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New results on the quasi-commuting inverses

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Abstract

A matrix has an ordinary inverse only if it is square, and even then only if it is nonsingular or, in other words, if its columns (or rows) are linearly independent. In recent years needs have been felt in numerous areas of applied mathematics for some kind of partial inverse of a matrix that is singular or even rectangular. In this paper, some results on the Quasi-commuting inverses, are given and the effect of them in solving the case of linear system of equations where the coefficient matrix is a singular matrix, is illustrated.

Keywords: Index of matrix; Generalized inverse; Singular linear system; Quasi-commuting inverses.
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1. Introduction

Consider the following linear system of equations

$$Ax = b, \tag{1.1}$$

wherein $A \in C^{n\times n}$ is a singular matrix. These systems arise in many different scientific applications such as linear regression, constrained linear systems and fuzzy singular linear systems [1, 7, 8]. The consistent singular linear system has a set of solutions and the inconsistent singular linear system has leat-squares solutions [4].

During the last two decades, the study of generalized inversion of linear transformations and related applications has grown to became an important topic of interest to researchers engaged in

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the study of linear mathematical problems as well as to practitioners concerned with applications of linear mathematics. Unlike the case of the nonsingular matrix, which has a single unique inverse for all purposes, there are different generalized inverses for different purposes. For some purposes, as in the examples of solutions of linear systems, there is not a unique inverse, but any matrix of a certain class will do. The $\{1\}-inverse$ in analyzing the solution of the linear system (1.1) is studied [1]. For finding least-square solutions of linear system the $\{1,3\}-inverse$ and pseudoinverse are used [7, 8]. Drazin inverse, one of the generalized inverse, play major role in solving consistent or inconsistent singular linear system [2, 5, 6].

In this paper, the effect of Quasi-commuting inverse is extended and in finding least-square solution for singular linear system is illustrated. We hope that this paper will be a new path to improve solving singular linear system.

In section 2, we present some preliminaries. Some new results on the Quasi-commuting inverses in section 3, are given. In section 4, the effect of Quasi-commuting inverse in finding least-square solution for singular linear system, is illustrated.

2. Preliminaries and Basic Definitions

We first present some preliminaries and basic definitions which are needed in this paper.

Definition 2.1. *Let $A \in C^{n\times n}$. We say the nonnegative integer number k to be the index of matrix A, if k is the smallest nonnegative integer number such that*

$$rank(A^{k+1}) = rank(A^k).$$

It is equivalent to the dimension of largest Jordan block corresponding to the zero eigenvalue of A. The index of matrix A, is denoted by $ind(A)$.

Definition 2.2. *Let $A \in C^{n\times n}$, with $ind(A) = k$. The matrix X of order n is the Drazin inverse of A, denoted by A^D, if X satisfies the following conditions*

$$AX = XA, XAX = X, A^k XA = A^k.$$

Theorem 2.3. *([1, 2]) Let $A \in C^{n\times n}$, with $ind(A) = k$, $rank(A^k) = r$. Assume that the Jordan normal form of A has the form as follows*

$$A = P \begin{pmatrix} D & o \\ o & N \end{pmatrix} P^{-1},$$

where P is a nonsingular matrix, D is a nonsingular matrix of order r, and N is a nilpotent matrix that $N^k = \bar{o}$. Then we can write the Drazin inverse of A in the form

$$A^D = P \begin{pmatrix} D^{-1} & o \\ o & 0 \end{pmatrix} P^{-1}.$$

When $ind(A) = 1$, it is obvious that $N = o$.

When $ind(A) = 1$, A^D is called the group inverse of A, and denoted by A_g.

Theorem 2.4. *([2]) $A^D b$ is a solution of*

$$Ax = b, k = ind(A) \tag{2.1}$$

if and only if $b \in Range(A^k)$, and $A^D b$ is an unique solution of (2.1) provided that $x \in Range(A^k)$.

Definition 2.5. *([1]) Let $A \in C^{n\times n}$, with $ind(A) = k$. The matrix X of order $n \times n$ is the Quasi-commuting inverse of A, denoted by A^Q, if X satisfies the following conditions*

$$AXA = A, \quad XAX = X, \quad A^kX = XA^k, \quad AX^k = X^kA.$$

Definition 2.6. *([4]) Let $A \in C^{m\times n}$. The matrix X of order $n \times m$ is the pseudoinverse of A denoted by A^+, if X satisfies the following conditions*

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.$$

3. Some New Results

In this section, we give some new results on the Quasi-commuting inverses.

