A fixed point result for a new class of set-valued contractions

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Dedicated to the Memory of Charalambos J. Papaioannou
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Abstract

In this paper, we introduce a new class of set-valued contractions and obtain a fixed point theorem for such mappings in complete metric spaces. Our main result generalizes and improves many well-known fixed point theorems in the literature.

Keywords: Fixed point, Set-valued contraction.

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1. Introduction and preliminaries

Let $(X,d)$ be a metric space. We denote the family of all nonempty closed and bounded subsets of $X$ by $CB(X)$. Let $\mathcal{H}$ denotes the Hausdorff metric on $CB(X)$ induced by $d$, that is,

$$\mathcal{H}(A,B) = \max\{\sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A)\},$$

for all $A, B \in CB(X)$,

where $d(x, B) = \inf_{y \in B} d(x, y)$.

In 1989, Mizoguchi and Takahashi [10] proved the following generalization of Nadler’s fixed point theorem [12].

Theorem 1.1. ([10]) Let $(X,d)$ be a complete metric space and $T : X \to CB(X)$ be a set-valued mapping. Assume that

$$\mathcal{H}(Tx,Ty) \leq \alpha(d(x,y))d(x,y)$$

for all $x, y \in X$,

where $\alpha : [0, \infty) \to [0, 1)$ satisfies $\limsup_{s \to t^+} \alpha(s) < 1$ for each $t \in [0, \infty)$. Then $T$ has a fixed point.
In 2011, the second author [1] gave the following fixed point theorem for set-valued quasi-contraction mappings in metric spaces.

**Theorem 1.2. ([1])** Let \((X, d)\) be a complete metric space. Let \(T : X \to CB(X)\) be a \(k\)-set-valued quasi-contraction mapping with \(k < \frac{1}{2}\), that is,
\[
\mathcal{H}(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},
\]
for any \(x, y \in X\). Then \(T\) has a fixed point.

He raised the following question.

**Question 1.3.** Does the conclusion of Theorem 1.2 remain true for any \(k \in \left[\frac{1}{2}, 1\right)\)?

Up to our knowledge, this question is still open.

**Theorem 1.4. ([8])** Let \((X, d)\) be a complete metric space. Let \(T : X \to CB(X)\) be a set-valued mapping such that for any \(x, y \in X\),
\[
\mathcal{H}(Tx, Ty) \leq k \max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},
\]
(1.1)
where \(0 < k < 1\). Then \(T\) has a fixed point.

In recent years, the existence of fixed points for various set-valued contractive mappings have been studied by many authors under different conditions, see [1-12] and references therein.

In this paper, we introduce a new class of set-valued contractions and then we give a fixed point result for such mappings.

**2. Main results**

We denote by \(\Phi\) the set of all functions \(\phi : \mathbb{R}_+^5 \to \mathbb{R}_+\) satisfy the following conditions:

\(C_1\) \(\phi(t_1, t_2, t_3, t_4, t_5)\) is non-decreasing in \(t_2, t_3, t_4\) and \(t_5\),

\(C_2\) \(t_{n+1} \leq \phi(t_n, t_n, t_{n+1}, t_n + t_{n+1}, 0)\) implies \(\sum_{n=1}^{\infty} t_n \leq \infty\), for each positive sequence \(\{t_n\}\),

\(C_3\) If \(t_n, s_n \to 0\) and \(u_n \to \gamma\) for some \(\gamma > 0\) as \(n \to \infty\), then \(\limsup_{n \to \infty} \phi(t_n, s_n, \gamma, u_n, t_{n+1}) < \gamma\).

Now, we are ready to state our main result.

**Theorem 2.1.** Let \((X, d)\) be a complete metric space and let \(T : X \to CB(X)\) be a set-valued map which satisfying:
\[
\mathcal{H}(Tx, Ty) < \phi(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)),
\]
(2.1)
for each \(x, y \in X\) with \(x \neq y\), where \(\phi \in \Phi\). Then, \(T\) has a fixed point.

**Proof.** Let \(x_1 \in X\) and \(x_2 \in Tx_1\). If \(x_1 = x_2\), then \(x_1 \in T(x_1)\) and we are done. So, we may assume \(x_1 \neq x_2\). Then, by (2.1) and \((C_1)\), we have
\[
d(x_2, Tx_2) \leq \mathcal{H}(Tx_1, Tx_2)
< \phi(d(x_1, x_2), d(x_1, Tx_1), d(x_2, Tx_2), d(x_1, Tx_2), d(x_2, Tx_1))
\]
\[
\leq \phi(d(x_1, x_2), d(x_1, x_2), d(x_2, Tx_2), d(x_1, x_2) + d(x_2, Tx_2), 0).
\]

