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Tripled partially ordered sets

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Dedicated to the Memory of Charalambos J. Papaioannou

(Communicated by Th. M. Rassias)

Abstract

In this paper, we introduce tripled partially ordered sets and monotone functions on tripled partially ordered sets. Some basic properties on these new defined sets are studied and some examples for clarifying are given.

Keywords: partially ordered set, upper bound, lower bound, monotone function.
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1. Introduction and Preliminaries

Partially ordered sets (or poset) are generalizations of ordered sets. A partially ordered set is a set together with a binary relation that indicates that, for some pairs of elements in the set, one of the elements precedes the other. A set X is a partially ordered set if it has a binary relation $x \preceq y$ defined on it that satisfies

1. Reflexivity: $x \preceq x$ for all $x \in X$;
2. Antisymmetry: If $x \preceq y$, and $y \preceq x$, then $x = y$ for all $x, y \in X$;
3. Transitivity: If $x \preceq y$ and $y \preceq z$, then $x \preceq z$ for all $x, y, z \in X$.

Examples of partially ordered sets include the integers and real numbers with their ordinary ordering, subsets of a given set ordered by inclusion and natural numbers ordered by divisibility. For more details about partially ordered sets and their properties we refer the reader to [1] and [2].

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Definition 1.1. *A partially ordered set X is a totally ordered set if any two elements are comparable.*

If X is a partially ordered set and Y is totally ordered, there is a natural interpretation of the term monotonic as applied to functions with domain X and range in Y. Let $(X, \leq)$ be a partially ordered set and let Y be totally ordered. The function $f : X \to Y$ is said to be increasing if $f(x_1) \leq f(x_2)$ whenever $x_1, x_2 \in X$ and $x_1 \leq x_2$ and the function $f : X \to Y$ is said to be decreasing if $f(x_1) \geq f(x_2)$ whenever $x_1, x_2 \in X$ and $x_1 \leq x_2$. In [3] and [**?**], examples have given of monotone functions. The aim of this paper is to introduce tripled ordered sets and monotone functions on tripled ordered sets.

2. Tripled Partially Ordered sets

The following definition is the main definition of our paper.

Definition 2.1. *Let X be a nonempty set. A* ***tripled partial order*** *relation is a triple relation $\preceq_3$ on X (i.e. $\preceq_3 \subseteq X \times X \times X$) which satisfies the following conditions:*

(i) (*reflexivity*) $(x, x, x) \in \preceq_3$,

(ii) (*antisymmetry*) *if* $(x, y, z) \in \preceq_3$, $(y, z, x) \in \preceq_3$, $(z, x, y) \in \preceq_3$, $(x, z, y) \in \preceq_3$, $(y, x, z) \in \preceq_3$ *and* $(z, y, x) \in \preceq_3$, *then* $x = y = z$,

(iii) (*transitivity*) *if* $(x, y, z) \in \preceq_3$, $(y, z, t) \in \preceq_3$ *and* $(z, t, w) \in \preceq_3$, *then* $(x, z, w) \in \preceq_3$,

for all $x, y, z, t, w \in X$. A set with a triple partial order $\preceq_3$ is called a ***tripled partially ordered set****. We denote this tripled partially ordered set by $(X, \preceq_3)$.*

Lemma 2.2. *We can replace the hypotheses* (ii) *of Definition 2.1 with*

$$(ii)' \qquad (x, y, z) \in \preceq_3, \quad (y, z, x) \in \preceq_3 \quad \text{and} \quad (z, x, y) \in \preceq_3 . \tag{2.1}$$

Proof . According to the transitivity property of $\preceq_3$, the hypotheses (2.1) implies

$$(x, z, y) \in \preceq_3, \quad (y, x, z) \in \preceq_3 \quad \text{and} \quad (z, y, x) \in \preceq_3 . \tag{2.2}$$

□

There are many examples for tripled partially ordered sets that we give some of them as follows:

Example 2.3. 1. *Let $X = \{1, 2, 3, 4\}$. Define $\preceq_3 = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (4, 1, 1), (4, 2, 2), (4, 3, 3), (4, 4, 1), (4, 4, 2), (4, 4, 3)\}$. Then $\preceq$ is a tripled partial order on X, and $(X, \preceq)$ is a tripled partially ordered set.*

