Fixed points for Banach and Kannan contractions in modular spaces with a graph

Aris Aghanians\textsuperscript{a}, Kourosh Nourouzi\textsuperscript{a,}\textsuperscript{*}

\textsuperscript{a}Faculty of Mathematics, K. N. Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran.

(Communicated by M.B. Ghaemi)

Abstract

In this paper, we discuss the existence and uniqueness of fixed points for Banach and Kannan contractions defined on modular spaces endowed with a graph. We do not impose the $\Delta_2$-condition or the Fatou property on the modular spaces to give generalizations of some recent results. The given results play as a modular version of metric fixed point results.

Keywords: Complete modular space, Fixed point, Banach contraction, Kannan contraction.

2010 MSC: Primary 47H10; Secondary 46A80, 05C40.

1. Introduction and preliminaries

To control the pathological behavior of a modular in modular spaces the conditions $\Delta_2$ and Fatou property are usually assumed (see, e.g., \cite{1,5,7,8,11,12}. For instance, in \cite{1}, Banach fixed point theorem is given in modular spaces that their modular satisfy both the $\Delta_2$-condition and the Fatou property. In \cite{7}, Khamis established some fixed point theorems for quasi-contractions in modular spaces satisfying only the Fatou property.

In \cite{6}, Jachymski investigated Banach fixed point theorem in metric spaces with a graph and his idea followed by the authors in uniform spaces (see, e.g., \cite{2,3}).

In this paper motivated by the ideas given in \cite{1,6}, we aim to discuss the fixed points of Banach and Kannan contractions in modular spaces endowed with a graph without $\Delta_2$-condition and Fatou property. We also clarify the independence of these contractions in modular spaces.

We first commence some basic concepts about modular spaces as formulated by Musielak and Orlicz \cite{10}. For more details, the reader is referred to \cite{9}.
Definition 1.1. A real-valued function $\rho$ defined on a real vector space $X$ is called a modular on $X$ if it satisfies the following conditions:

1. $\rho(x) \geq 0$ for all $x \in X$; 
2. $\rho(x) = 0$ if and only if $x = 0$; 
3. $\rho(x) = \rho(-x)$ for all $x \in X$; 
4. $\rho(ax + by) \leq \rho(a) \rho(x) + \rho(y)$ for all $x, y \in X$ and all $a, b \geq 0$ with $a + b = 1$.

If $\rho$ satisfies (M1)-(M4), then the pair $(X, \rho)$, shortly denoted by $X$, is called a modular space.

The modular $\rho$ is called convex if Condition (M4) is strengthened by replacing with

$\rho(ax + by) \leq a\rho(x) + b\rho(y)$ for all $a, b \geq 0$ with $a + b = 1$.

It is easy to obtain the following two immediate consequences of Condition (M4) which we need in the sequel:

1. If $a$ and $b$ are real numbers with $|a| \leq |b|$, then $\rho(ax) \leq \rho(bx)$ for all $x \in X$; 
2. If $a_1, \ldots, a_n$ are nonnegative numbers with $\sum_{i=1}^n a_i = 1$, then

$$\rho\left(\sum_{i=1}^n a_i x_i\right) \leq \sum_{i=1}^n \rho(x_i) \quad (x_1, \ldots, x_n \in X).$$

Definition 1.2. Let $(X, \rho)$ be a modular space.

1. A sequence $\{x_n\}$ in $X$ is said to be $\rho$-convergent to a point $x \in X$, denoted by $x_n \xrightarrow{\rho} x$, if $\rho(x_n - x) \to 0$ as $n \to \infty$. 
2. A sequence $\{x_n\}$ in $X$ is said to be $\rho$-Cauchy if $\rho(x_m - x_n) \to 0$ as $m, n \to \infty$. 
3. The modular space $X$ is called $\rho$-complete if each $\rho$-Cauchy sequence in $X$ is $\rho$-convergent to a point of $X$. 
4. The modular $\rho$ is said to satisfy the $\Delta_2$-condition if $2x_n \xrightarrow{\rho} 0$ as $n \to \infty$ whenever $x_n \xrightarrow{\rho} 0$ as $n \to \infty$. 
5. The modular $\rho$ is said to have the Fatou property if

$$\rho(x - y) \leq \liminf_{n \to \infty} \rho(x_n - y_n)$$

whenever

$$x_n \xrightarrow{\rho} x \quad \text{and} \quad y_n \xrightarrow{\rho} y \quad \text{as} \quad n \to \infty.$$

Conditions (M2) and (M4) ensure that each sequence in a modular space can be $\rho$-convergent to at most one point. In other words, the limit of a $\rho$-convergent sequence in a modular space is unique.

