



# A Companion of Ostrowski's Inequality for Functions of Bounded Variation and Applications

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*Dedicated to the Memory of Charalambos J. Papaioannou*

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## Abstract

A companion of Ostrowski's inequality for functions of bounded variation and applications are given.

*Keywords:* Ostrowski's Inequality, Trapezoid Rule, Midpoint Rule.

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## 1. Introduction

In [11], the author has proved the following inequality of Ostrowski type [24] for functions of bounded variation.

**Theorem 1.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function of bounded variation on  $[a, b]$ . Denote by  $V_a^b(f)$  its total variation on  $[a, b]$ . Then, for any  $x \in [a, b]$ , one has the inequality:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] V_a^b(f). \quad (1.1)$$

The constant  $\frac{1}{2}$  is best possible in the sense that it cannot be replaced by a smaller quantity.

The above inequality (1.1) has as a remarkable particular case, the *mid-point inequality*, namely

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{2} V_a^b(f).$$

Here  $\frac{1}{2}$  is a best constant as well.

The corresponding version for the generalized trapezoid inequality was obtained in [4].

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**Theorem 1.2.** *With the assumptions in Theorem 1.1, one has the inequality*

$$\left| \frac{(x-a)f(a) + (b-x)f(b)}{b-a} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] V_a^b(f) \quad (1.2)$$

for any  $x \in [a, b]$ .

Here the constant  $\frac{1}{2}$  is also best possible.

The trapezoid inequality

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{2} V_a^b(f)$$

is the best inequality one can derive from (1.2). Here the constant  $\frac{1}{2}$  is also sharp.

Recently, Guessab and Schmeisser [23], in the effort of incorporating together the mid-point and trapezoid inequality, have proved amongst others, the following companion of Ostrowski's inequality.

**Theorem 1.3.** *Assume that the function  $f : [a, b] \rightarrow \mathbb{R}$  is of  $H - r$ -Hölder type with  $r \in (0, 1]$ , i.e.,*

$$|f(t) - f(s)| \leq H |t - s|^r \text{ for any } t, s \in [a, b]. \quad (1.3)$$

Then, for each  $x \in [a, \frac{a+b}{2}]$ , one has the inequality

$$\left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{2^{r+1}(x-a)^{r+1} + (a+b-2x)^{r+1}}{2^r(r+1)(b-a)} \right] H. \quad (1.4)$$

This inequality is sharp for each admissible  $x$ . Equality is obtained if and only if  $f = \pm H f_* + c$ , with  $c \in \mathbb{R}$  and

$$f_*(t) = \begin{cases} (x-t)^r, & \text{for } a \leq t \leq x \\ (t-x)^r, & \text{for } x \leq t \leq \frac{1}{2}(a+b) \\ f_*(a+b-t), & \text{for } \frac{1}{2}(a+b) \leq t \leq b. \end{cases} \quad (1.5)$$

**Remark 1.4.** *For  $r = 1$ , i.e.,  $f$  is Lipschitzian with the constant  $L > 0$ , and since*

$$\frac{4(x-a)^2 + (a+b-2x)^2}{4(b-a)} = \left[ \frac{1}{8} + 2 \left( \frac{x - \frac{3a+b}{4}}{b-a} \right)^2 \right] (b-a)$$

then, by (1.4), we get the following companion of Ostrowski's inequality for Lipschitzian functions

$$\left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{8} + 2 \left( \frac{x - \frac{3a+b}{4}}{b-a} \right)^2 \right] (b-a) L, \quad (1.6)$$

for any  $x \in [a, \frac{a+b}{2}]$ .

The constant  $\frac{1}{8}$  is best possible in (1.6) in the sense that it cannot be replaced by a smaller constant.

By substituting  $x = \frac{3a+b}{4}$  into the above inequality, we obtain the following trapezoid type inequality, which is the best in the class,

$$\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{8} (b-a) L. \quad (1.7)$$

The constant  $\frac{1}{8}$  here is also best possible in the above sense.