2. *Greater than or equal relation "$\geq$" is a tripled partial ordering on the set of integers. Thus $(\mathbb{Z}, \geq)$ is a tripled partially ordered set.*
3. *The division symbol "$|$" is a tripled partial ordering on the set of positive integers.*
4. *Consider $\mathbb{R}^2$ with the following tripled partial order*
$$((a_1, b_1), (a_2, b_2), (a_3, b_3)) \in \preceq_3 \text{ if and only if } a_1 \leq a_2 \leq a_3,\ b_1 \leq b_2 \leq b_3.$$
5. *Consider $\mathbb{R}^2$ with the following tripled partial order*
$$((a_1, b_1), (a_2, b_2), (a_3, b_3)) \in \preceq_3 \text{ if and only if } a_1 \leq a_2 \leq a_3,\ b_1 \geq b_2 \geq b_3.$$
6. *Consider $\mathbb{R}^2$ with the following tripled partial order*
$$((a_1, b_1), (a_2, b_2), (a_3, b_3)) \in \preceq_3 \text{ if and only if } a_1 \geq a_2 \geq a_3,\ b_1 \leq b_2 \leq b_3.$$
7. *Consider $\mathbb{R}^2$ with the following tripled partial order*
$$((a_1, b_1), (a_2, b_2), (a_3, b_3)) \in \preceq_3 \text{ if and only if } a_1 \geq a_2 \geq a_3,\ b_1 \geq b_2 \geq b_3.$$

3. Well-Ordered Relation

One of the most important tools in (binary) posets are comparable elements. We define the comparable elements in tripled partially ordered sets as follows.

Definition 3.1. *Let $(X, \preceq_3)$ be a tripled partially ordered set and $x, y, z \in X$. Elements x, y and z are said to be* ***comparable elements*** *of X if one of the following cases holds*

1. $(x, y, z) \in \preceq_3$;
2. $(y, z, x) \in \preceq_3$;
3. $(z, x, y) \in \preceq_3$;
4. $(x, z, y) \in \preceq_3$;
5. $(y, x, z) \in \preceq_3$;
6. $(z, y, x) \in \preceq_3$.

When x, y and z are elements of X such that neither of the above relations hold, they are called ***incomparable****.*

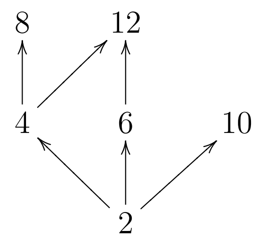
Example 3.2.
1. *Consider the tripled partially ordered set $(\mathbb{Z}, |)$ (see Example 2.3). The integers 3, 9, 27 are comparable and 3, 9, 10 are incomparable.*
2. *Consider the tripled partially ordered set $\mathbb{R}^2$ with defined tripled order in Example 2.3 (5). Elements $(1, 4), (2, 3)$ and $(3, 1)$ are comparable and $(2, 2), (3, 3)$ and $(4, 4)$ are incomparable.*

Definition 3.3. *A tripled partially ordered set X is a* ***chain*** *if any three elements are comparable.*

Example 3.4.
1. *The tripled partially ordered set $(\mathbb{Z}, \geq)$ is a chain.*
2. *The tripled partially ordered set $(\mathbb{Z}, |)$ is not a chain.*
3. *$\mathbb{R}^2$ with the defined tripled partially order in Example 2.3 is not a chain.*

Definition 3.5. *Let $(X, \preceq_3)$ be a tripled partially ordered set. An element $\alpha \in X$ is called* ***smallest element*** *if $(\alpha, x, y) \in \preceq_3$, for all $x, y \in X$.*

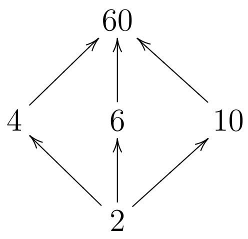
Example 3.6.
1. *Consider the natural numbers set $\mathbb{N}$ with greater than or equal relation. Then 1 is the smallest element of $(\mathbb{N}, \leq)$.*
2. *Let $X = \{2, 4, 6, 8, 10, 12\}$. Consider X with division relation. According to the following diagram 2 is the smallest element.*



Definition 3.7. *Let $(X, \preceq_3)$ be a tripled partially ordered set. An element $\beta \in X$ is called* ***biggest element*** *if $(x, y, \beta) \in \preceq_3$, for all $x, y \in X$.*