We next review some notions in graph theory. All of them can be found in, e.g., [4].

Let $X$ be a modular space. Consider a directed graph $G$ with $V(G) = X$ and $E(G) \supseteq \{(x, x) : x \in X\}$, i.e., $E(G)$ contains all loops. Suppose further that $G$ has no parallel edges. With these assumptions, we may denote $G$ by the pair $(V(G), E(G))$. In this way, the modular space $X$ is endowed with the graph $G$. The notation $\tilde{G}$ is used to denote the undirected graph obtained from $G$ by deleting the directions of the edges of $G$. Thus,

$$V(G) = X \quad \text{and} \quad E(G) = \{(x, y) \in X \times X : (x, y) \in E(G) \lor (y, x) \in E(G)\}.$$
2. Main results

Let $X$ be a modular space endowed with a graph $G$ and $f : X \to X$ be any mapping. The set of all fixed points for $f$ is denoted by $\text{Fix}(f)$, and by $C_f$, we mean the set of all elements $x$ of $X$ such that $(f^nx, f^m x) \in E(\tilde{G})$ for $m, n = 0, 1, \ldots$.

We begin with introducing Banach and Kannan $G$-$\rho$-contractions.

**Definition 2.1.** Let $X$ be a modular space with a graph $G$ and $f : X \to X$ be a mapping. We call $f$ a Banach $G$-$\rho$-contraction if

- (B1) $f$ preserves the edges of $G$, i.e., $(x, y) \in E(G)$ implies $(fx, fy) \in E(G)$ for all $x, y \in X$;
- (B2) there exist positive numbers $k, a$ and $b$ with $k < 1$ and $a < b$ such that
  \[\rho(b fx - fy) \leq k \rho(a x - y)\]
  for all $x, y \in X$ with $(x, y) \in E(G)$.

The numbers $k, a$ and $b$ are called the constants of $f$. And we call $f$ a Kannan $G$-$\rho$-contraction if

- (K1) $f$ preserves the edges of $G$;
- (K2) there exist positive numbers $k, l, a_1, a_2$ and $b$ with $k + l < 1$, $a_1 \leq \frac{b}{2}$ and $a_2 \leq b$ such that
  \[\rho(b fx - fy) \leq k \rho(a_1(x - x)) + l \rho(a_2(y - y))\]
  for all $x, y \in X$ with $(x, y) \in E(G)$.

The numbers $k, l, a_1, a_2$ and $b$ are called the constants of $f$.

It might be valuable if we discuss these contractions a little. Our first proposition follows immediately from Condition (M3) and Definition 2.1.

**Proposition 2.2.** Let $X$ be a modular space with a graph $G$. If a mapping from $X$ into itself satisfies (B1) (respectively, (B2)) for $G$, then it satisfies (B1) (respectively, (B2)) for $\tilde{G}$. In particular, a Banach $G$-$\rho$-contraction is also a Banach $\tilde{G}$-$\rho$-contraction. Similar statements are true for Kannan $G$-$\rho$-contractions provided that $a_2 \leq \frac{b}{2}$.

We also have the following remark about Kannan $G$-$\rho$-contractions.

**Remark 2.3.** For a Kannan $\tilde{G}$-$\rho$-contraction $f : X \to X$, we can interchange the roles of $x$ and $y$ in (K2) since $E(\tilde{G})$ is symmetric. Having done this, we find

\[
\begin{align*}
\rho(b fx - fy) &= \rho(b fy - fx) \\
&\leq k \rho(a_1(fy - y)) + l \rho(a_2(fx - x)) \\
&= l \rho(a_2(fx - x)) + k \rho(a_1(fy - y)).
\end{align*}
\]

Therefore, no matter $a_1 \leq \frac{b}{2}$ or $a_2 \leq \frac{b}{2}$ whenever we are faced with Kannan $\tilde{G}$-$\rho$-contractions. Nevertheless, both $a_1$ and $a_2$ must be not more than $b$.

We now give some examples.