For a monograph devoted to Ostrowski type inequalities, see [18].  
 For research papers on Ostrowski's inequality see [1]-[17], [19]-[21] and [22].  
 The main aim of this paper is to provide a sharp bound for the difference

$$\frac{f(x) + f(a + b - x)}{2} - \frac{1}{b - a} \int_a^b f(t) dt,$$

where  $f$  is assumed to be of bounded variation. Some applications for cumulative distribution function and quadrature rules are also given.

## 2. Some Integral Inequalities

The following identity holds.

**Lemma 2.1.** *Assume that the function  $f : [a, b] \rightarrow \mathbb{R}$  is of bounded variation on  $[a, b]$ . Then we have the equality*

$$\begin{aligned} & \frac{1}{2} [f(x) + f(a + b - x)] - \frac{1}{b - a} \int_a^b f(t) dt \\ &= \frac{1}{b - a} \left[ \int_a^x (t - a) df(t) + \int_x^{a+b-x} \left( t - \frac{a + b}{2} \right) df(t) + \int_{a+b-x}^b (t - b) df(t) \right] \end{aligned} \tag{2.1}$$

for any  $x \in [a, \frac{a+b}{2}]$ .

**Proof .** Obviously, all the Riemann-Stieltjes integrals from the right hand side of (2.1) exist because the functions  $(\cdot - a)$ ,  $(\cdot - \frac{a+b}{2})$  and  $(\cdot - b)$  are continuous on these intervals and  $f$  is of bounded variation.

Using the integration by parts formula for Riemann-Stieltjes integrals, we have, for any  $x \in [a, \frac{a+b}{2}]$ , that

$$\int_a^x (t - a) df(t) = f(x)(x - a) - \int_a^x f(t) dt,$$

$$\int_x^{a+b-x} \left( t - \frac{a + b}{2} \right) df(t) = f(a + b - x) \left( \frac{a + b}{2} - x \right) - f(x) \left( x - \frac{a + b}{2} \right) - \int_x^{a+b-x} f(t) dt$$

and

$$\int_{a+b-x}^b (t - b) df(t) = (x - a) f(a + b - x) - \int_{a+b-x}^b f(t) dt.$$

Summing the above equalities we deduce (2.1).  $\square$

**Remark 2.2.** *A version of this identity for piecewise continuously differentiable functions has been obtained in [23, Lemma 3.2].*

The following companion of Ostrowski's inequality holds.

**Theorem 2.3.** *Assume that the function  $f : [a, b] \rightarrow \mathbb{R}$  is of bounded variation on  $[a, b]$ . Then we have the inequalities:*

$$\left| \frac{1}{2} [f(x) + f(a + b - x)] - \frac{1}{b - a} \int_a^b f(t) dt \right| \tag{2.2}$$

$$\leq \frac{1}{b-a} \left[ (x-a) \mathbb{V}_a^x(f) + \left(\frac{a+b}{2} - x\right) \mathbb{V}_x^{a+b-x}(f) + (x-a) \mathbb{V}_{a+b-x}^b(f) \right]$$

$$\leq \begin{cases} [14 + |x - \frac{3a+b}{4} - a|]_a^b(f) \\ [2(x-ab-a)^\alpha + (\frac{a+b}{2} - xb - a)^\alpha]^{\frac{1}{\alpha}} \\ \times \left[ \left[ \mathbb{V}_a^x(f) \right]^\beta + \left[ \mathbb{V}_x^{a+b-x}(f) \right]^\beta + \left[ \mathbb{V}_{a+b-x}^b(f) \right]^\beta \right]^{\frac{1}{\beta}}, \text{ if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1 \text{ for any } x \in [a, \frac{a+b}{2}], \\ [x-a + \frac{b-a}{2} - a] \max \left\{ \mathbb{V}_a^x(f), \mathbb{V}_x^{a+b-x}(f), \mathbb{V}_{a+b-x}^b(f) \right\} \end{cases}$$

where  $\mathbb{V}_c^d(f)$  denotes the total variation of  $f$  on  $[c, d]$ . The constant  $\frac{1}{4}$  is best possible in the first branch of the second inequality in (2.2).

