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Coincident Point and Fixed Point Results For Three Self Mappings in Cone Metric Spaces

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Abstract

In this attempt we proved results on points of coincidence and common fixed points for three self mappings satisfying generalized contractive type conditions in cone metric spaces. Our results generalizes some previous known results in the literature (eg. [5], [6])

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1. Introduction and preliminaries

In 2007, Huang and Zhang ([3]) introduced the concept of cone metric spaces which is a generalization of metric spaces, by replacing the set of real numbers by ordered Banach space and proves some fixed point theorems for some contractive maps in normal cone metric spaces. Subsequently some other authors (see [1], [2], [6]) studied properties of cone metric spaces and fixed point results of mappings satisfying contractive type condition in cone metric spaces.

Abbas and Jungck ([1]) studied common fixed points for non-commuting mappings in normal cone metric spaces. However, there exists non normal cone metric spaces [4].

Recently, Stojan Radenovic [5] has obtained coincidence point results for two mappings in cone metric spaces which satisfies new contractive conditions. The same concept was further extended by M. Rangamma and K. Prudhvi ([6]) and proved coincidence point results and common fixed point results for three self mappings. The aim of this paper is to generalize, extend and improves the results of ([5], [6]).

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The following definitions and results will be needed in the sequel. Also in the following we always suppose $\mathbb{R}$ as a set of real numbers and $\mathbb{N}$ as a set of natural number.

Definition 1.1. *[3] Let E be a real Banach space and P be the subset of E. Then, P is called cone if and only if*

(*i*) *P is closed, non empty and $P \neq \{0\}$;*

(*ii*) $a, b \in \mathbb{R}$, $a, b \geq 0$; $x, y \in P \Rightarrow ax + by \in P$

(*iii*) $x \in P$ *and* $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, $intP$ denotes the interior of P.

Definition 1.2. *[3] The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $||x|| \leq K||y||$. The least positive number satisfying above is called the normal constant of P.*

In the following we always suppose E is a Banach Space, P is a cone in E with $intP \neq \phi$ and $\leq$ is partial ordering with respect to P.

Definition 1.3. *[3] Let X be a non empty set. Suppose the mapping $d : X \times X \to E$ satisfies:*

(*i*) *$0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;*

(*ii*) *$d(x, y) = d(y, x)$ for all $x, y \in X$;*

(*iii*) *$d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y \in X$.*

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 1.4. *Let $E = \mathbb{R}^2$ and $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2, X = \mathbb{R}^2$ and suppose that $d : X \times X \to E$ is defined by $d(x, y) = d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1| + |x_2 - y_2|, \alpha \max\{|x_1 - y_1|, |x_2 - y_2|\})$ where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space. It is easy to see that d is a cone metric, and hence (X, d) becomes a cone metric space over (E, P).*

Definition 1.5. *[3] Let (X, d) be a cone metric space, let $\{x_n\}$ be a sequence in X and $x \in X$. If for any $c \in E$ with $c \gg 0$, there is N such that for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x. (i.e. $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.)*

Definition 1.6. *[3] Let (X, d) be a cone metric space, let $\{x_n\}$ be a sequence in X, if for any $c \in E$ with $c \gg 0$, there is N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.*

Definition 1.7. *[3] Let (X, d) be a cone metric, if every Cauchy sequence is convergent in X then X is called a complete cone metric space.*

Definition 1.8. *[5] Let $f, g : X \to X$. Then the pair (f, g) is said to be (IT)-commuting at $z \in X$ if $f(g(z)) = g(f(z))$ with $f(z) = g(z)$.*

Definition 1.9. *For the mapping $f, g : X \to X$ let $C(f, g)$ denotes set of coincidence points of f, g that is $C(f, g) = \{z \in X : f(z) = g(z)\}$.*

Lemma 1.10. *[3] Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then*

(i) $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ $(n \to \infty)$.

(ii) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0, (n, m \to \infty)$.

2. Common Fixed Point Theorems

In this section we obtain coincidence points and common fixed point theorems for three maps in cone metric spaces.

Theorem 2.1. *Let (X, d) be a cone metric space, and P be a normal cone with normal constant L. Suppose the self maps $f, g, h : X \to X$ satisfy the condition*

$$d(fx, gy) \leq ad(hx, hy) + b[d(hx, fx) + d(hy, gy)] \tag{2.1}$$

where a, b are non-negative reals with $a + 2b < 1$. If $f(X) \cup g(X) \subseteq h(X)$ and $h(X)$ is a complete subspace of X. Then the maps f, g and h have a coincidence point p in X. Moreover, if (f, h) and (g, h) are (IT)-commuting at p, then f, g and h have a unique common fixed point.