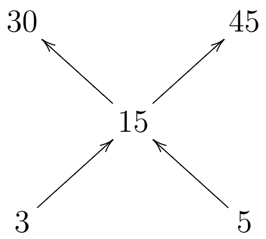
Example 3.8.
1. *Let $X = \{1, 2, \cdots, 10\}$ with greater than or equal relation. Then 10 is the biggest element of $(X, \leq)$.*

2. *Let $X = \{2, 4, 6, 10, 60\}$. Consider X with division relation. According to the following diagram 60 is the biggest element.*



Definition 3.9. *Let $(X, \preceq_3)$ be a tripled partially ordered set. An element $\gamma \in X$ is called **mid most element** if $(x, \gamma, y) \in \preceq_3$, for all $x, y \in X$.*

Example 3.10. *Let $X = \{3, 5, 15, 30, 45\}$ with division relation. According to the following diagram 15 is the mid most element.*



Theorem 3.11. *Let $(X, \preceq_3)$ be a tripled partially ordered set. Then the smallest, biggest and mid most elements of X are unique if they exist.*

Proof . Let α, $\acute{\alpha}$ both be smallest elements of X. Since α is the smallest element of X and $\acute{\alpha} \in X$, then

$$(\alpha, \acute{\alpha}, \acute{\alpha}) \in \preceq_3, \quad (\alpha, \alpha, \acute{\alpha}) \in \preceq_3 \quad \text{and} \quad (\alpha, \acute{\alpha}, \alpha) \in \preceq_3 . \tag{3.1}$$

Similarly, since $\acute{\alpha}$ is smallest element of X and $\alpha \in X$, then

$$(\acute{\alpha}, \alpha, \alpha) \in \preceq_3, \quad (\acute{\alpha}, \acute{\alpha}, \alpha) \in \preceq_3 \quad \text{and} \quad (\acute{\alpha}, \alpha, \acute{\alpha}) \in \preceq_3 . \tag{3.2}$$

Since $\preceq_3$ is a tripled partially order on X, by transitivity property of $\preceq_3$, we have $\alpha = \acute{\alpha}$. By the similar argument we can obtain uniqueness of the biggest and midmost elements of X. □

Definition 3.12. *A tripled partially ordered set X is **well-ordered** if every nonempty subset S of X contains the smallest element.*

Example 3.13. *Set of all pairs $X = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0,\ y \geq 0\}$ with tripled partial order $\preceq_3$ that is*

$$((x_1, y_1), (x_2, y_2), (x_3, y_3)) \in \preceq_3 \text{ if and only if } x_1 \leq x_2 \leq x_3,\ y_1 \leq y_2 \leq y_3. \tag{3.3}$$

Then $(0, 0)$ is the smallest element of $(X, \preceq_3)$. Thus, $(X, \preceq_3)$ is a well-ordered set.

Proposition 3.14. *Every subset Y of a well-ordered set X is itself well-ordered.*

Proof . If S is a nonempty subset of Y, then it is also a subset of X and, as any nonempty subset of X, it contains a smallest element. Therefore, Y is well-ordered. □

Proposition 3.15. *Let X is a well-ordered set. If $x, y, z \in X$, then X is chain.*

Proof . The subset $s = \{x, y, z\}$ has a smallest element, which is x or y or z. In the first case, if x is smallest element, then we have

$$(x, y, z) \in \preceq_3 \quad \text{or} \quad (x, z, y) \in \preceq_3 . \tag{3.4}$$

In the second case, if y is the smallest element, then we have

$$(y, x, z) \in \preceq_3 \quad \text{or} \quad (y, z, x) \in \preceq_3 . \tag{3.5}$$

In the third case, if z is the smallest element, then we have

$$(z, x, y) \in \preceq_3 \quad \text{or} \quad (z, y, x) \in \preceq_3, \tag{3.6}$$

therefore, X is a chain. □

4. Maximal, Minimal and Middimal Elements

The maximal and minimal elements in (binary) partially order sets are very useful. We define maximal, minimal and middimal elements in tripled partially ordered sets as follows.

Definition 4.1. *Let X be a tripled partially ordered set and let $E \subseteq X$. An element $m \in X$ is a **minimal element** for E if there is no $x, y \in E$ for which*

$$(x, y, m) \in \preceq_3 \quad \text{and} \quad (x, m, y) \in \preceq_3, \tag{4.1}$$

that is, if $x, y \in E$ and if $(x, y, m) \in \preceq_3$ and $(x, m, y) \in \preceq_3$, then $x = m$ and $y = m$.