**Proof .** We use the fact that for a continuous function  $p : [c, d] \rightarrow \mathbb{R}$  and a function  $v : [a, b] \rightarrow \mathbb{R}$  of bounded variation, one has the inequality

$$\left| \int_c^d p(t) dv(t) \right| \leq \sup_{t \in [c, d]} |p(t)| \mathbb{V}_c^d(v). \quad (2.3)$$

Taking the modulus in (2.1) we have

$$\begin{aligned} & \left| \frac{1}{2} [f(x) + f(a+b-x)] - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{b-a} \left[ \left| \int_a^x (t-a) df(t) \right| + \left| \int_x^{a+b-x} (t - \frac{a+b}{2}) df(t) \right| \right. \\ & \left. + \left| \int_{a+b-x}^b (t-b) df(t) \right| \right] \\ & \leq \frac{1}{b-a} \left[ (x-a) \mathbb{V}_a^x(f) + \left(\frac{a+b}{2} - x\right) \mathbb{V}_x^{a+b-x}(f) + (x-a) \mathbb{V}_{a+b-x}^b(f) \right] =: M(x) \end{aligned}$$

and the first inequality in (2.2) is obtained.

Now, observe that

$$\begin{aligned} M(x) & \leq \frac{1}{b-a} \max \left\{ x-a, \frac{a+b}{2} - x \right\} \left[ \mathbb{V}_a^x(f) + \mathbb{V}_x^{a+b-x}(f) + \mathbb{V}_{a+b-x}^b(f) \right] \\ & = \frac{1}{b-a} \left[ \frac{1}{4}(b-a) + |x - \frac{3a+b}{4}| \right] \mathbb{V}_a^b(f) \end{aligned}$$

and the first branch in the second inequality in (2.2) is proved.

Using Hölder's discrete inequality we have (for  $\alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1$ ) that

$$\begin{aligned} M(x) & \leq \frac{1}{b-a} \left[ (x-a)^\alpha + \left(\frac{a+b}{2} - x\right)^\alpha + (x-a)^\alpha \right]^{\frac{1}{\alpha}} \\ & \quad \times \left[ \left[ \mathbb{V}_a^x(f) \right]^\beta + \left[ \mathbb{V}_x^{a+b-x}(f) \right]^\beta + \left[ \mathbb{V}_{a+b-x}^b(f) \right]^\beta \right]^{\frac{1}{\beta}} \end{aligned}$$

giving the second branch in the second inequality.

Finally, we have

$$\begin{aligned} M(x) & \leq \frac{1}{b-a} \max \left\{ \mathbb{V}_a^x(f), \mathbb{V}_x^{a+b-x}(f), \mathbb{V}_{a+b-x}^b(f) \right\} \\ & \quad \times [(x-a) + (\frac{a+b}{2} - x) + (x-a)], \end{aligned}$$

which is equivalent with the last inequality in (2.2).

The sharpness of the constant  $\frac{1}{4}$  in the first branch of the second inequality in (2.2) will be proved in a particular case later.  $\square$

**Corollary 2.4.** *With the assumptions in Theorem 2.3, one has the trapezoid inequality*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{2} \bigvee_a^b(f). \tag{2.4}$$

The constant  $\frac{1}{2}$  is best possible in (2.4).

**Proof .** Follows from the first inequality in (2.2) on choosing  $x = a$ . For the sharpness of the constant, assume that (2.4) holds with a constant  $A > 0$ , i.e.,

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq A \bigvee_a^b(f). \tag{2.5}$$

If we choose  $f : [a, b] \rightarrow \mathbb{R}$  with

$$f(x) = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{if } x \in (a, b), \\ 1 & \text{if } x = b, \end{cases}$$

then  $f$  is of bounded variation on  $[a, b]$  and

$$\frac{f(a) + f(b)}{2} = 1, \quad \int_a^b f(t) dt = 0, \quad \text{and} \quad \bigvee_a^b(f) = 2,$$

giving in (2.5)  $1 \leq 2A$ , thus  $A \geq \frac{1}{2}$  and the corollary is proved.  $\square$

**Remark 2.5.** *The inequality (2.4) was first proved in a different manner in [8].*

**Corollary 2.6.** *With the assumptions in Theorem 2.3, one has the midpoint inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{2} \bigvee_a^b(f). \tag{2.6}$$

The constant  $\frac{1}{2}$  is best possible in (2.6).