Proof . Let x_0 be an arbitrary point in X and define the sequence $\{y_n\}$ in X such that $y_{2n} = fx_{2n} = hx_{2n+1}$ and $y_{2n+1} = gx_{2n+1} = hx_{2n+2}$ for all $n = 0, 1, 2, ...$ By 2.1, we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(fx_{2n}, gx_{2n+1}) \\ &\leq ad(hx_{2n}, hx_{2n+1}) + b[d(hx_{2n}, fx_{2n}) + d(hx_{2n+1}, gx_{2n+1})] \\ &\leq ad(gx_{2n-1}, fx_{2n}) + b[d(gx_{2n-1}, fx_{2n}) + d(fx_{2n}, gx_{2n+1})] \\ &\leq ad(y_{2n-1}, y_{2n}) + b[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]. \\ &\leq \frac{a+b}{1-b} d(y_{2n}, y_{2n-1}) \\ &\leq \lambda d(y_{2n}, y_{2n-1}). \end{aligned}$$

Similarly we can show that

$$d(y_{2n+1}, y_{2n+2}) \leq \lambda d(y_{2n}, y_{2n+1}).$$

Therefore, for all $n \in \mathbb{N}$ we can get,

$$d(y_{n+1}, y_{n+2}) \leq \lambda d(y_n, y_{n+1}) \leq .... \leq \lambda^{n+1} d(y_0, y_1).$$

Now for any $m > n$,

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d(y_{n-1}, y_m) \\ &\leq [\lambda^n + \lambda^{n+1} + ...\lambda^{m-1}] d(y_1, y_0) \\ &\leq \frac{\lambda^n}{1-\lambda} d(y_1, y_0) \end{aligned}$$

From definition 1.2, we have

$$d(y_n, y_m) \leq \frac{\lambda^n}{1-\lambda} K d(y_1, y_0)$$

Thus $d(y_n, y_m) \to 0$ as $n, m \to \infty$, (since $0 < \lambda < 1$). Hence $\{y_n\}$ is a Cauchy sequence, where $y_n = \{hx_n\}$. Therefore $\{hx_n\}$ is a Cauchy sequence. Since $h(X)$ is complete subspace of X, then there exist q in $h(X)$ such that $\{hx_n\} \to q$ as $n \to \infty$. Consequently, we can find $p \in X$ such that $h(p) = q$. We shall show that $hp = fp = gp$. Note that $d(hp, fp) = d(q, fp)$. Let us estimate $d(hp, fp)$, for this by the triangle inequality, we have,

$$d(hp, fp) \leq d(hp, hx_{2n+2}) + d(hx_{2n+2}, fp) = d(q, hx_{2n+2}) + d(fp, gx_{2n+1}) \tag{2.2}$$

By the Contractive condition 2.1, we get

$$\begin{aligned} d(fp, gx_{2n+1}) &\leq ad(hp, hx_{2n+1}) + b[d(hp, fp) + d(hx_{2n+1}, gx_{2n+1})] \\ &\leq ad(hp, hx_{2n+1}) + b[d(hp, hx_{2n+1}) + d(hx_{2n+1}, fp) \\ &\quad + d(hx_{2n+1}, gx_{2n+1})] \\ &\leq [ad(hp, hx_{2n+1}) + bd(hp, hx_{2n+1})] + bd(fp, gx_{2n+1}) \\ &\leq \frac{a+b}{1-b} d(hp, hx_{2n+1}) \\ &\leq \lambda d(hp, hx_{2n+1}) = \lambda d(q, hx_{2n+1}) \to 0, \quad \text{as} \quad n \to \infty. \end{aligned}$$

Therefore, for large n, we get

$$d(hp, fp) \leq d(q, hx_{2n+2}) \leq d(q, q) = 0$$

which leads to $d(hp, fp) = 0$ and hence $hp = q = fp$, similarly we can show $hp = q = gp$. Therefore we have

$$q = hp = fp = gp \tag{2.3}$$

i.e. p is a coincidence point of mappings f, g and h. Since (f, h) and (g, h) are (IT)-commuting at point p therefore by 2.3 and contractive condition 2.1 we have,

$$\begin{aligned} d(ffp, fp) = d(ffp, gp) &\leq \lambda d(hfp, hp) \\ &\leq d(hfp, hp) = d(fhp, fp) = d(ffp, fp) \end{aligned}$$

i.e. $d(ffp, fp) < d(ffp, fp)$, a contradiction (since $\lambda < 1$ and $fp = hp$). Therefore, $ffp = fp$. Hence $fp = ffp = fhp = hfp \Rightarrow ffp = hfp = fp = q$. Therefore, $fp(= q)$ is a common fixed point of f and h. Similarly we can show that $gp = ggp = ghp = hgp \Rightarrow ggp = hgp = gp = q$. Therefore, $gp = fp(= q)$ is a common fixed point of g and h. Hence by the above discussion we conclude that f, g and h have a common fixed point q. The uniqueness of the common fixed point q follows contractive condition 2.1. Indeed, let q_1 be any other fixed point of f, g and h. Consider