Definition 4.2. *Let X be a tripled partially ordered set and let $E \subseteq X$. An element $\rho \in X$ is a **maximal element** for E if there is no $x, y \in E$ for which*

$$(\rho, x, y) \in \preceq_3 \quad \text{and} \quad (x, \rho, y) \in \preceq_3, \tag{4.2}$$

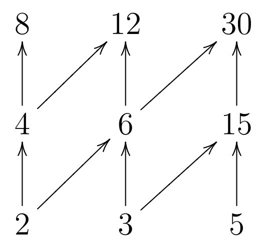
that is, if $x, y \in E$ and if $(\rho, x, y) \in \preceq_3$ and $(x, \rho, y) \in \preceq_3$, then $x = \rho$ and $y = \rho$.

Definition 4.3. *Let X be a tripled partially ordered set and let $E \subseteq X$. An element $l \in X$ is a **middimal element** for E if there is no $x, y \in E$ for which*

$$(l, x, y) \in \preceq_3 \quad \text{and} \quad (x, y, l) \in \preceq_3, \tag{4.3}$$

that is, if $x, y \in E$ and if $(l, x, y) \in \preceq_3$ and $(x, y, l) \in \preceq_3$, then $x = l$ and $y = l$.

Example 4.4. *Let $X = \{2, 3, 4, 5, 6, 8, 12, 15, 30\}$. Consider X with division relation $|$ as a tripled partially ordered set. Then consider the following diagram*



2, 3 and 5 are minimal elements for X, 4, 6 and 15 are middimal elements for X and 8, 12 and 30 are maximal elements for X.

Theorem 4.5. *Let $(X, \preceq_3)$ be a tripled partially ordered set. Then the following statements hold:*

(i) if α is smallest element of X, then α is unique minimal element of X.

(ii) if β is biggest element of X, then β is unique maximal element of X.

(iii) if l is mid most element of X, then l is unique middimal element of X.

Proof . Suppose α is the smallest element of X, then there is no $x, y \in X$ for which

$$(x, y, \alpha) \in \preceq_3 \quad \text{and} \quad (x, \alpha, y) \in \preceq_3 . \tag{4.4}$$

Thus α is a minimal element of X. Now suppose $\acute{\alpha}$ is an another minimal element of X. Note that α is the smallest element of X, we have

$$(\alpha, \alpha, \acute{\alpha}) \in \preceq_3 \quad \text{and} \quad (\alpha, \acute{\alpha}, \alpha) \in \preceq_3 . \tag{4.5}$$

Therefore, from definition of minimal element, we deduce $\alpha = \acute{\alpha}$. By the similar argument, we can prove (ii) and (iii). □

Theorem 4.6. *Let $(X, \preceq_3)$ be a chain. Then the following statements hold:*

(i) α is smallest element of X if and only if α is a minimal element of X.

(ii) β is biggest element of X if and only if β is a maximal element of X.

(iii) l is midmost element of X if and only if l is a middimal element of X.

Proof . (ii) If l is midmost element of X, by previous theorem, l is middimal element of X. Suppose l be a middimal element of X, we show that l is midmost element of X. Note that X is a chain, for any $x \in X$, we have one of the following

$$(l, x, x) \in \preceq_3, \ (x, l, x) \in \preceq_3, \ (x, x, l) \in \preceq_3 . \tag{4.6}$$

On the other hand, from the fact that l is a middimal element of X, it follows that there is no $X \ni x \neq l$ for which $(l, x, x) \in \preceq_3$ or $(x, x, l) \in \preceq_3$. Therefore, for any $x \in X$, we deduce $(x, l, x) \in \preceq_3$, that is, l is midmost element of X. □

5. Upper, Lower and Middle Bounds

In this section we define upper, lower and middle bounds for subsets of a tripled partially ordered set.

