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Bhaskar-Lakshmikantham type results for monotone mappings in partially ordered metric spaces

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Abstract

In this paper, coupled fixed point results of Bhaskar-Lakshmikantham type [T. Gnana Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis 65 (2006) 1379-1393] are extend, generalized, unify and improved by using monotone mappings instead mappings with mixed monotone property. Also, an example is given to support these improvements.

Keywords: coupled fixed point, uniqueness of the coupled fixed point, monotone mappings, partially ordered set.
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1. Introduction and preliminaries

It is well known that the metric fixed point theory is still very actual, important and useful in all area of Mathematics. It can be applied, for instance in variational inequalities, optimization, dynamic programing, approximation theory, etc.

The fixed point theorems in partially ordered metric spaces play a major role to prove the existence and uniqueness of solutions for some differential, integral equations or matrix equations ([4], [10]). One of the most interesting fixed point theorems in ordered metric spaces was investigated by T. Gnana Bhaskar and V. Lakshmikantham [1] applied their result to the existence and uniqueness of solution for a periodic boundary value problem. For some questions from linear and nonlinear differential equations as well as matrix equations the reader can see the recent papers of J. Nieto and R. R. Lopez [4] and Ran and Reurings [10]. Then many authors obtained several interesting results in ordered metric spaces.

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We start out with listing some notation and preliminaries that we shall need to express our results. In this paper $(X, d, \preceq)$ denotes a partially ordered metric space where $(X, \preceq)$ is a partially ordered set and (X, d) is a metric space.

In this paper we do not use mappings with the mixed monotone property as in [1]. Therefore, similar as in the [2], we introduce:

Definition 1.1. *Let $(X, \preceq)$ be a partially ordered set and let $F : X \times X \to X$. We say that F is a monotone if $F(x, y)$ is monotone nondecreasing in x and y, that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),$$

and

$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_1) \preceq F(x, y_2).$$

An element $(x, y) \in X \times X$ is called a coupled fixed point of F if $F(x, y) = x, F(y, x) = y$. It is clear that (x, y) is a coupled fixed point of F if and only if $(y, x)$is a coupled fixed point of F.

Definition 1.2. *[1] Let $(X, \preceq)$ be an ordered set and d be a metric on X. We say that $(X, d, \preceq)$ is regular if it has the following properties:*

(i) if for non-decreasing sequence $\{x_n\}$ holds $d(x_n, x) \to 0$, then $x_n \preceq x$ for all n,

(i) if for non-increasing sequence $\{y_n\}$holds $d(x_n, x) \to 0$, then $y_n \succeq y$ for all n.

The proof of the following Lemma is immediately.

Lemma 1.3. *(**1**) Let $(X, d, \preceq)$ be a partially ordered metric space. If relation $\sqsubseteq$ is defined on $X^2 = X \times X$ by*

$$Y \sqsubseteq V \Leftrightarrow x \preceq u \wedge y \preceq v,\ Y = (x, y), V = (u, v) \in X^2,$$

and $d_+ : X^2 \times X^2 \to \mathbb{R}^+$ is given by

$$d_+(Y, V) = d(x, u) + d(y, v), Y = (x, y), V = (u, v) \in X^2,$$

then $(X^2, \sqsubseteq, d_+)$ is an ordered metric spaces. The space (X^2, d_+) is a complete if and only if (X, d) is a complete. Also, the space $(X^3, d_+, \sqsubseteq)$ is a regular if and only if $(X, d, \preceq)$ is a such.

(**2**) If $F : X \times X \to X$, then the mapping $T_F : X \times X \to X \times X$ given by

$$T_F(Y) = (F(x, y), F(y, x)),\ Y = (x, y) \in X^2$$

is non-decreasing with respect to $\sqsubseteq$, that is,

$$Y \sqsubseteq V \Rightarrow T_F(Y) \sqsubseteq T_F(V).$$

(**3**) The mapping F is a continuous if and only if T_F is a continuous.