**Example 2.4.** Let $X$ be a modular space with any arbitrary graph $G$. Since $E(G)$ contains all loops, each constant mapping $f : X \to X$ is both a Banach and a Kannan $G$-$\rho$-contraction. In fact, $E(G)$ should contain all loops if we want any constant mapping to be either a Banach or a Kannan $G$-$\rho$-contraction.
Example 2.5. Let $X$ be a modular space and $G_0$ be the complete graph $(X, X \times X)$. Then Banach (Kannan) $G_0$-contractions are precisely the Banach (Kannan) contractions in modular spaces.

Example 2.6. Let $\leq$ be a partial order on a modular space $X$ and consider a poset graph $G_1$ by $V(G_1) = X$ and $E(G_1) = \{(x, y) \in X \times X : x \leq y\}$. Then Banach $G_1$-contractions are precisely the nondecreasing ordered $\rho$-contractions. A similar statement is true for Kannan $G_1$-contractions.

Finally, we show that Banach and Kannan $G$-contractions are independent of each other. More precisely, we construct two mappings on $\mathbb{R}$ such that one of them satisfies (B2) but not (K2), and the other, (K2) but not (B2) for the complete graph $G_0$.

Example 2.7. Let $\rho$ be the usual Euclidean norm on $\mathbb{R}$, i.e., $\rho(x) = |x|$ for all $x \in \mathbb{R}$. Define a mapping $f : \mathbb{R} \to \mathbb{R}$ by $fx = \frac{x}{3}$ for all $x \in \mathbb{R}$. Then $f$ is a Banach $G_0$-contraction with the constants $k = \frac{2}{3}$, $a = \frac{1}{2}$ and $b = 1$. Indeed, given any $x, y \in \mathbb{R}$, we have

$$\rho(b(fx-fy)) = \frac{1}{3} |x-y| = k\rho(a(x-y)).$$

On the other hand, if $k, l, a_1, a_2$ and $b$ are any arbitrary positive numbers satisfying $k+l < 1$, $a_1 \leq \frac{b}{2}$ and $a_2 \leq b$, then for $y = 0$ and any $x \neq 0$ we see that

$$\rho(b(fx-f0)) = \frac{b|x|}{3} > \frac{2a_1k|x|}{3} = k\rho(a_1(fx-x)) + l\rho(a_2(f0-0)).$$

Therefore, (K2) fails to hold and $f$ is not a Kannan $G_0$-contraction.

Example 2.8. It is easy to verify that the function $\rho(x) = x^2$ defines a modular on $\mathbb{R}$ and $(\mathbb{R}, \rho)$ is $\rho$-complete because $(\mathbb{R}, |\cdot|)$ is a Banach space. Now, consider a mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $fx = \frac{1}{2}$ if $x \neq 1$ and $f1 = \frac{1}{10}$. Then $f$ is a $G_0$-Kannan contraction with the constants $k = \frac{64}{81}$, $l = \frac{16}{81}$, $a_1 = \frac{1}{2}$ and $a_2 = b = 1$. Indeed, given any $x, y \in \mathbb{R}$, we have the following three possible cases:

**Case 1:** If $x = y = 1$ or $x, y \neq 1$, then (K2) holds trivially since $fx = fy$;

**Case 2:** If $x = 1$ and $y \neq 1$, then

$$\rho(b(fx-fy)) = \frac{4}{25} \leq \frac{4}{25} + \frac{16}{81} \left(\frac{1}{2} - y\right)^2 = k\rho(a_1(fx-x)) + l\rho(a_2(fy-y));$$

**Case 3:** Finally, if $x \neq 1$ and $y = 1$, then

$$\rho(b(fx-fy)) = \frac{4}{25} \leq \frac{16}{81} \left(\frac{1}{2} - x\right)^2 + \frac{4}{25} = k\rho(a_1(fx-x)) + l\rho(a_2(fy-y)).$$

Note that $k + l = \frac{80}{81} < 1$, $a_1 \leq \frac{b}{2}$ and $a_2 \leq b$. But $f$ is not a Banach $G_0$-contraction; for if $k$, $a$ and $b$ are any arbitrary positive numbers satisfying $k < 1$ and $a < b$, then putting $x = 1$ and $y = \frac{2}{3}$ yields

$$\rho(b(fx-fy)) = \frac{4b^2}{25} > \frac{4a^2k}{25} = k\rho(a(x-y)).$$

Now we are going to prove our fixed point results. The first one is about the existence and uniqueness of fixed points for Banach $G$-contractions.
Theorem 2.9. Let $X$ be a $\rho$-complete modular space endowed with a graph $G$ and the triple $(X, \rho, G)$ have the following property:

(\ast) If $\{x_n\}$ is a sequence in $X$ such that $\beta x_n \xrightarrow{\rho} \beta x$ for some $\beta > 0$ and $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $(x_{n_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.