Thus there exists \( x_3 \in Tx_2 \), such that
\[
d(x_2, x_3) \leq \phi(d(x_1, x_2), d(x_1, x_2), d(x_2, Tx_2), d(x_1, x_2) + d(x_2, Tx_2), 0)
\]
\[
\leq \phi(d(x_1, x_2), d(x_1, x_2), d(x_2, x_3), d(x_1, x_2) + d(x_2, x_3), 0).
\]

Therefore, by induction we can find a sequence \( \{x_n\} \) in \( X \) such that for each \( n \in \mathbb{N} \), \( x_{n+1} \in Tx_n \) and
\[
d(x_{n+1}, x_{n+2}) \leq \phi(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}), 0) \tag{2.2}
\]

Then from (C_3) and (2.2), we have \( \sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty \). Thus \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is complete, there exists \( x^* \in X \) such that \( \lim_{n \to \infty} x_n = x^* \). We show that \( x^* \) is a fixed point of \( T \). Assume that \( x^* \not\in Tx^* \), that is, \( d(x^*, Tx^*) > 0 \). Then, by (2.1) and (C_1), we have (without loss of generality, we may assume that \( x_n \neq x^* \) for each \( n \in \mathbb{N} \))
\[
d(x_{n+1}, Tx^*) \leq \mathcal{H}(Tx_n, Tx^*)
\]
\[
< \phi(d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), d(x_n, Tx^*), d(x^*, Tx_n))
\]
\[
\leq \phi(d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), d(x_n, Tx^*), d(x^*, x_{n+1})). \tag{2.3}
\]

Then, from (2.3) and (C_3), we get
\[
d(x^*, Tx^*) = \lim_{n \to \infty} d(x_{n+1}, Tx^*)
\]
\[
\leq \limsup_{n \to \infty} \phi(d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), d(x_n, Tx^*), d(x^*, x_{n+1}))
\]
\[
< d(x^*, Tx^*),
\]
a contradiction. This implies that \( d(x^*, Tx^*) = 0 \), and since \( Tx^* \) is closed, then we have \( x^* \in Tx^* \).

\[\square\]

**Remark 2.2.** In 2011, Chen obtained a fixed point theorem ([6], Theorem 4) for a class of set-valued mappings satisfy a contractive condition similar to (2.1) under some different conditions. But the proof of his main result seems to be incorrect. Indeed, in page 3, line 15, he used the inequality \( d(x_m, x_n) \leq \mathcal{H}(Tx_{m-1}, Tx_{n-1}) \), where \( x_m \in Tx_{m-1} \) and \( x_n \in Tx_{n-1} \), which is false in general.

Now, we get the following generalization of the above mentioned Theorem 1.1 of Mizoguchi and Takahashi [10], Theorem 4 of Berinde and Berinde [5] and Theorem 2.2 of Berinde [4].

**Theorem 2.3.** Let \( (X, d) \) be a complete metric space and \( T : X \to CB(X) \) be a set-valued mapping such that for all \( x, y \in X \),
\[
\mathcal{H}(Tx, Ty) \leq \alpha(d(x, y)) \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2} [d(x, Ty) + d(y, Tx)] \right\} + Ld(y, Tx),
\]
where \( L \geq 0 \) and \( \alpha : [0, \infty) \to [0, 1) \) satisfies \( \limsup_{s \to t^+} \alpha(s) < 1 \) for all \( t \in [0, \infty) \). Then, \( T \) has a fixed point in \( X \).
**Proof.** Define $\phi : \mathbb{R}^5_+ \to \mathbb{R}_+$ by

$$
\phi(t_1, t_2, t_3, t_4, t_5) = \alpha(t_5) \cdot \max\{t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)\} + Lt_5.
$$

We claim that $\phi \in \Phi$. Indeed $(C_1)$ obviously holds. To show $(C_2)$, let $\{t_n\}$ be a positive sequence such that $t_{n+1} = \phi(t_n, t_n, t_{n+1}, t_n + t_{n+1}, 0) = \alpha(t_n) \cdot \max\{t_n, t_{n+1}, \frac{1}{2}(t_n + t_{n+1})\}$ for all $n$. If for some $n_0 \in \mathbb{N}$, $t_{n_0+1} \geq t_{n_0}$, then from the above $t_{n_0+1} \leq \alpha(t_{n_0})t_{n_0+1} < t_{n_0+1}$, a contradiction. Hence, $t_{n+1} \leq t_n$, for all $n \in \mathbb{N}$. So

$$
t_{n+1} \leq \alpha(t_n)t_n, \quad \text{for all } n \in \mathbb{N}.
$$

Thus $\{t_n\}_{n \in \mathbb{N}}$ is a non-negative non-increasing sequence and so, is convergent. Let $\lim_{n \to \infty} t_n = r_0$. Since $\limsup t_n < 1$, there exist $0 < k < 1$ and $N \in \mathbb{N}$ such that $\alpha(t_n) < k$, for all $n \geq N$. Consequently,