(**4**) $F(X^2)$ is a complete in the metric spaces (X, d) if and only if $T_F(X^2)$ is a complete in the space (X^2, d_+).

(**5**) Mapping $F : X^2 \to X$ has a coupled fixed point if and only if mapping T_F has a fixed point in X^2.

Assertions similar to the following lemma were used in the frame of metric spaces in the course of proofs of several fixed point results in various papers.

Lemma 1.4. *[5] Let (X,d) be a metric space and let $\{y_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

If $\{y_n\}$ is not a Cauchy sequence in (X,d), then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $m(k) > n(k) > k$ and the following four sequences tend to ε^+ when $k \to \infty$:

$$d\left(y_{m(k)}, y_{n(k)}\right), d\left(y_{m(k)}, y_{n(k)+1}\right), d\left(y_{m(k)-1}, y_{n(k)}\right), d\left(y_{m(k)-1}, y_{n(k)+1}\right).$$

In [1] Blaskar and Lakshmikantham proved the following theorem and formulated as Theorem 2.1.

Theorem 1.5. *Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist a $k \in [0,1)$ with*

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)], \forall x \geq u, y \preceq v. \tag{1.1}$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$x = F(x,y) \text{ and } y = F(y,x).$$

Also, in [1] Blaskar and Lakshmikantham proved the following theorem and formulated as Theorem 2.2.

Theorem 1.6. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Assume that X has the following property:*

(i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x, \forall n$;
(ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y \preceq y_n, \forall n$.
Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0,1)$ with

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)], \forall x \geq u, y \preceq v. \tag{1.2}$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$x = F(x,y) \text{ and } y = F(y,x).$$

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2. Main results

Our first result generalizes Theorem 2.1. from [1].

Theorem 2.1. *Let $F : X \times X \to X$ be a continuous monotone mapping on X. Assume that there exist a $k \in [0,1)$ such that*

$$d\left(F\left(x,y\right),F\left(u,v\right)\right) \leq \frac{k}{2}\left[d\left(x,u\right)+d\left(y,v\right)\right], \tag{2.1}$$

for all $x,y,u,v \in X$ for which $x \preceq u$ and $y \preceq v$. If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F\left(x_0,y_0\right) \text{ and } y_0 \preceq F\left(y_0,x_0\right),$$

then there exist $x,y \in X$ such that

$$x = F\left(x,y\right) \text{ and } y = F\left(y,x\right).$$

Proof . Let $x_0, y_0 \in X$ be such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \preceq F(y_0,x_0)$. Since $F : X \times X \to X$ we can choose $x_1, y_1 \in X$ such that $x_1 = F(x_0,y_0)$ and $y_1 = F(y_0,x_0)$. Again from $F : X \times X \to X$ we can choose $x_2, y_2 \in X$ such that $x_2 = F(x_1,y_1)$ and $y_2 = F(y_1,x_1)$. Continuing this process we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_{n+1} = F\left(x_n,y_n\right) \text{ and } y_{n+1} = F\left(y_n,x_n\right)$$

for all $n \geq 0$.

We shall show that

$$x_n \preceq x_{n+1} \text{ and } y_n \preceq y_{n+1} \tag{2.2}$$

for all $n \geq 0$.

We will use the mathematical induction. For $n = 0$, since $x_0 \preceq F(x_0,y_0)$ and $y_0 \preceq F(y_0,x_0)$, and as $x_1 = F(x_0,y_0)$ and $y_1 = F(y_0,x_0)$, we have that $x_0 \preceq x_1$ and $y_0 \preceq y_1$. Thus (2.2) holds for $n = 0$.