Definition 5.1. *Let X be a tripled partially ordered set. A subset A of X is **bounded from above** if there exists a $u \in X$, called an **upper bound** of A, such that $(x, y, u) \in \preceq_3$ for all $x, y \in X$.*

Definition 5.2. *Let X be a tripled partially ordered set. A subset A of X is **bounded from below** if there exists a $v \in X$, called a **lower bound** of A, such that $(v, x, y) \in \preceq_3$ for all $x, y \in X$.*

Definition 5.3. *Let X be a tripled partially ordered set. A subset A of X is* ***bounded from middling*** *if there exists a $h \in X$, called a* ***middle bound*** *of A, such that $(x, h, y) \in \preceq_3$ for all $x, y \in X$.*

Example 5.4.
1. *The set of natural numbers with greater than or equal relation is bounded from below that* 1 *is the lower bound of it.*
2. *Let X be the tripled partially ordered set defined in Example 3.8* (2)*, then it is bounded from below and bounded from above. The lower bound of X is* 2 *and the upper bound of X is* 60.
3. *Let X be the tripled partially ordered set defined in Example 3.10, then it is bounded from middling with middle bound* 15.

Definition 5.5. *Let X be a tripled partially ordered set. A subset A of X is called* ***bounded*** *if it is bounded from above, bounded from below and bounded from middle.*

Proposition 5.6. *Let X be a tripled partially ordered set and $\alpha, \beta, \gamma \in X$. Then the following statements hold:*

(i) α is smallest element of X if and only if α is a lower element of X.

(ii) β is biggest element of X if and only if β is an upper element of X.

(iii) γ is midmost element of X if and only if γ is a middling element of X.

Proof . (i) α is the smallest element of X if and only if $(\alpha, x, y) \in \preceq_3$ for all $x, y \in X$, if and only if α is a lower element of X.

(ii) β is the biggest element of X if and only if $(x, y, \beta) \in \preceq_3$, for all $x, y \in X$ if and only if β is an upper element of X.

(iii) γ is the midmost element of X if and only if $(x, \gamma, y) \in \preceq_3$, for all $x, y \in X$ if and only if γ is a middlling element of X. □

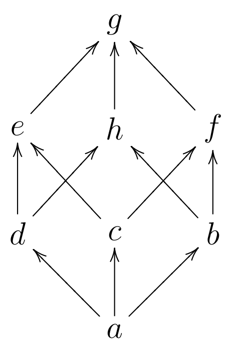
6. Supremum, Infimum and Middlmum

Definition 6.1. *Let $(X, \preceq_3)$ be a tripled partially ordered set and $A \subset X$. If $u \in X$ is an upper bound of A such that $(x, y, u) \in \preceq_3$ for all upper bounds x, y of A, then u is called the* ***supremum*** *of A, denoted by $u = \sup A$.*

Definition 6.2. *Let $(X, \preceq_3)$ be a tripled partially ordered set and $A \subset X$. If $v \in X$ is an lower bound of A such that $(v, x, y) \in \preceq_3$, for all lower bounds x, y of A, then v is called the* ***infimum*** *of A, denoted by $v = \inf A$.*

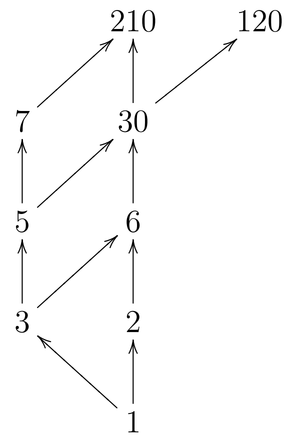
Definition 6.3. *Let $(X, \preceq_3)$ be a tripled partially ordered set and $A \subset X$. If $h \in X$ is a middle bound of A such that $(x, h, y) \in \preceq_3$, for all middle bounds x, y of A, then h is called the* ***Middlmum*** *of A, denoted by $h = midA$.*

Example 6.4. *Let $X = \{a, b, c, d, e, f, g, h\}$. Consider the following cubic diagram that we show by vector for comparable elements:*



1. *Set $A_1 = \{a, b, c, d\}$. g is an upper bound for A_1 and A_1 has not another upper bound. Then* $\sup A_1 = g$. *A lower bound for A_1 is a and A_1 has not another lower bound. Thus* $\inf A_1 = a$.
2. *Set $A_2 = \{a, c, d, e\}$. The upper bounds of A_2 are e and g. Then* $\sup A_2 = e$. *A_2 has only one lower bound. It is a, thus* $\inf A_2 = a$.
3. *Set $A_3 = \{a, b, c, f\}$. The upper bounds of A_3 are f and g. Then* $\sup A_3 = f$. *A lower bound of A_3 is a and A_3 has not another lower bound. Then* $\inf A_3 = a$.

