding the smaller terms)

$$\begin{aligned}&M(x,x,y)\le Q(x,y):=\max\{d(x,Tx)+d(Tx,T^2x),\\&d(x,Tx)+d(Tx,y),d(T^2x,Ty)+d(y,Ty),\\&d(Tx,T^2x)+d(T^2x,Ty),d(x,y),d(x,Ty)\}.\end{aligned}\tag{5.6}$$

Summing up, we therefore have

$$d(Tx,Ty)\le\gamma\max\{P(x,y),Q(x,y)\},\ \forall x,y\in X.\tag{5.7}$$

In other words, T is $(d,P,Q;\gamma)$-contractive (according to a preceding convention). But then, the metrical fixed point result (involving anticipative contractions) we just evoked gives us the conclusion in terms of d. The remaining conclusion (in terms of G) is a direct consequence of it, by the properties of the Mustafa-Sims convergence we already sketched. □

Note, finally, that this reduction process comprises as well another fixed point result over Mustafa-Sims metric spaces given by Karapinar and Agarwal [20]; we do not give details. Further aspects may be found in Samet et al [34].

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