Theorem 3.1. *A^T and $(A^Q)^T$ are quasi-commuting inverses of each other.*

Proof . From [4], we can get

$$(AA^QA)^T = (A^QA)^TA^T = A^T(A^Q)^TA^T = A^T,$$
$$(A^QAA^Q)^T = (AA^Q)^T(A^Q)^T = (A^Q)^TA^T(A^Q)^T = (A^Q)^T,$$

and

$$(A^kA^Q)^T = (A^Q)^T(A^k)^T = (A^k)^T(A^Q)^T = (A^QA^k)^T,$$
$$(A^QA^k)^T = (A^k)^T(A^Q)^T = (A^Q)^T(A^k)^T = (A^kA^Q)^T.$$

From [4] $(A^k)^T = (A^T)^k$. Therefore A^T and $(A^Q)^T$ are quasi-commuting inverses of each other. □

Theorem 3.2. *If $\lambda \neq 0$ be an eigenvalue of $A \in C^{n\times n}$, then $\frac{1}{\lambda}$ is an eigenvalue of A^Q.*

Proof . From $Ax = \lambda x, (x \neq 0)$ we have

$$AA^Qx = \lambda A^Qx,$$
$$A^QAA^Qx = \lambda A^QA^Qx.$$

From [4] we can get $A^Qx = \lambda A^QA^Qx$. Now if we set $A^Qx = y$ we have $\frac{1}{\lambda}y = A^Qy$. Therefore $\frac{1}{\lambda}$ is an eigenvalue of A^Q. □

Theorem 3.3. *Let $A \in C^{n\times n}$, then $rank(A) = rank(A^Q)$.*

Proof . We can get [3]

$$rank(A) = rank(AA^QA) \leq rank(A^QA) \leq rank(A^Q),$$
$$rank(A^Q) = rank(A^QAA^Q) \leq rank(AA^Q) \leq rank(A).$$

Therefore $rank(A) = rank(A^Q)$. □

Corollary 3.4. *If A is a singular matrix, then A^Q is a singular matrix.*

Theorem 3.5. *The matrix $S = \begin{pmatrix} B & C \\ C & B \end{pmatrix}$, and*

$$S^Q = \begin{pmatrix} D & E \\ E & D \end{pmatrix}, \tag{3.1}$$

where

$$D = \frac{1}{2}[(B + C)^Q + (B - C)^Q], \quad E = \frac{1}{2}[(B + C)^Q - (B - C)^Q],$$

are Quasi-commuting inverse of each other.

Proof . We know $SS^QS = S$, hence

$$\begin{pmatrix} B & C \\ C & B \end{pmatrix}\begin{pmatrix} D & E \\ E & D \end{pmatrix}\begin{pmatrix} B & C \\ C & B \end{pmatrix} = \begin{pmatrix} B & C \\ C & B \end{pmatrix},$$

and get

$$BD + CE = DB + EC, \quad CD + BE = EB + DC. \tag{3.2}$$

By adding and then by subtracting the two parts of (3.2), we obtain

$$(B + C)(D + E)(B + C) = (B + C), \quad (B - C)(D - E)(B - C) = (B - C),$$

also, we can show

$$\begin{array}{rl} (D + E)(B + C)(D + E) = (D + E), & (D - E)(B - C)(D - E) = (D - E), \\ (B + C)^k(D + E) = (D + E)(B + C)^k, & (B - C)^k(D - E) = (D - E)(B - C)^k, \\ (B + C)(D + E)^k = (D + E)^k(B + C), & (B - C)(D - E)^k = (D - E)^k(B - C). \end{array}$$

Thus S^Q must have the structure given by (3.1). In order to calculates E and D, we have

$$(B + C)^Q = (D + E,) \ \ (B - C)^Q = (D - E).$$

and consequently,

$$D = \frac{1}{2}[(B + C)^Q + (B - C)^Q], \quad E = \frac{1}{2}[(B + C)^Q - (B - C)^Q].$$

□

Theorem 3.6. *Let $r \in R - \{0\}$. (rA) and $(\frac{1}{r}A^Q)$ are quasi-commuting inverses of each other.*