$$\begin{aligned} d(q, q_1) &= d(fq, gq_1) \\ &\leq ad(hq, hq_1) + b[d(hq, fq) + d(hq_1, gq_1)] \\ &\leq ad(hq, hq_1) + b[d(hq, hq_1) + d(hq_1, fq) + d(hq_1, gq_1)] \\ &\leq (a+b)d(hq, hq_1) + bd(fq, gq_1) \\ &\leq \frac{a+b}{1-b} d(hq, hq_1) \leq \lambda d(hq, hq_1) = \lambda d(q, q_1), \end{aligned}$$

a contradiction. Therefore f, g and h have a unique common fixed point. □

Corollary 2.2. *Let (X, d) be a cone metric space, P be a normal cone with normal constant L. Suppose the self maps f, g and h on X satisfy the condition*

$$d(fx, gy) \leq \alpha d(hx, hy) + \beta d(hx, fx) + \gamma d(hy, gy) \tag{2.4}$$

for all $x, y \in X$ where α, β, γ are non-negative reals with $\alpha + \beta + \gamma < 1$. If $f(X) \cup g(X) \subseteq h(X)$ and $h(X)$ is complete subspace of X. Then the maps f, g and h have a coincidence point p in X. Moreover, if (f, h) and (g, h) are (IT)-commuting at p, then f, g and h have a unique common fixed point.

Proof . The symmetric property of d and the above inequality imply that

$$d(fx, gy) \leq \alpha d(hx, hy) + \frac{\beta + \gamma}{2}[d(hx, fx) + d(hy, gy)] \tag{2.5}$$

By substituting $\alpha = a$ and $\frac{\beta+\gamma}{2} = b$ in 2.5, we obtain the required result as given in Theorem 2.1. It is also a generalization of the Theorem 1 of [8]. □

Theorem 2.3. *Let (X, d) be a cone metric space, and P be a normal cone with normal constant L. Suppose the self maps f, g and h on X satisfy the condition*

$$d(fx, gy) \leq ad(hx, hy) + b[d(hx, gy) + d(hy, fx)] \tag{2.6}$$

for all $x, y \in X$ where a, b are non-negative reals with $a + 2b < 1$. If $f(X) \cup g(X) \subset h(X)$ and $h(X)$ is complete subspace of X. Then the maps f, g and h have a coincidence point p in X. Moreover, if (f, h) and (g, h) are (IT)-commuting at p, then f, g and h have a unique common fixed point.

Proof . Suppose x_0 is an arbitrary point of X and define the sequence $\{y_n\}$ in X such that $y_{2n} = fx_{2n} = hx_{2n+1}$ and $y_{2n+1} = gx_{2n+1} = hx_{2n+2}$ for all $n = 0, 1, 2, ...$ By 2.6, we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(fx_{2n}, gx_{2n+1}) \\ &\leq ad(hx_{2n}, hx_{2n+1}) + b[d(hx_{2n}, gx_{2n+1}) + d(hx_{2n+1}, fx_{2n})] \\ &\leq ad(gx_{2n-1}, fx_{2n}) + b[d(gx_{2n-1}, gx_{2n+1}) + d(fx_{2n}, fx_{2n})] \\ &\leq ad(y_{2n-1}, y_{2n}) + b[d(y_{2n-1}, y_{2n+1})] \\ &\leq ad(y_{2n-1}, y_{2n}) + b[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] \\ &\leq \frac{a+b}{1-b} d(y_{2n}, y_{2n-1}) \leq \lambda d(y_{2n}, y_{2n-1}) \end{aligned}$$

Similarly we can show that

$$d(y_{2n+1}, y_{2n+2}) \leq \lambda d(y_{2n}, y_{2n+1}).$$

Therefore, for all $n \in \mathbb{N}$ we can get,

$$d(y_{n+1}, y_{n+2}) \leq \lambda d(y_n, y_{n+1}) \leq .... \leq \lambda^{n+1} d(y_0, y_1).$$

Now for any $m > n$,

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d(y_{n-1}, y_m) \\ &\leq [\lambda^n + \lambda^{n+1} + ...\lambda^{m-1}] d(y_1, y_0) \\ &\leq \frac{\lambda^n}{1-\lambda} d(y_1, y_0) \end{aligned}$$