$$
t_{n+1} \leq kt_n, \quad n \geq N,
$$

and so $\sum_{n=1}^{\infty} t_n < \infty$. To show $(C_3)$, assume that $t_n, s_n \to 0$ and $u_n \to \gamma$ for some $\gamma > 0$ as $n \to \infty$. Then

$$
\limsup_{n \to \infty} \phi(t_n, s_n, \gamma, u_n, t_{n+1})
= \limsup_{n \to \infty} \left( \alpha(t_n) \cdot \max\{t_n, s_n, \gamma, \frac{1}{2}(u_n + t_{n+1})\} + Lt_{n+1} \right)
= \limsup_{n \to \infty} \alpha(t_n)\gamma < \gamma.
$$

Hence all of the assumptions of Theorem 2.1 are satisfied and so, $T$ has a fixed point. □

**Remark 2.4.** If we define the function $\phi : \mathbb{R}^5_+ \to \mathbb{R}_+$ by

$$
\phi(t_1, t_2, t_3, t_4, t_5) = k \max\{t_1, t_2, t_3, t_4, t_5\}, \quad \text{where } 0 < k < \frac{1}{2},
$$

then $\phi \in \Phi$ and Theorem 2.1 reduces to the above mentioned Theorem 1.2 of the second author [1].

**Corollary 2.5.** Let $(X, d)$ be a complete metric space and let $T : X \to CB(X)$ be a set-valued mapping such that for any $x, y \in X$

$$
\mathcal{H}(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\},
$$

where $k \in (0, 1)$. Then, $T$ has a fixed point in $X$.

**Proof.** Define the function $\phi : \mathbb{R}^5_+ \to \mathbb{R}_+$ by

$$
\phi(t_1, t_2, t_3, t_4, t_5) = k \max\{t_1, t_2, t_3, t_5\},
$$

and apply Theorem 2.1 □
Corollary 2.6. Let \((X,d)\) be a complete metric space and let \(T : X \to CB(X)\) be a set-valued mapping such that for any \(x,y \in X\),

\[
\mathcal{H}(Tx,Ty) \leq k \max \left\{ d(x,y), d(x,Tx), \frac{1}{2}(d(y,Ty) + d(x,Ty)), d(y,Tx) \right\},
\]

where \(0 < k < \frac{2}{3}\). Then, \(T\) has a fixed point in \(X\).

**Proof.** Define \(\phi : \mathbb{R}_+^5 \to \mathbb{R}_+\) by

\[
\phi(t_1,t_2,t_3,t_4,t_5) = k \max \{t_1,t_2,\frac{1}{2}(t_3 + t_4),t_5\}.
\]

If we show that \(\phi \in \Phi\) the the conclusion follows from Theorem 2.1. The condition \((C_1)\) and \((C_3)\) obviously hold. To show \((C_2)\), let \(\{t_n\}\) be a positive sequence satisfying

\[
t_{n+1} \leq \phi(t_n,t_n,t_{n+1},t_n + t_{n+1},0)
= k \max \left\{ t_n,t_n,\frac{1}{2}(t_n + 2t_{n+1}),0 \right\}
= k \max \left\{ t_n,\frac{1}{2}(t_n + 2t_{n+1}) \right\}
\]

for each \(n \in \mathbb{N}\). Let \(c = \max \left\{ \frac{1}{2}, \frac{k}{2(1-k)} \right\}\). Then \(0 < c < 1\) (note that \(0 < k < \frac{2}{3}\)). Now, we prove that

\[
t_{n+1} \leq ct_n, \quad \text{for each } n \in \mathbb{N}. \quad (2.5)
\]

If for some \(n \in \mathbb{N}\), \(\max \{t_n,\frac{1}{2}(t_n + 2t_{n+1})\} = t_n\), then \(\frac{1}{2}(t_n + 2t_{n+1}) \leq t_n\) implies \(t_{n+1} \leq \frac{1}{2}t_n \leq ct_n\).

Now, if \(\max \{t_n,\frac{1}{2}(t_n + 2t_{n+1})\} = \frac{1}{2}(t_n + 2t_{n+1})\), then from (2.4), we have \(t_{n+1} \leq k\left(\frac{1}{2}(t_n + 2t_{n+1})\right)\), and so \(t_{n+1} \leq k2(1-k)t_n \leq ct_n\). From (2.5), we get \(\sum_{n=1}^{\infty} t_n < \infty\). \(\Box\)

Now, we illustrate our main result by the following examples.