Proof . We have

$$(rA)(\frac{1}{r}A^Q)(rA) = (rA), \quad (\frac{1}{r}A^Q)(rA)(\frac{1}{r}A^Q) = (\frac{1}{r}A^Q).$$

We can get

$$(r^kA^k)(\frac{1}{r}A^Q) = (rA)^k(\frac{1}{r}A^Q) = (\frac{1}{r}A^Q)(rA)^k,$$

and

$$(r^kA^k)(\frac{1}{r}A^Q) = (rA)^k(\frac{1}{r}A^Q) = (\frac{1}{r}A^Q)(rA)^k.$$

Therefore (rA) and $(\frac{1}{r}A^Q)$ are quasi-commuting inverses of each other. □

Theorem 3.7. *A and A_g are quasi-commuting inverses of each other.*

Proof . For any singular matrices with $ind(A) = 1$ the group inverse exist and is unique [1]. Therefore for any singular matrices with index one $A^Q = A_g$. □

Theorem 3.8. *Let A be a symmetric matrix with index one. A and A^+ are quasi-commuting inverses of each other.*

Proof . For any singular matrices with $ind(A) = 1$ the pseudoinverse exist and is unique [4]. A^+ is pseudoinverse of A , then we can get

$$AA^+A = A, \quad A^+AA^+ = A^+.$$

In this case $(A^+)^T = A^+$ then

$$(AA^+)^T = (A^+)^T A^T = A^+A,$$
$$(A^+A)^T = A^T(A^+)^T = AA^+.$$

□

Corollary 3.9. *Let A be an $n \times n$ matrix, then*

1. *A and A^Q are quasi-commuting inverses of each other, ie. $(A^Q)^Q = A$[1].*
2. *For any nonsingular matrix A, $A^Q = A^{-1}$.*

4. Numerical Examples

We now give the following example to explain the present results.

Example 4.1. *Consider the following symmetric matrix*

$$A = \begin{pmatrix} -1 & -1 & 2 & 2 \\ 3 & 3 & -5 & -5 \\ 2 & 2 & -1 & -1 \\ -5 & -5 & 3 & 3 \end{pmatrix}.$$

Jordan normal form of matrix C has the following form

$$A = P \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P^{-1}, \quad P = \begin{pmatrix} \frac{3}{10} & \frac{3}{10} & -\frac{3}{10} & -\frac{3}{10} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{11}{5} & \frac{34}{45} & -\frac{149}{45} & -\frac{59}{45} \\ 2 & \frac{7}{9} & -\frac{14}{9} & - -\frac{5}{9} \end{pmatrix}.$$

We have

$$A^Q = P^{-1} \begin{pmatrix} \frac{1}{5} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P = \begin{pmatrix} \frac{11}{25} & \frac{11}{25} & \frac{14}{25} & \frac{14}{25} \\ -\frac{21}{25} & -\frac{21}{25} & -\frac{29}{25} & -\frac{29}{25} \\ \frac{14}{25} & \frac{14}{25} & \frac{11}{25} & \frac{11}{25} \\ -\frac{29}{25} & -\frac{29}{25} & -\frac{21}{25} & -\frac{21}{25} \end{pmatrix}.$$

Consider the following singular linear system of equation with index one

$$\begin{cases} -x_1 - x_2 + 2x_3 + 2x_4 & = 1, \\ 3x_1 + 3x_2 - 5x_3 - 5x_4 & = -1, \\ 2x_1 + 2x_2 - x_3 - x_4 & = 4, \\ -5x_1 - 5x_2 + 3x_3 + 3x_4 & = -9. \end{cases} \tag{4.1}$$

Since $b = \begin{pmatrix} 1 \\ -1 \\ 4 \\ -9 \end{pmatrix} \in R(A)$, the solution of (4.1) is

$$x = A^Q b = \begin{pmatrix} -\frac{14}{5} \\ \frac{29}{5} \\ -\frac{11}{5} \\ \frac{21}{5} \end{pmatrix}.$$

5. Conclusions and Suggestions

Quasi-commuting inverse in solving singular consistent linear system and finding least-square solution of inconsistent singular linear system, is applied. Computing Quasi-commuting inverses for any arbitrary square matrix, is suggested.

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