Suppose now that (2.2) holds for some fixed $n \geq 0$. Then, since $x_n \preceq x_{n+1}$ and $y_n \preceq y_{n+1}$, and so F is a monotone, we obtain

$$x_{n+1} = F\left(x_n,y_n\right) \preceq F\left(x_{n+1},y_n\right) \preceq F\left(x_{n+1},y_{n+1}\right) = x_{n+2},$$

and

$$y_{n+1} = F\left(y_n,x_n\right) \preceq F\left(y_{n+1},x_n\right) \preceq F\left(y_{n+1},x_{n+1}\right) = y_{n+2},$$

that is., by mathematical induction follows that (2.2) holds for all $n \geq 0$.

Denote,

$$\delta_n = d\left(x_n,x_{n+1}\right)+d\left(y_n,y_{n+1}\right).$$

We show that $\delta_n \to 0$, from which it follows that $d(x_n,x_{n+1}) \to 0$ and $d(y_n,y_{n+1}) \to 0$. Indeed, by (2.1) since $x_{n-1} \preceq x_n$ and $y_{n-1} \preceq y_n$ we have

$$\begin{aligned}
\delta_n &= d\left(F\left(x_{n-1},y_{n-1}\right),F\left(x_n,y_n\right)\right)+d\left(F\left(y_{n-1},x_{n-1}\right),F\left(y_n,x_n\right)\right)\\
&\leq \frac{k}{2}\left[d\left(x_{n-1},x_n\right)+d\left(y_{n-1},y_n\right)\right]+\frac{k}{2}\left[d\left(y_{n-1},y_n\right)+d\left(x_{n-1},x_n\right)\right]\\
&= k\delta_{n-1} \leq k^2\delta_{n-2} \leq ... \leq k^n\delta_0 = k^n\left(d\left(x_0,x_1\right)+d\left(y_0,y_1\right)\right) \to 0 \ \ (n \to \infty).
\end{aligned}$$

Now, we prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{x_n\}$ and $\{y_n\}$ is not a Cauchy sequence. Then (by Lemma 1.5) there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $m(k) > n(k) > k$ and the following four sequences tend to ε^+ when $k \to \infty$:

$$d_+\left(z_{m(k)}, z_{n(k)}\right), d_+\left(z_{m(k)}, z_{n(k)+1}\right), d_+\left(z_{m(k)-1}, z_{n(k)}\right), d_+\left(z_{m(k)-1}, z_{n(k)+1}\right), \tag{2.3}$$

where $z_n = (x_n, y_n)$ is a sequence in (X^2, d_+). Putting, $(x, y) = \left(x_{m(k)-1}, y_{m(k)-1}\right)$ and $(u, v) = \left(x_{n(k)}, y_{n(k)}\right)$ in (2.1) we have

$$d\left(F\left(x_{m(k)-1}, y_{m(k)-1}\right), F\left(x_{n(k)}, y_{n(k)}\right)\right) \leq \frac{k}{2}\left[d\left(x_{m(k)-1}, x_{n(k)}\right) + d\left(y_{m(k)-1}, y_{n(k)}\right)\right],$$

i.e.,

$$d\left(x_{m(k)}, x_{n(k)+1}\right) \leq \frac{k}{2}\left[d\left(x_{m(k)-1}, x_{n(k)}\right) + d\left(y_{m(k)-1}, y_{n(k)}\right)\right]. \tag{2.4}$$

Similar, putting $(y, x) = \left(y_{m(k)-1}, x_{m(k)-1}\right)$ and $(v, u) = \left(y_{n(k)}, x_{n(k)}\right)$ in (2.1) we obtain

$$d\left(F\left(y_{m(k)-1}, x_{m(k)-1}\right), F\left(y_{n(k)}, x_{n(k)}\right)\right) \leq \frac{k}{2}\left[d\left(y_{m(k)-1}, y_{n(k)}\right) + d\left(x_{m(k)-1}, x_{n(k)}\right)\right],$$

i.e.,

$$d\left(y_{m(k)}, y_{n(k)+1}\right) \leq \frac{k}{2}\left[d\left(y_{m(k)-1}, y_{n(k)}\right) + d\left(x_{m(k)-1}, x_{n(k)}\right)\right]. \tag{2.5}$$