From definition 1.2, we have

$$d(y_n, y_m) \leq \frac{\lambda^n}{1-\lambda} K d(y_1, y_0)$$

Thus $d(y_n, y_m) \to 0$ as $n, m \to \infty$, (since $0 < \lambda < 1$). Hence $\{y_n\}$ is a Cauchy sequence, where $y_n = \{hx_n\}$. Therefore $\{hx_n\}$ is a Cauchy sequence. Since $h(X)$ is complete subspace of X, then there exist q in $h(X)$ such that $\{hx_n\} \to q$ as $n \to \infty$. Consequently, we can find $p \in X$ such that $h(p) = q$. We shall show that $hp = fp = gp$. Note that $d(hp, fp) = d(q, fp)$. Let us estimate $d(hp, fp)$, for this by the triangle inequality we have,

$$d(hp, fp) \leq d(hp, hx_{2n+2}) + d(hx_{2n+2}, fp) = d(q, hx_{2n+2}) + d(fp, gx_{2n+1}) \tag{2.7}$$

By the contractive condition 2.6, we get

$$\begin{aligned} d(fp, gx_{2n+1}) &\leq ad(hp, hx_{2n+1}) + b[d(hp, gx_{2n+1}) + d(hx_{2n+1}, fp)] \\ &\leq ad(hp, hx_{2n+1}) + b[d(hp, hx_{2n+1}) + d(hx_{2n+1}, gx_{2n+1}) \\ &\quad + d(hx_{2n+1}, fp)] \\ &\leq [ad(hp, hx_{2n+1}) + bd(hp, hx_{2n+1})] + bd(fp, gx_{2n+1}) \\ &\leq \frac{a+b}{1-b} d(hp, hx_{2n+1}) \\ &\leq \lambda d(hp, hx_{2n+1}) = \lambda d(q, hx_{2n+1}) \to 0, \ as\ n \to \infty. \end{aligned}$$

Therefore, for large n, we get

$$d(hp, fp) \leq d(q, hx_{2n+2}) \leq d(q, q) = 0,$$

which leads to $d(hp, fp) = 0$ and hence $hp = q = fp$. Similarly, we can show $hp = q = gp$. Therefore p is a coincidence point of f, g and h. Since (f, h) and (g, h) are (IT)-commuting at p and with the help of contractive condition 2.6 we can show that $fp = gp = q$ is a common fixed point of f, g and h. Uniqueness of common fixed point q can easily be verified by using contractive condition 2.6. Hence f, g and h have unique common fixed point. □

Corollary 2.4. *Let (X,d) be a cone metric space, P be a normal cone with normal constant L. Suppose the self maps f, g and h on X satisfy the condition*

$$d(fx,gy) \leq \alpha d(hx,hy) + \beta d(hx,gy) + \gamma d(hy,fx) \quad (2.8)$$

for all $x,y \in X$ where α,β,γ are non-negative reals with $\alpha+\beta+\gamma < 1$. If $f(X)\cup g(X) \subseteq h(X)$ and $h(X)$ is complete subspace of X. Then the maps f, g and h have a coincidence point p in X. Moreover, if (f,h) and (g,h) are (IT)-commuting at p, then f, g and h have a unique common fixed point.

Proof . The symmetric property of d and the above inequality 2.8 imply that

$$d(fx,gy) \leq \alpha d(hx,hy) + \frac{\beta+\gamma}{2}[d(hx,gy) + d(hy,fx)] \quad (2.9)$$

By substituting $\alpha = a$ and $\frac{\beta+\gamma}{2} = b$ in 2.9, we obtain the required result as given in Theorem 2.3.

If $\alpha = 0$ in the above corollary 2.4, then we have a generalized result of Theorem 5 of [3], Theorem 2.5 of [1], and Theorem 2.7 of [7]. □

Example 2.5. *Let $X = \{a,b,c\}$ and $E = \mathbb{R}^2$ and $P = \{(x,y) \in E : x,y \geq 0\}$. Define mapping $d : X \times X \to E$ as follows*

$$d(x,y) = \begin{cases} (0,0) & \text{if} \quad x = y \\ (\frac{5}{9},5) & \text{if} \quad x \neq y, x,y \in X-\{b\}, \\ (1,9) & \text{if} \quad x \neq y, x,y \in X-\{c\}, \\ (\frac{4}{9},4) & \text{if} \quad x \neq y, x,y \in X-\{a\}, \end{cases}$$

Define mapping $f,g,h : X \times X$ as $hx = x$, $fx = c$ and

$$g(x) = \begin{cases} c & \text{if} \quad x \neq b, \\ a & \text{if} \quad x = b \end{cases}$$

and , then for $\alpha = 0 = \beta, \gamma = \frac{5}{9}$ we have

$$d(fx,gy) = \begin{cases} (0,0) & \text{if} \quad y \neq b \\ (\frac{5}{9},5) & \text{if} \quad y = b \end{cases}$$

$\alpha d(hx,hy) + \beta d(hx,fx) + \gamma d(hy,gy) = (\frac{5}{9},5)$ if $y = b$. It follows that all conditions of corollary 2.2 are satisfied for $\alpha = 0 = \beta$, $\gamma = \frac{5}{9}$ and so, f, g and h have a unique common fixed point c.

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