Then a Banach $\tilde{G}$-$\rho$-contraction $f : X \to X$ has a fixed point if and only if $C_f \neq \emptyset$. Moreover, this fixed point is unique if $G$ is weakly connected.

Proof. (\Rightarrow) It is trivial since $\text{Fix}(f) \subseteq C_f$.

(\Leftarrow) Let $k, a$ and $b$ be the constants of $f$ and let $\alpha > 1$ be the exponential conjugate of $\frac{b}{a}$, i.e., $\frac{a}{b} + \frac{1}{\alpha} = 1$. Choose an $x \in C_f$ and keep it fixed. We are going to show that the sequence $\{bf^n x\}$ is $\rho$-Cauchy in $X$. To this end, note first if $n$ is a positive integer, then by (B2) we have

\[
\rho(a(f^nx - x)) = \rho(a(f^n x - fx) + a(fx - x)) = \rho\left(\frac{a}{b}b(f^n x - fx) + \frac{1}{\alpha} a(fx - x)\right) \leq \rho(b(f^n x - fx)) + \rho\left(\alpha a(fx - x)\right)
\]

\[
\leq k \rho(a(f^{n-1} x - x)) + r,
\]

where $r = \rho(\alpha a(fx - x))$. Hence using the mathematical induction, we get

\[
\rho(a(f^nx - x)) \leq k \rho(a(f^{n-1} x - x)) + r \leq k \left[k \rho(a(f^{n-2} x - x)) + r\right] + r = k^2 \rho(a(f^{n-2} x - x)) + kr + r
\]

\[
\vdots
\]

\[
\leq k^{n-1} \rho(a(f x - x)) + k^{n-2} r + \ldots + r
\]

for all $n \geq 1$. Since $\alpha > 1$, it follows that $\rho(a(fx - x)) \leq k \rho(\alpha a(fx - x)) = r$ and therefore,

\[
\rho(a(f^nx - x)) \leq k^{n-1} r + \ldots + r = \frac{(1 - k^n)r}{1 - k} \leq \frac{r}{1 - k} \quad n = 1, 2, \ldots . \tag{2.1}
\]

Now using (B2) once more, we find

\[
\rho(b(f^mx - f^n x)) \leq k \rho(a(f^{m-1} x - f^{n-1} x)) \leq k \rho(b(f^{m-1} x - f^{n-1} x)) \leq k^n \rho(a(f^{m-n} x - x))
\]

\[
\tag{2.2}
\]

for all $m$ and $n$ with $m > n \geq 1$. Consequently, by combining (2.1) and (2.2), it is seen that for all $m > n \geq 1$ we have

\[
\rho(b(f^mx - f^n x)) \leq k^n \rho(a(f^{m-n} x - x)) \leq \frac{k^n r}{1 - k}.
\]

Therefore, $\rho(b(f^mx - f^n x)) \to 0$ as $m, n \to \infty$, and so $\{bf^n x\}$ is a $\rho$-Cauchy sequence in $X$ and because $X$ is $\rho$-complete, it is $\rho$-convergent. On the other hand, $X$ is a real vector space and $b \geq 0$. Thus, there exists an $x^* \in X$ such that $bf^n x \xrightarrow{\rho} bx^*$.
We next show that $x^*$ is a fixed point for $f$. Since $x \in C_f$, it follows that $(f^n x, f^{n+1} x) \in E(\widetilde{G})$ for all $n \geq 0$, and so by Property $(\ast)$, there exists a strictly increasing sequence $\{n_i\}$ of positive integers such that $(f^{n_i} x, x^*) \in E(\widetilde{G})$ for all $i \geq 1$. Hence using $(B2)$ we get

