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On approximate dectic mappings in non-Archimedean spaces: a fixed point approach

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Abstract

In this paper, we investigate the Hyers-Ulam stability for the system of additive, quadratic, cubic and quartic functional equations with constants coefficients in the sense of dectic mappings in non-Archimedean normed spaces.

Keywords: Hyers-Ulam stability; non-Archimedean normed space; dectic functional equation; fixed point method.
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1. Introduction and Preliminaries

A classical equation in the theory of functional equations is the following: "when is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?". If the problem accepts a solution, we say that the equation is stable. The first problem concerning group homomorphisms was raised by Ulam [32] in 1940. In the next year Hyers [14] gave a first affirmative answer to the question of Ulam in context of Banach spaces. Subsequently, the result of Hyers was generalized by Aoki [2] for additive mapping and by Rassias [27] for linear mapping by considering an unbounded Cauchy difference. The result of Rassias has provided a lot of influence during the last three decades in the development of generalization of Hyers-Ulam stability concept. Furthermore, in 1994, Găvruţa [11] provided a further generalization of Rassias' theorem in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. The stability problems

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of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 9, 10, 15, 28]). In 1897, Hensel [13] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they done not satisfy the Archimedean axiom: for all $x, y > 0$, there exists an integer n such that $x < ny$.

Fix a prime number p. For any nonzero rational number x, there exists a unique integer n_x such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, and it is called the p-adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k\geq n}^{\infty} a^k p_k$, where $|a_k| \leq p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k\geq n}^{\infty} a^k p_k|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field [12, 29]. Note that if $p \geq 3$, then $|2^n|_p = 1$ for each integer n.

During the last three decades theory of non-Archimedean spaces has gained the interest of physicists for their research, in particular the problems that emerge in quantum physics, p-adic strings and superstrings [21]. Although many results in the classical normed space theory have a non-Archimedean counterpart, their proofs are essentially different and require an entirely new kind of intuition. One may note that for $|n| \leq 1$ in each valuation field, every triangle is isosceles and there many be no unit vector in a non-Archimedean normed space [21]. These facts show that the non-Archimedean framework is of special interest. It turned out that non-Archimedean spaces have many nice applications [12, 29, 30, 33]. In 2007, Moslehian and Rassias [23] proved the generalized Hyers-Ulam stability of the Cauchy and quadratic functional equation in non-Archimedean normed spaces.

A valuation is a function $|.|$ from a field $\mathbb{K}$ into $[0, \infty)$ such that 0 is the unique element having the 0 , $|ab| = |a||b|$, and the triangle inequality holds, that is, for all $a, b \in \mathbb{K}$, we have $|a + b| \leq |a| + |b|$. A field $\mathbb{K}$ is called a valued field if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|.| : \mathbb{K} \to \mathbb{R}$ such that , for any $a, b \in \mathbb{K}$, we have, $|a| \geq 0$ and equality holds if and only if $a = 0$,, $|ab| = |a||b|$, $|a + b| \leq max\{|a|, |b|\}$ (the strict triangle inequality). Note that $|1| = |-1| = 1$ and $|n| \leq 1$ for each integer n. We always assume, in addition, that $|.|$ is non-trivial, i.e., there exists an $a_0 \in \mathbb{K}$ such that $|a_0| \notin \{0, 1\}$.

Definition 1.1. *Let X be a linear space over a scaler field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|.|$. A function $\|.\| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:*
$(N1)$ $\|x\| = 0$ *if and only if* $x = 0$,
$(N2)$ $\|rx\| = |r|\|x\|$,
$(N3)$ $\|x + y\| \leq max\{\|x\|, \|y\|\}$ *(the strict triangle inequality (ultrametric))*

for all $x, y \in X$. Then $(X, \|.\|)$ is called a non-Archimedean space.
It follows from (N3) that

$$\|x_n - x_m\| \leq \max\{\|x_{i+1} - x_i\| : m \leq i \leq n - 1\} \quad (n > m).$$

Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. The sequence $\{x_n\}$ is called a Cauchy sequence if for any $\varepsilon > 0$, there is a positive integer N such that $\|x_n - x_m\| < \varepsilon$ for all

$n, m \geq N$. If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a non-Archimedean Banach space. For more detailed definition of non-Archimedean Banach space, we refer to [30].