Example 6.5. *Let $X = \{1, 2, 3, 5, 6, 7, 30, 120, 210\}$. Consider the following diagram that we show by vector for comparable elements:*



1. *Set $A_1 = \{1, 2, 3\}$. The set of all upper bounds for A_1 is $\{6, , 30, 120, 210\}$ and so* $\sup A_1 = 6$. *On the other hand, the only lower bound for A_1 is 1, thus* $\inf A_1 = 1$.
2. *Set $A_2 = \{1, 2, 3, 5\}$. Then $\{30, 120, 210\}$ is the set of all upper bounds for A_2, then* $\sup A_2 = 30$. *Also, the only lower bound for A_2 is 1, thus* $\inf A_2 = 1$.
3. *Set $A_3 = \{1, 2, 3, 6\}$. Then $\{6, 30, 120, 210\}$ is the set of all upper bounds for A_3, then* $\sup A_3 = 6$. *1 is a lower bound for A_3 and there is not another lower bound for A_3. Then* $\inf A_3 = 1$.
4. *Set $A_4 = \{1, 2, 3, 5, 6\}$. The set of all upper bounds of A_4 is $\{30, 120, 210\}$. Then* $\sup A_4 = 30$. *A_4 has only one lower bound. Its lower bound is 1, thus* $\inf A_4 = 1$.
5. *Set $A_5 = \{1, 2, 3, 5, 6, 7\}$. A_5 has only one upper bound. Its upper bound is 210, then* $\sup A_3 = 210$. *A lower bound for A_5 is 1 (there is not another lower bound for A_5), thus* $\inf A_5 = 1$.
6. *Set $A_6 = \{1, 2, 3, 5, 6, 30\}$. Then $\{30, 120, 210\}$ is the set of all upper bounds of A_6. It follows that* $\sup A_6 = 30$. *The only lower bound of A_6 is 1, thus* $\inf A_6 = 1$.

7. *Set* $A_7 = \{1, 2, 3, 5, 6, 30, 120\}$. *Then* A_7 *has only one upper bound. It is* 120, *then* $\sup A_7 = 120$. *A lower bound for* A_7 *is* 1 *and there is not another lower bound for* A_7, *thus* $\inf A_7 = 1$.
8. *Set* $A_8 = \{1, 2, 3, 5, 6, 7, 30\}$. *Then* A_8 *has only one upper bound. It is* 210, *then* $\sup A_8 = 210$. A_8 *has only one lower bound. It is* 1, *thus* $\inf A_8 = 1$.
9. *Set* $A_9 = \{1, 2, 3, 5, 6, 7, 30, 120\}$. *Then* A_9 *has not upper bound, therefore it has not supremum. A lower bound for* A_9 *is* 1 *and there is not another lower bound for* A_9, *thus* $\inf A_9 = 1$.
10. *Set* $A_{10} = \{1, 2, 3, 5, 6, 7, 30, 210\}$. *Then* A_{10} *has only one upper bound. It is* 210, *then* $\sup A_{10} = 210$. *A lower bound for* A_{10} *is* 1 *and there is not another lower bound for* A_{10}, *thus* $\inf A_{10} = 1$.
11. *Set* $A_{11} = \{2, 3, 5\}$. *Then* $\{30, 120, 210\}$ *is the set of all upper bounds of* A_{11}. *Then* $\sup A_{11} = 30$. *A lower bound for* A_{11} *is* 1 *and there is not another lower bound for it, thus* $\inf A_{11} = 1$.
12. *Set* $A_{12} = \{2, 3, 5, 6\}$. *Then* $\{30, 120, 210\}$ *is the set of all upper bounds of* A_{12}. *Then* $\sup A_{12} = 30$. *A lower bound for* A_{12} *is* 1 *and there is not another lower bound for it, thus* $\inf A_{12} = 1$.

Proposition 6.6. *Let* X *be a tripled partially ordered set and* $A \subset X$. *The supremum (infimum or Middlmum) of* A *is unique if it exists.*

Proof . Suppose that u, $\acute{u}$ are two supremums for A. Since $\acute{u}$ is an upper bound of A and u is a smallest upper bound, then

$$(u, u, \acute{u}) \in \preceq_3, \ (u, \acute{u}, \acute{u}) \in \preceq_3 \quad \text{and} \quad (\acute{u}, u, \acute{u}) \in \preceq_3 . \tag{6.1}$$

Similarly

$$(u, \acute{u}, \acute{u}) \in \preceq_3, \ (\acute{u}, \acute{u}, u) \in \preceq_3 \quad \text{and} \quad (\acute{u}, u, u) \in \preceq_3 . \tag{6.2}$$

On the other hand, $\preceq_3$ is a tripled partially ordered on X. Then by transitivity property of $\preceq_3$, we deduce $u = \acute{u}$.