Adding (2.4) and (2.5) we get

$$d\left(x_{m(k)}, x_{n(k)+1}\right) + d\left(y_{m(k)}, y_{n(k)+1}\right) \leq k\left[d\left(x_{m(k)-1}, x_{n(k)}\right) + d\left(y_{m(k)-1}, y_{n(k)}\right)\right],$$

or equivalently,

$$d_+\left(z_{m(k)}, z_{n(k)+1}\right) \leq k d_+\left(z_{m(k)-1}, z_{n(k)}\right). \tag{2.6}$$

Letting $k \to \infty$ in (2.6) we obtain that $\varepsilon \leq k\varepsilon < \varepsilon$, a contradiction. Hence, both sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in complete metric space (X, d).

Since, (X, d) is a complete, there exist $x, y \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} F(x_{n-1}, y_{n-1}) = x$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} F(y_{n-1}, x_{n-1}) = y$. Further, from the continuity of mapping F we have that $F(x, y) = x$ and $F(y, x) = y$. Theorem is proved. □

We note that previous result is still valid for F not necessarily continuous. We have the following result.

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that $(X, d \preceq)$ is a regular.*

Let $F : X \times X \to X$ be a monotone mapping on X. Assume that there exists $k \in [0, 1)$ such that

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}\left[d(x, u) + d(y, v)\right],$$

for all $x, y, u, v \in X$ for which $x \preceq u$ and $y \preceq v$. If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \preceq F(y_0, x_0),$$

then there exist $x, y \in X$ *such that*

$$x = F(x, y) \text{ and } y = F(y, x).$$

Proof . Following the proof of Theorem 2.1 we only have to show that $x = F(x, y)$ and $y = F(y, x)$. First, we have

$$d(F(x, y), x) \leq d(F(x, y), F(x_n, y_n)) + d(x_{n+1}, x) \tag{2.7}$$

and

$$d(F(y, x), y) \leq d(F(y, x), F(y_n, x_n)) + d(y_{n+1}, y). \tag{2.8}$$

Since $x_n \to x$ and $y_n \to y$ as $n \to \infty$ then $x_n \preceq x$ and $y_n \preceq y$ for all n (because $(X, d, \preceq)$ is a regular) from (2.7) and (2.8) follows respectively

$$d(F(x, y), x) \leq \frac{k}{2}[d(x, x_n) + d(y, y_n)] + d(x_{n+1}, x) \to 0 + 0 = 0$$

and

$$d(F(y, x), y) \leq \frac{k}{2}[d(y, y_n) + d(x, x_n)] + d(y_{n+1}, y) \to 0 + 0 = 0.$$

Hence, $x = F(x, y)$and $y = F(y, x)$, that is F has a couple fixed point (x, y). □

One can prove that the coupled fixed point is in fact unique, provided that the product space $X \times X$ endowed with the partial order mentioned earlier has the following property:

Each pair of elements has either a lower bound or an upper bound. It is known that this condition is equivalent to:

For every $(x, y), (x^*, y^*) \in X \times X$, there exists a $(z_1, z_2) \in X \times X$ that is comparable to (x, y) and (x^*, y^*).

We now the following result:

Theorem 2.3. *Adding previous condition to the hypothesis of Theorem (2.1), we obtain the uniqueness of the coupled fixed point of F.*

Proof . If (x^*, y^*) is another coupled fixed point of F, then we show that

$$d_+((x, y), (x^*, y^*)) = 0,$$

where

$$\begin{aligned} x &= F(x, y) = \lim_{n\to\infty} F^n(x_0, y_0) = \lim_{n\to\infty} F\left(F^{n-1}(x_0, y_0), F^{n-1}(y_0, x_0)\right) = \lim_{n\to\infty} x_n \\ \text{and } y &= F(y, x) = \lim_{n\to\infty} F^n(y_0, x_0) = \lim_{n\to\infty} F\left(F^{n-1}(y_0, x_0), F^{n-1}(x_0, y_0)\right) = \lim_{n\to\infty} y_n. \end{aligned}$$