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall the a fundamental result in fixed point theory.

Theorem 1.2. *(see.[6, 26]) Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipshitz constant $L < 1$. Then, for each given $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \quad \textit{for all} \quad n \geq 0,$$

or there exists a natural number n_0 such that
(1) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0}, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, Isac and Rassias [16] were the first to provide applications of stability theory of functional equations for the proof of new fixed-point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 25]).

Khodaei and Rassias [20] investigated the solution and stability of the n-dimensional additive functional equations such that in the special case $n = 2$,

$$f(ax + by) + f(ax - by) = 2af(x)$$

where $a, b \in \mathbb{Z} \setminus \{0\}$ with $a \neq \pm 1, \pm b$.

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.1}$$

is called quadratic functional equation and every solution of quadratic equation (1.1) is said to be a quadratic function. The function $f(x) = x^2$ satisfies the functional equation (1.1). The Hyers-Ulam stability problem for the quadratic functional equation was solved by Skof [31] and, independently, by Cholewa [5]. In Czerwik [3] proved the generalized Hyers-Ulam stability for the functional equation. Eshaghi Gordji and Khodaei [8] investigated the solution and the Hyers- Ulam stability for the quadratic functional equation

$$f(ax + by) + f(ax - by) = 2a^2 f(x) + 2b^2 f(y),$$

where $a, b \in \mathbb{Z} \setminus \{0\}$ with $a \neq \pm 1, \pm b$. Jun and Kim [17] introduced the following functional equation

$$f(2x + y) + f(2x - y) = 2(f(x + y) + f(x - y)) + 12f(x), \tag{1.2}$$

and established the general solution and the Hyers-Ulam stability for this functional equation. Functional equation (1.2) is called cubic functional equation and every solution of cubic equation (1.2) is said to be a cubic function. Obviously, the function $f(x) = x^3$ satisfies the functional equation

(1.2). Jun et al. [18] investigated the solution and the Hyers-Ulam stability for the cubic functional equation

$$f(ax+by)+f(ax-by)=ab^2(f(x+y)+f(x-y))+2a(a^2-b^2)f(x)$$

where $a,b\in\mathbb{Z}\setminus\{0\}$ with $a\neq\pm1,\pm b$.

Lee et al. [22] considered the following functional equation

$$f(2x+y)+f(2x-y)=4(f(x+y)+f(x-y))+24f(x)-6f(y). \tag{1.3}$$

and established the general solution and the Hyers-Ulam stability for this functional equation. Functional equation (1.3) is called quartic functional equation and every solution of quartic equation (1.3) is said to be a quartic function. Obviously, the function $f(x)=x^4$ satisfies the functional equation (1.3). Kang [19] investigated the solution and the Hyers-Ulam stability for the quartic functional equation

$$f(ax+by)+f(ax-by)=a^2b^2(f(x+y)+f(x-y))+2a^2(a^2-b^2)f(x)-2b^2(a^2-b^2)f(y)$$

where $a,b\in\mathbb{Z}\setminus\{0\}$ with $a\neq\pm1,\pm b$.

Ebadian et al. [7] considered the Hyers-Ulam stability of the system of additive-quartic functional equations and the system of quadratic-cubic functional equations. Recently, Park et al. [24] considered the Hyers-Ulam stability of the system of additive-quadratic-quartic functional equations.