If v, $\acute{v}$ are two infimums of A, then $\acute{v}$ is an upper bound of A and v is a biggest upper bound, then

$$(\acute{v}, \acute{v}, v) \in \preceq_3, \ (\acute{v}, v, v) \in \preceq_3, \quad \text{and} \quad (\acute{v}, v, \acute{v}) \in \preceq_3, \tag{6.3}$$

and similarly

$$(v, v, \acute{v}) \in \preceq_3, \ (v, \acute{v}, \acute{v}) \in \preceq_3 \quad \text{and} \quad (v, \acute{v}, v) \in \preceq_3 . \tag{6.4}$$

Since $\preceq_3$ is a tripled partially ordered on X, by transitivity property of relation $\preceq_3$ we deduce $v = \acute{v}$.

By the similar argument, we can show the Middlmum of A, is unique. □

7. Monotone Functions

One of the most important class of functions on partially ordered sets are monotone functions. We define the monotone functions on tripled partially ordered sets as follows.

Definition 7.1. *Let* $(X, \preceq_3)$ *be a tripled partially ordered set and let* $f : X \longrightarrow X$ *be a mapping. Then*

1. f *is monotone of type 1, if* $(f(a), f(b), f(c)) \in \preceq_3$ *if* $(a, b, c) \in \preceq_3$.
2. f *is monotone of type 2, if* $(f(a), f(c), f(b)) \in \preceq_3$ *if* $(a, b, c) \in \preceq_3$.
3. f *is monotone of type 3, if* $(f(c), f(b), f(a)) \in \preceq_3$ *if* $(a, b, c) \in \preceq_3$.

4. *f is monotone of type 4, if $(f(c), f(a), f(b)) \in \preceq_3$ if $(a, b, c) \in \preceq_3$.*
5. *f is monotone of type 5, if $(f(b), f(a), f(c)) \in \preceq_3$ if $(a, b, c) \in \preceq_3$.*
6. *f is monotone of type 6, if $(f(b), f(c), f(a)) \in \preceq_3$ if $(a, b, c) \in \preceq_3$.*

Example 7.2. 1. *Let $(\mathbb{R}, \preceq_3)$ be a tripled partially ordered set and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be identity function. Then f is a monotone function of type 1.*

2. *Let $(\mathbb{R}^2, \preceq_3)$ be a tripled partially ordered set such that*

$$((a_1, b_1), (a_2, b_2), (a_3, b_3)) \in \preceq_3 \text{ if and only if } a_1 \leq a_2 \leq a_3,\ b_1 \leq b_2 \leq b_3.$$

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be identity function. Then f is a monotone function of type 1.

3. *Consider the set $S = \{(2n, 2n+1, 2n+2) : n \in \mathbb{N}\}$ with order " $\leq$". Define $f : S \longrightarrow \mathbb{N} \cup \{0\}$ by $f(x) =$remainder of division of x to 2. Then f is a monotone of type 2.*

4. *Let $(\mathbb{R}^2, \preceq_3)$ be a tripled partially ordered set such that*

$$((a_1, b_1), (a_2, b_2), (a_3, b_3)) \in \preceq_3 \text{ if and only if } a_1 \leq a_2 \leq a_3,\ b_1 \leq b_2 \leq b_3.$$

Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $f(x) = -x$, for all $x \in \mathbb{R}^2$. Then f is a monotone function of type 3.

5. *Consider the set $S = \{(3n, 3n+1, 3n+3) : n \in \mathbb{N}\}$ with order " $\leq$". Define $f : S \longrightarrow \mathbb{N} \cup \{0\}$ by $f(x) =$remainder of division of x to 3. Then f is a monotone of type 4.*

6. *Consider the set $S = \{(2n+1, 2n+2, 2n+3) : n \in \mathbb{N}\}$ with order " $\leq$". Define $f : S \longrightarrow \mathbb{N} \cup \{0\}$ by $f(x) =$remainder of division of x to 2. Then f is a monotone function of type 5.*

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