$$\rho\left(\frac{b}{2}(f^* x - x^*)\right) = \rho\left(\frac{b}{2}(f^* x - f^{n_i+1} x) + \frac{b}{2}(f^{n_i+1} x - x^*)\right) \leq \rho(b(f^* x - f^{n_i+1} x)) + \rho(b(f^{n_i+1} x - x^*)) = \rho(b(f^{n_i+1} x - f^* x)) + \rho(b(f^{n_i+1} x - x^*)) \leq k\rho(a(f^{n_i} x - x^*)) + \rho(b(f^{n_i+1} x - x^*)) \leq k\rho(b(f^* x - x^*)) + \rho(b(f^{n_i+1} x - x^*)) \rightarrow 0$$

as $i \rightarrow \infty$. So $\rho(\frac{b}{2}(f^* x - x^*)) = 0$, and since $b > 0$, it follows that $f^* x = x^*$ or equivalently, $f x^* = x^*$, i.e., $x^*$ is a fixed point for $f$.

Finally, to prove the uniqueness of the fixed point, suppose that $G$ is weakly connected and $y^* \in X$ is a fixed point for $f$. Then there exists a path $(x_s)_{s=0}^N$ in $\widetilde{G}$ from $x^*$ to $y^*$, i.e., $x_0 = x^*$, $x_N = y^*$, and $(x_{s-1}, x_s) \in E(\widetilde{G})$ for $s = 1, \ldots, N$. Thus, by $(B1)$, we have

$$(f^n x_{s-1}, f^n x_s) \in E(\widetilde{G}) \quad (n \geq 0 \text{ and } s = 1, \ldots, N).$$

And using $(B2)$ and the mathematical induction we get

$$\rho\left(\frac{b}{N}(x^* - y^*)\right) = \rho\left(\frac{b}{N}(x^* - f^1 x_1) + \cdots + \frac{b}{N}(f^n x_{N-1} - y^*)\right) \leq \rho(b(x^* - f^1 x_1)) + \cdots + (b(f^n x_{N-1} - y^*)) = \sum_{s=1}^{N} \rho(b(f^s x_{s-1} - f^s x_s)) \leq k \sum_{s=1}^{N} \rho(a(f^{n-1} x_{s-1} - f^{n-1} x_s)) \leq k \sum_{s=1}^{N} \rho(b(f^{n-1} x_{s-1} - f^{n-1} x_s)) \vdots \leq k^n \sum_{s=1}^{N} \rho(b(x_{s-1} - x_s)) \rightarrow 0$$

as $n \rightarrow \infty$. So $\frac{b}{N}(x^* - y^*) = 0$, and since $b > 0$, it follows that $x^* = y^*$. Consequently, the fixed point of $f$ is unique. \(\square\)

Setting $G = G_0$ and $G = G_1$, we get the following consequences of Theorem 2.9 in modular and partially ordered modular spaces, respectively.

**Corollary 2.10.** Let $X$ be a $\rho$-complete modular space and a mapping $f : X \rightarrow X$ satisfies

$$\rho(b(fx - fy)) \leq k\rho(a(x - y)) \quad (x, y \in X),$$

where $0 < k < 1$ and $0 < a < b$. Then $f$ has a unique fixed point $x^* \in X$ and $bf^n x \xrightarrow{\rho} bx^*$ for all $x \in X$. 

Corollary 2.11. Let $\preceq$ be a partial order on a $\rho$-complete modular space $X$ such that the triple $(X, \rho, \preceq)$ has the following property:

(**) If $\{x_n\}$ is a sequence in $X$ with successive comparable terms such that $\beta x_n \overset{\rho}{\longrightarrow} \beta x$ for some $\beta > 0$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \preceq x$ for all $i \geq 1$.

Assume that a nondecreasing mapping $f : X \to X$ satisfies

$$\rho(b(fx - fy)) \leq kp(a(x - y)) \quad (x, y \in X \text{ and } x \preceq y),$$

where $0 < k < 1$ and $0 < a < b$. Then $f$ has a fixed point if and only if there exists an $x \in X$ such that $T^n x$ is comparable to $T^n x$ for all $m, n \geq 0$. Moreover, this fixed point is unique if the following condition holds:

For all $x, y \in X$, there exists a finite sequence $(x_i)_{i=0}^N$ in $X$ with comparable successive terms such that $x_0 = x$ and $x_N = y$.