In this paper, we investigate the Hyers-Ulam stability for the system of additive-quadratic-quartic-cubic functional equations

$$\begin{cases} f(ax_1+bx_2,y,z,w)+f(ax_1-bx_2,y,z,w)=2af(x_1,y,z,w),\\ f(x,ay_1+by_2,z,w)+f(x,ay_1-by_2,z,w)=2a^2f(x,y_1,z,w)+2b^2f(x,y_2,z,w),\\ f(x,y,az_1+bz_2,w)+f(x,y,az_1-bz_2,w)=a^2b^2(f(x,y,z_1+z_2,w)\\ \quad+f(x,y,z_1-z_2,w))+2a^2(a^2-b^2)f(x,y,z_1,w)-2b^2(a^2-b^2)f(x,y,z_2,w),\\ f(x,y,z,aw_1+bw_2)+f(x,y,z,aw_1-bw_2)=ab^2(f(x,y,z,w_1+w_2)\\ \quad+f(x,y,z,w_1-w_2))+2a(a^2-b^2)f(x,y,z,w_1) \end{cases} \tag{1.4}$$

where $a,b\in\mathbb{Z}\setminus\{0\}$ with $a\neq\pm1,\pm b$. Also by a example we show that approximation in non-Archimedean normed spaces is better than the approximation in (Archimedean) normed spaces.

The function $f:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ given by $f(x,y,z,w)=cxy^2z^4w^3$ is solution of (1.4). In particular, putting $x=y=z=w$, we get a dectic function $g:\mathbb{R}\to\mathbb{R}$ in one variable given by $g(x):=f(x,x,x,x)=cx^{10}$. The proof of the following proposition is evident, and we omit the details.

Proposition 1.3. *Let X and Y be real linear spaces. If a mappingf $:X\times X\times X\times X\to Y$ satisfies system (1.4), then $f(\lambda x,\mu y,\eta z,\gamma w)=\lambda\mu^2\eta^4\gamma^3f(x,y,z,w)$ for all $x,y,z,w\in X$, and all rational numbers λ,μ,η,γ.*

2. Approximation of dectic mappings

From now on, unless otherwise stated, we will assume that X is a non-Archimedean normed space and Y is a non-Archimedean Banach space. Utilizing the fixed point alternative, we investigate the Hyers-Ulam stability problem for the system of functional equations (1.4) in non-Archimedean Banach spaces.

Theorem 2.1. *Let $\beta \in \{-1, 1\}$ be fixed. Let $\psi_1, \psi_2, \psi_3, \psi_4 : X \times X \times X \times X \times X \to [0, \infty)$ be functions such that*

$$\begin{aligned}\Psi(x, y, z, w) := |\tfrac{1}{2}| \max\{&|a^{-5\beta+4}|\psi_1(a^{\frac{\beta-1}{2}}x, 0, a^{\frac{\beta-1}{2}}y, a^{\frac{\beta-1}{2}}z, a^{\frac{\beta-1}{2}}w),\\ &|a^{-5\beta+2}|\psi_2(a^{\frac{\beta+1}{2}}x, a^{\frac{\beta-1}{2}}y, 0, a^{\frac{\beta-1}{2}}z, a^{\frac{\beta-1}{2}}w),\\ &|a^{-5\beta-2}|\psi_3(a^{\frac{\beta+1}{2}}x, a^{\frac{\beta+1}{2}}y, a^{\frac{\beta-1}{2}}z, 0, a^{\frac{\beta-1}{2}}w),\\ &|a^{-5\beta-5}|\psi_4(a^{\frac{\beta+1}{2}}x, a^{\frac{\beta+1}{2}}y, a^{\frac{\beta+1}{2}}z, a^{\frac{\beta-1}{2}}w, 0)\}\end{aligned} \tag{2.1}$$

for all $x, y, z, w \in X$, and for some $0 < L < 1$,

$$\Psi(a^\beta x, a^\beta y, a^\beta z, a^\beta w) \le L|a^{10\beta}|\Psi(x, y, z, w) \tag{2.2}$$