**Example 2.7.** Let \(X = [0,2]\) and \(d(x,y) = |x - y|\) for each \(x,y \in X\). Define \(T : X \to CB(X)\) by \(Tx = [1, \frac{5}{4}]\) whenever \(x \in [0, \frac{3}{4}]\), \(Tx = [\frac{5}{4}, \frac{9}{5}]\) whenever \(x \in (\frac{3}{4}, \frac{5}{4})\) and \(Tx = [\frac{5}{4}, 1]\) whenever \(x \in \left[\frac{5}{4}, 2\right]\). \(T\) does not satisfy (1.1) for any \(0 < k < 1\) (see Example 2.1 in [11]) and so we cannot invoke the above mentioned Theorem 1.4 of Haghi et al [8] to show the existence of a fixed point for \(T\).

Now, we show that

\[
\mathcal{H}(Tx,Ty) \leq \frac{1}{2} \max \{d(x,y), d(x,Tx), d(y,Ty), d(y,Tx)\}, \quad (2.6)
\]

for each \(x,y \in X\). Obviously (2.6) holds whenever either \(x,y \in [0, \frac{3}{4}]\) or \(x,y \in (\frac{3}{4}, \frac{5}{4})\) or \(x,y \in \left[\frac{5}{4}, 2\right]\).

If \(x \in [0, \frac{3}{4}]\) and \(y \in \left[\frac{5}{4}, 2\right]\), then \(\mathcal{H}(Tx,Ty) = \frac{1}{4}\) and \(d(x,y) \geq \frac{5}{4} - \frac{3}{4} = \frac{1}{2}\). Hence,

\[
\mathcal{H}(Tx,Ty) = \frac{1}{4}
\leq \frac{1}{2} d(x,y) \leq \frac{1}{2} \max \{d(x,y), d(x,Tx), d(y,Ty), d(y,Tx)\}.
\]

If \(x \in [0, \frac{3}{4}]\) and \(y \in (\frac{3}{4}, \frac{5}{4})\), then \(\mathcal{H}(Tx,Ty) = \frac{1}{8}\) and \(d(x,Tx) \geq \frac{1}{4}\). Hence,

\[
\mathcal{H}(Tx,Ty) = \frac{1}{8}
\]
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\[ \frac{1}{2} d(x, Tx) \leq \frac{1}{2} \max \{ d(x, y), d(x, Tx), d(y, Ty), d(y, Tx) \}. \]

If \( x \in \left[ \frac{5}{4}, 2 \right] \) and \( y \in \left( \frac{3}{4}, \frac{5}{4} \right) \), then \( \mathcal{H}(Tx, Ty) = \frac{1}{8} \) and \( d(x, Tx) \geq \frac{1}{4} \). Hence,

\[ \mathcal{H}(Tx, Ty) = \frac{1}{8} \]

\[ \leq \frac{1}{2} d(x, Tx) \leq \frac{1}{2} \max \{ d(x, y), d(x, Tx), d(y, Ty), d(y, Tx) \}. \]

Therefore, (2.6) holds and then by Corollary 2.5, \( T \) has a fixed point.

**Example 2.8.** Let \( X = [0, 1] \) and let \( d(x, y) = |x - y| \) for each \( x, y \in X \). Define \( T : X \to CB(X) \) by \( Tx = \left[ \frac{x}{2}, \frac{3x}{4} \right] \) whenever \( x \in [0, 1) \), and \( Tx = \{ 1 \} \) whenever \( x = 1 \). It is straightforward to show that

\[ \mathcal{H}(Tx, Ty) \leq \frac{1}{2} \max \{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2} |d(x, Ty) + d(y, Tx)| \} + 2d(y, Tx), \]

for each \( x, y \in [0, 1] \). Then by theorem 2.3, \( T \) has a fixed point. Since \( H(T \frac{1}{2}, T1) = \frac{3}{4} > \frac{1}{2} - 1 \) then \( T \) does not satisfy Mizoguchi-Takahashi contractive condition. Now we show that \( T \) does not satisfy the contractive condition of the above mentioned Theorem 1.4 of Haghi et al. On the contrary, assume that there exists \( 0 < k < 1 \) such that

\[ \mathcal{H}(Tx, Ty) \leq k \max \{ d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx) \}; \]

for each \( x, y \in [0, 1] \). Let \( x \in [0, 1) \) and let \( y = 1 \). Then, we have

\[ \mathcal{H}(Tx, T1) = 1 - \frac{x}{2} \leq k \max \{ \frac{x}{4}, 1 - x, 1 - \frac{3x}{4} \} = k(1 - \frac{3x}{4}). \]

Thus \( \frac{1 - \frac{3x}{4}}{1 - \frac{x}{2}} \leq k \) for each \( x \in [0, 1) \). Letting \( x \to 0 \), we get \( 1 \leq k \), a contradiction.

Acknowledgments

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References