We consider two cases:

(**i**) If (x, y) is comparable to (x^*, y^*) with respect to the ordering in $X \times X$, then, for every $n = 0, 1, 2, ... (x, y) = (F^n(x, y), F^n(y, x))$ is comparable to $(x^*, y^*) = (F^n(x^*, y^*), F^n(y^*, x^*))$.

In this case we have

$$\begin{aligned} d_+((x, y), (x^*, y^*)) &= d(x, x^*) + d(y, y^*) \\ &= d(F^n(x, y), F^n(x^*, y^*)) + d(F^n(y, x), F^n(y^*, x^*)) \\ &\leq k^n[d(x, x^*) + d(y, y^*)] = k^n d_+((x, y), (x^*, y^*)). \end{aligned}$$

This implies that $d_+((x, y), (x^*, y^*)) = 0$.

(**ii**) If (x,y) is not comparable to (x^*,y^*), then there exists an upper bound or lower bound $(u,v)\in X\times X$ of $(x,y),(x^*,y^*)$. Then, for all $n=0,1,2,\ldots$ $(F^n(u,v),F^n(v,u))$ is comparable to $(x,y)=(F^n(x,y),F^n(y,x))$ and $(x^*,y^*)=(F^n(x^*,y^*),F^n(y^*,x^*))$.

Further, we have

$$\begin{aligned} d_+((x,y),(x^*,y^*)) &= d_+((F^n(x,y),F^n(y,x)),(F^n(x^*,y^*),F^n(y^*,x^*))) \\ &\le d_+((F^n(x,y),F^n(y,x)),(F^n(u,v),F^n(v,u))) \\ &\quad +d_+((F^n(u,v),F^n(v,u)),(F^n(x^*,y^*),F^n(y^*,x^*))) \\ &\le k^n([d(x,u)+d(y,v)]+[d(u,x^*)+d(v,y^*)]) \\ &= k^n[d_+((x,y),(u,v))+d_+((u,v),(x^*,y^*))]\to 0\ \ (n\to\infty), \end{aligned}$$

so that $d_+((x,y),(x^*,y^*))=0$. $\square$

Assuming that every pair of elements of X have either an upper bound or a lower bound in X, one can in fact show that even the components of the coupled fixed points are equal. The following theorem establishes this fact.

Theorem 2.4. *In addition to the hypothesis of Theorem 2.1, suppose that every pair of elements of X has an upper bound or a lower bound in X. Then $x=y$.*

Proof . It is clear that (y,x) is a coupled fixed point of F if and only if (x,y) is coupled fixed point. Therefore, by previous Theorem we obtain that $(x,y)=(y,x)$, that is $x=y$. $\square$

Example 2.5. *Let $X=\mathbb{R}, d(x,y)=|x-y|, x\preceq y$ if and only if $x\le y$ and $F:X\times X\to X$, defined by $F(x,y)=\frac{2x+y}{12}$. It is easy to check that all the conditions of Theorems 2.1. and 2.3. are satisfied for $k\in[\frac{1}{2},1)$ and that $(0,0)$ is a unique coupled fixed point of F. We note that the function F has not mixed monotone property, but F is a monotone, that is $F(x,y)$ is monotone nondecreasing in x and y. Hence coupled fixed point $(0,0)$ of F cannot be obtained by Theorem 1.6. that is, by results from [1].*

As shown in [1] for the case of the functions with mixed monotone property, such kind of results can be used to investigate a large class of problems, like periodic boundary value problems (see [1], section 3). A similar approach for monotone functions instead the functions with mixed monotone property is also possible and will be done in some other paper.

For similar approach also see [3], [6], [7] and [8].

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