Corollary 2.12. Let $(X, \rho)$ be a $\rho$-complete modular space endowed with a graph $G$, where $\rho$ is a convex modular, and the triple $(X, \rho, G)$ have Property $(\ast)$. Assume that $f : X \to X$ is a mapping which preserves the edges of $\tilde{G}$ and satisfies

$$\rho(b(fx - fy)) \leq kp(a(x - y)) \quad (x, y \in X \text{ and } (x, y) \in E(\tilde{G})),$$

where $k$, $a$ and $b$ are positive numbers with $b > \max\{a, ak\}$. Then $f$ has a fixed point if and only if $C_f \neq \emptyset$. Moreover, this fixed point is unique if $G$ is weakly connected.

Proof. Set $c = \max\{a, ak\}$ and choose any $a_0 \in (c, b)$. Then by the hypothesis and convexity of $\rho$, we have

$$\rho(b(fx - fy)) \leq kp(a(x - y))$$

$$= kp\left(\frac{a}{a_0}a_0(x - y) + (1 - \frac{a}{a_0})0\right)$$

$$\leq \frac{ak}{a_0} \rho(a_0(x - y))$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$. Since $a_0 < b$, and $\frac{ak}{a_0} < 1$, it follows that $f$ satisfies (B2) for the graph $\tilde{G}$ with the constants $k$ and $a$ replaced with $\frac{ak}{a_0}$ and $a_0$, respectively, and $b$ kept fixed. Since $f$ preserves the edges of $\tilde{G}$, it follows that $f$ is a Banach $\tilde{G}$-$\rho$-contraction and the results are concluded immediately from Theorem 2.9.

Our next result is about the existence and uniqueness of fixed points for Kannan $\tilde{G}$-$\rho$-contractions.

Theorem 2.13. Let $X$ be a $\rho$-complete modular space endowed with a graph $G$ and the triple $(X, \rho, G)$ have Property $(\ast)$. Then a Kannan $\tilde{G}$-$\rho$-contraction $f : X \to X$ has a fixed point if and only if $C_f \neq \emptyset$. Moreover, this fixed point is unique if $k < \frac{1}{2}$ and $X$ satisfies the following condition:

$(\ast)$ For all $x, y \in X$, there exists a $z \in X$ such that $(x, z), (y, z) \in E(\tilde{G})$. 

Proof. ($\Rightarrow$) It is trivial since $\text{Fix}(f) \subseteq C_f$.

($\Leftarrow$) Let $k$, $l$, $a_1$, $a_2$ and $b$ be the constants of $f$. Choose an $x \in C_f$ and keep it fixed. We are going to show that the sequence $\{bf^n x\}$ is $\rho$-Cauchy in $X$. Given any integer $n \geq 2$, by (K2) we have

$$\rho(b(f^n x - f^{n-1} x)) \leq k \rho(a_1(f^n x - f^{n-1} x)) + l \rho(a_2(f^n x - f^{n-1} x))$$

which yields

$$\rho(b(f^n x - f^{n-1} x)) \leq \delta \rho(b(f^n x - f^{n-2} x)),$$

where $\delta = \frac{l}{1-k} \in (0, 1)$. Hence using the mathematical induction, we get

$$\rho(b(f^n x - f^{n-1} x)) \leq \delta^n \rho(b(f x - x)) \quad n = 1, 2, \ldots.$$

Now using (K2) once more, we find

$$\rho(b(f^{m+n} x - f^n x)) \leq k \rho(a_1(f^m x - f^{m-1} x)) + l \rho(a_2(f^m x - f^{m-1} x))$$

for all $m, n \geq 1$. Therefore, $\rho(b(f^n x - f^m x)) \to 0$ as $m, n \to \infty$, and so $\{bf^n x\}$ is a $\rho$-Cauchy sequence in $X$ and because $X$ is $\rho$-complete, it is $\rho$-convergent. Thus, there exists an $x^* \in X$ such that $bf^n x \xrightarrow{\rho} bx^*$.