**Proof .** Follows from the first inequality in (2.2) on choosing  $x = \frac{a+b}{2}$ . For the sharpness of the constant, assume that (2.6) holds with a constant  $B > 0$ , i.e.,

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq B \bigvee_a^b(f). \tag{2.7}$$

If we choose  $f : [a, b] \rightarrow \mathbb{R}$  with

$$f(x) = \begin{cases} 0 & \text{if } x \in [a, \frac{a+b}{2}), \\ 1 & \text{if } x = \frac{a+b}{2}, \\ 0 & \text{if } x \in (\frac{a+b}{2}, b], \end{cases}$$

then  $f$  is of bounded variation on  $[a, b]$ , and  $f(\frac{a+b}{2}) = 1, \int_a^b f(t) dt = 0,$  and  $\bigvee_a^b(f) = 2,$  giving in (2.7),  $1 \leq 2B$ , thus  $B \geq \frac{1}{2}$ .  $\square$

**Remark 2.7.** *The inequality (2.6) was firstly proved in a different manner in [9].*

The best inequality we may get from Theorem 2.3 on using the bound provided by the first branch in the second inequality in (2.2) is incorporated in the following corollary.

**Corollary 2.8.** *With the assumptions in Theorem 2.3, one has the inequality:*

$$\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{4} \bigvee_a^b(f). \tag{2.8}$$

The constant  $\frac{1}{4}$  is best possible.

**Proof .** Follows by Theorem 2.3 on choosing  $x = \frac{3a+b}{4}$ .

To prove the sharpness of the constant  $\frac{1}{4}$ , assume that (2.8) holds with a constant  $C > 0$ , i.e.,

$$\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq C \bigvee_a^b(f). \tag{2.9}$$

Consider the function  $f : [a, b] \rightarrow \mathbb{R}$ , given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \left\{ \frac{3a+b}{4}, \frac{a+3b}{4} \right\}, \\ 0 & \text{if } x \in [a, b] \setminus \left\{ \frac{3a+b}{4}, \frac{a+3b}{4} \right\}. \end{cases}$$

Then  $f$  is of bounded variation on  $[a, b]$ ,

$$\frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} = 1, \quad \int_a^b f(t) dt = 0$$

and

$$\bigvee_a^b(f) = 4,$$

giving in (2.9)  $4C \geq 1$ , thus  $C \geq \frac{1}{4}$ .

This example can be used to prove the sharpness of the constant  $\frac{1}{4}$  in (2.2) as well.  $\square$

### 3. Applications for P.D.F.'s

Let  $X$  be a random variable taking values in the finite interval  $[a, b]$ , with the probability density function  $f : [a, b] \rightarrow [0, \infty)$  and with the cumulative distribution function  $F(x) = \Pr(X \leq x) = \int_a^x f(t) dt$ .

We may state the following theorem.

**Theorem 3.1.** *With the above assumptions, we have the inequality*

$$\left| \frac{1}{2} [F(x) + F(a+b-x)] - \frac{b - E(X)}{b-a} \right| \tag{3.1}$$

$$\begin{aligned} &\leq \frac{1}{b-a} \left\{ \left(2x - \frac{3a+b}{4}\right) [F(x) - F(a+b-x)] + (x-a) \right\} \\ &\leq \frac{1}{4} + \left| x - \frac{3a+b}{4} \right|, \end{aligned}$$

for any  $x \in \left[ a, \frac{a+b}{2} \right]$ , where  $E(X)$  denotes the expectation of  $X$ , namely  $E(X) = \int_a^b t dF(t)$ .