We next show that $x^*$ is a fixed point for $f$. Since $x \in C_f$, it follows that $(f^n x, f^{n+1} x) \in E(\tilde{G})$ for all $n \geq 0$, and so by Property ($*$), there exists a strictly increasing sequence $\{n_i\}$ of positive integers such that $(f^{-n_i} x, x^*) \in E(\tilde{G})$ for all $i \geq 1$. Hence using (K2), we get

$$\rho\left(\frac{b}{2}(fx^* - x^*)\right) = \rho\left(\frac{b}{2}(fx^* - f^{n_i+1} x) + \frac{b}{2}(f^{n_i+1} x - x^*)\right)$$

$$\leq \rho(b(fx^* - f^{n_i+1} x)) + \rho((f^{n_i+1} x - x^*))$$

$$\leq k \rho(a_1(f^{n_i} x - x^*)) + l \rho(a_2(f^{n_i} x - f^{n_i+1} x)) + \rho(b(f^{n_i+1} x - x^*))$$

$$\leq k \rho\left(\frac{b}{2}(fx^* - x^*)\right) + l \rho(b(f^{n_i+1} x - f^{n_i} x)) + \rho(b(f^{n_i+1} x - x^*))$$

for all $k \geq 1$. Hence

$$\rho\left(\frac{b}{2}(fx^* - x^*)\right) \leq \delta \rho(b(f^{n_i+1} x - f^{n_i} x)) + \frac{1}{1-k} \rho(b(f^{n_i+1} x - x^*)) \to 0$$

as $i \to \infty$. So $\rho\left(\frac{b}{2}(fx^* - x^*)\right) = 0$, and since $b > 0$, it follows that $fx^* - x^* = 0$ or equivalently, $fx^* = x^*$, i.e., $x^*$ is a fixed point for $f$.

Finally, to prove the uniqueness of the fixed point, suppose that Condition ($*$) holds and $y^* \in X$ is a fixed point for $f$. We consider the following two cases:

**Case 1**: $(x^*, y^*)$ is an edge of $\tilde{G}$.

In this case, using (K2), we find

$$\rho(b(x^* - y^*)) = \rho(b(fx^* - fy^*)) \leq k \rho(a_1(fx^* - x^*)) + l \rho(a_2(fy^* - y^*)) = 0.$$
Therefore, \( \rho(b(x^* - y^*)) = 0 \), and so \( x^* = y^* \) because \( b > 0 \).

**Case 2: \( (x^*, y^*) \) is not an edge of \( G \).**

In this case, by Condition \((*)\), there exists a \( z \in X \) such that both \( (x^*, z) \) and \( (y^*, z) \) are edges of \( \tilde{G} \). So by (K1), we have \( (x^*, f^n z), (y^*, f^n z) \in E(\tilde{G}) \) for all \( n \geq 0 \) since \( x^* \) is a fixed point for \( f \). Therefore, by (K2) we find

\[
\rho(b(f^n z - x^*)) = \rho(b(f^n z - f^n x^*)) \\
\leq k \rho(a_1(f^n z - f^{n-1} z)) + l \rho(a_2(f^n x^* - f^{n-1} x^*)) \\
\leq k \rho\left(\frac{b}{2}(f^n z - f^{n-1} z)\right) \\
= k \rho\left(\frac{b}{2}(f^n z - f^n x^*) + \frac{b}{2}(f^{n-1} x^* - f^{n-1} z)\right) \\
\leq k \rho(b(f^n z - f^n x^*)) + k \rho(b(f^{n-1} x^* - f^{n-1} z)) \\
= k \rho(b(f^n z - x^*)) + k \rho(b(f^{n-1} z - x^*))
\]

for all \( n \geq 1 \), which yields

\[
\rho(b(f^n z - x^*)) \leq \lambda \rho(b(f^{n-1} z - x^*)),
\]

where \( \lambda = \frac{k}{1-k} \in (0, 1) \) because \( k < \frac{1}{2} \). So by the mathematical induction, we get

\[
\rho(b(f^n z - x^*)) \leq \lambda^n \rho(b(z - x^*)) \quad n = 0,1,\ldots.
\]

Since \( \lambda < 1 \), it follows that \( bf^n z \xrightarrow{\rho} bx^* \). Similarly, one can show that \( bf^n z \xrightarrow{\rho} by^* \), and so \( bx^* = by^* \) because the limit of a \( \rho \)-convergent sequence in a modular space is unique. Thus, from \( b > 0 \), it follows that \( x^* = y^* \).

Consequently, the fixed point of \( f \) is unique. \( \square \)

Setting \( G = G_0 \) and \( G = G_1 \) once again, we get the following consequences of Theorem \ref{thm:fixed_points} in modular and partially ordered modular spaces, respectively.

**Corollary 2.14.** Let \( X \) be a \( \rho \)-complete modular space and a mapping \( f : X \to X \) satisfies

\[
\rho(b(f x - y)) \leq k \rho(a_1(f x - x)) + l \rho(a_2(f y - y)) \quad (x, y \in X),
\]

where \( k, l, a_1, a_2 \) and \( b \) are positive with \( k + l < 1 \), \( a_1 \leq \frac{b}{2} \) and \( a_2 \leq b \). Then \( f \) has a unique fixed point \( x^* \in X \) and \( bf^n x \xrightarrow{\rho} bx^* \) for all \( x \in X \).

**Corollary 2.15.** Let \( \preceq \) be a partial order on a \( \rho \)-complete modular space \( X \) such that the triple \( (X, \rho, \preceq) \) has Property \((\ast\ast)\). Assume that a nondecreasing mapping \( f : X \to X \) satisfies

\[
\rho(b(f x - y)) \leq k \rho(a_1(f x - x)) + l \rho(a_2(f y - y)) \quad (x, y \in X, \text{ and either } x \preceq y \text{ or } y \preceq x),
\]

where \( k, l, a_1, a_2 \) and \( b \) are positive with \( k + l < 1 \), \( a_1 \leq \frac{b}{2} \) and \( a_2 \leq b \). Then \( f \) has a fixed point if and only if there exists an \( x \in X \) such that \( T^m x \) is comparable to \( T^n x \) for all \( m, n \geq 0 \). Moreover, this fixed point is unique if \( k < \frac{1}{2} \) and each pair of elements of \( X \) has either an upper or a lower bound.

As another consequence of Theorem \ref{thm:fixed_points} we have the convex version of it as follows:
Corollary 2.16. Let \((X, \rho)\) be a \(\rho\)-complete modular space endowed with a graph \(G\), where \(\rho\) is a convex modular, and the triple \((X, \rho, G)\) have Property \((\ast)\). Assume that \(f : X \to X\) is a mapping which preserves the edges of \(G\) and satisfies
\[
\rho(b(fx - fy)) \leq k\rho(a_1(fx - x)) + l\rho(a_2(fy - y)) \quad (x, y \in X \text{ and } (x, y) \in E(\tilde{G})),
\]
where \(k, l, a_1, a_2\) and \(b\) are positive numbers with \(b > 4\max\{a_1, a_2, a_1k, a_2l\}\). Then \(f\) has a fixed point if and only if \(C_f \neq \emptyset\). Moreover, this fixed point is unique if \(X\) satisfies Condition \((\ast)\).

Proof. Set \(c = 2\max\{a_1, a_2, a_1k, a_2l\}\) and choose any \(a_0 \in (c, \frac{b}{2}]\). Then by the hypothesis and convexity of \(\rho\), we have
\[
\rho(b(fx - fy)) \leq k\rho(a_1(fx - x)) + l\rho(a_2(fy - y))
\]
\[
= k\rho\left(\frac{a_1}{a_0}(fx - x) + (1 - \frac{a_1}{a_0})0\right) + l\rho\left(\frac{a_2}{a_0}(fy - y) + (1 - \frac{a_2}{a_0})0\right)
\]
\[
\leq a_1k\frac{a_1}{a_0}\rho(x - x) + a_2l\frac{a_2}{a_0}\rho(y - y)
\]
for all \(x, y \in X\) with \((x, y) \in E(\tilde{G})\). Since \(a_0 \leq \frac{b}{2} < b\), and \(\frac{a_1}{a_0} + \frac{a_2}{a_0} < 1\), it follows that \(f\) satisfies (K2) for the graph \(\tilde{G}\) with the constants \(k, l, a_1\) and \(a_2\) replaced with \(\frac{a_1}{a_0}, \frac{a_2}{a_0}, a_0\) and \(a_0\), respectively, and \(b\) kept fixed. Since \(f\) preserves the edges of \(\tilde{G}\), it follows that \(f\) is a Kannan \(\tilde{G}\)-\(\rho\)-contraction and the first assertion is concluded immediately from Theorem 2.13.

On the other hand, since \(a_0 > c \geq 2a_1k\), it follows that \(\frac{a_1k}{a_0} < \frac{1}{2}\), and because \(X\) satisfies Condition \((\ast)\), Theorem 2.13 guarantees the uniqueness of the fixed point of \(f\). \(\square\)

References