**Proof .** If we apply Theorem 2.3 for  $F$ , which is monotonic nondecreasing, we get

$$\begin{aligned} & \left| \frac{1}{2} [F(x) + F(a + b - x)] - \frac{1}{b - a} \int_a^b F(t) dt \right| \tag{3.2} \\ & \leq \frac{1}{b - a} \left[ (x - a) F(x) + \left( \frac{a + b}{2} - x \right) (F(a + b - x) - F(x)) \right. \\ & \quad \left. + (x - a) (1 - F(a + b - x)) \right] \\ & \leq \frac{1}{4} + \left| x - \frac{3a + b}{4} - a \right|. \end{aligned}$$

Since

$$E(X) = \int_a^b t dF(t) = b - \int_a^b F(t) dt,$$

then by (3.2) we get (3.1) and the theorem is proved.  $\square$

In particular, we have:

**Corollary 3.2.** *With the above assumptions, we have:*

$$\left| \frac{1}{2} \left[ F\left(\frac{3a + b}{4}\right) + F\left(\frac{a + 3b}{4}\right) \right] - \frac{b - E(X)}{b - a} \right| \leq \frac{1}{4}.$$

#### 4. A Composite Quadrature Formula

Let  $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$  be a division of the interval  $[a, b]$  and  $h_i := x_{i+1} - x_i$  ( $i = 0, \dots, n - 1$ ) and  $\nu(I_n) := \max \{h_i | i = 0, \dots, n - 1\}$ .

Consider the composite quadrature rule

$$Q_n(I_n, f) = \frac{1}{2} \sum_{i=0}^{n-1} \left[ f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right) \right] h_i. \tag{4.1}$$

The following result holds.

**Theorem 4.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function of bounded variation on  $[a, b]$ . Then we have*

$$\int_a^b f(t) dt = Q_n(I_n, f) + R_n(I_n, f) \tag{4.2}$$

where  $Q_n(I_n, f)$  is defined in formula (4.1), and the remainder  $R_n(I_n, f)$  satisfies the estimate

$$|R_n(I_n, f)| \leq \frac{1}{4} \nu(I_n) \bigvee_a^b(f). \tag{4.3}$$

The constant  $\frac{1}{4}$  is best possible.

**Proof .** Applying Corollary 2.8 on the interval  $[x_i, x_{i+1}]$  we may state that

$$\left| \int_{x_i}^{x_{i+1}} f(t) dt - \frac{1}{2} \left[ f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right) \right] h_i \right| \leq \frac{1}{4} h_i \bigvee_{x_i}^{x_{i+1}}(f), \tag{4.4}$$

for any  $i \in \{0, \dots, n - 1\}$ .

Summing the inequality (4.4) over  $i$  from 0 to  $n - 1$ , and using the generalized triangle inequality we get

$$|R_n(I_n, f)| \leq \frac{1}{4} \sum_{i=0}^{n-1} h_i \bigvee_{x_i}^{x_{i+1}}(f) \leq \frac{1}{4} \nu(I_n) \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}}(f) = \frac{1}{4} \nu(I_n) \bigvee_a^b(f),$$

and the proof is completed.  $\square$

For the particular case when the division  $I_n$  is equidistant, i.e.,

$$I_n : x_i = a + i \cdot \frac{b-a}{n}, \quad i = 0, \dots, n,$$

we may consider the quadrature rule:

$$Q_n(f) := \frac{b-a}{2n} \sum_{i=0}^{n-1} \left\{ f \left[ a + \left( \frac{4i+1}{4n} \right) (b-a) \right] + f \left[ a + \left( \frac{4i+3}{4n} \right) (b-a) \right] \right\}.$$

The following corollary will be more useful in practice.

**Corollary 4.2.** *With the assumption of Theorem 4.1, we have*

$$\int_a^b f(t) dt = Q_n(f) + R_n(f), \quad (4.5)$$

where  $Q_n(f)$  is defined by (4) and the remainder  $R_n(f)$  satisfies the estimate

$$|R_n(f)| \leq \frac{1}{4} \cdot \frac{b-a}{n} \bigvee_a^b(f). \quad (4.6)$$

The constant  $\frac{1}{4}$  is sharp.

**Remark 4.3.** *If one is interested in finding the minimal number of points for the equidistant partition  $I_n$  so that the theoretical error in (4.6) is smaller than  $\varepsilon > 0$ , then this number  $n_\varepsilon$  is given by*

$$n_\varepsilon := \left[ \frac{1}{4} \cdot \frac{b-a}{\varepsilon} \bigvee_a^b(f) \right] + 1, \quad (4.7)$$

where  $[a]$  denotes the integer part of the positive number  $a$ .

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