and

$$\begin{aligned}&\lim_{n\to\infty} |a^{-10\beta n}|\psi_1(a^{\beta n}x_1, a^{\beta n}x_2, a^{\beta n}y, a^{\beta n}z, a^{\beta n}w) = 0,\\ &\lim_{n\to\infty} |a^{-10\beta n}|\psi_2(a^{\beta n}x, a^{\beta n}y_1, a^{\beta n}y_2, a^{\beta n}z, a^{\beta n}w) = 0,\\ &\lim_{n\to\infty} |a^{-10\beta n}|\psi_3(a^{\beta n}x, a^{\beta n}y, a^{\beta n}z_1, a^{\beta n}z_2, a^{\beta n}w) = 0,\\ &\lim_{n\to\infty} |a^{-10\beta n}|\psi_4(a^{\beta n}x, a^{\beta n}y, a^{\beta n}z, a^{\beta n}w_1, a^{\beta n}w_2) = 0\end{aligned} \tag{2.3}$$

for all $x, y, z, w, x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2 \in X$. If $f : X \times X \times X \times X \to Y$ is a mapping such that $f(x, 0, z, w) = f(x, y, 0, w) = 0$ for all $x, y, z, w \in X$, and

$$\|f(ax_1 + bx_2, y, z, w) + f(ax_1 - bx_2, y, z, w) - 2af(x_1, y, z, w)\| \le \psi_1(x_1, x_2, y, z, w), \tag{2.4}$$

$$\begin{aligned}&\|f(x, ay_1 + by_2, z, w) + f(x, ay_1 - by_2, z, w) - 2a^2 f(x, y_1, z, w) - 2b^2 f(x, y_2, z, w)\|\\ &\le \psi_2(x, y_1, y_2, z, w),\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\|f(x, y, az_1 + bz_2, w) + f(x, y, az_1 - bz_2, w) - a^2b^2(f(x, y, z_1 + z_2, w) + f(x, y, z_1 - z_2, w))\\ &- 2a^2(a^2 - b^2)f(x, y, z_1, w) + 2b^2(a^2 - b^2)f(x, y, z_2, w)\|\\ &\le \psi_3(x, y, z_1, z_2, w),\end{aligned} \tag{2.6}$$

$$\|f(x,y,z,aw_1+bw_2)+f(x,y,z,aw_1-bw_2)-ab^2(f(x,y,z,w_1+w_2)$$
$$+f(x,y,z,w_1-w_2))-2a(a^2-b^2)f(x,y,z,w_1)\| \quad (2.7)$$
$$\leq \psi_4(x,y,z,w_1,w_2)$$

for all $x,y,z,w,x_1,x_2,y_1,y_2,z_1,z_2,w_1,w_2 \in X$, then there exists a unique dectic mapping $D : X \times X \times X \times X \to Y$ satisfying (1.4) and

$$\|f(x,y,z,w)-D(x,y,z,w)\| \leq \frac{1}{1-L}\Psi(x,y,z,w) \quad (2.8)$$

for all $x,y,z,w \in X$.

Proof . Letting $x_2 = 0$ and replacing x_1,y,z,w by $2x,2y,2z,2w$ in (2.4), we get

$$\|f(2ax,2y,2z,2w)-af(2x,2y,2z,2w)\| \leq |\frac{1}{2}|\psi_1(2x,0,2y,2z,2w) \quad (2.9)$$

for all $x,y,z,w \in X$. Letting $y_2 = 0$ and replacing x,y_1,z,w by $2ax,2y,2z,2w$ in (2.5), we get

$$\|f(2ax,2ay,2z,2w)-a^2f(2ax,2y,2z,2w)\| \leq |\frac{1}{2}|\psi_2(2ax,2y,0,2z,2w) \quad (2.10)$$

for all $x,y,z,w \in X$. Letting and $z_2 = 0$ and replacing x,y,z_1,w by $2ax,2ay,2z,2w$ in (2.6), we get

$$\|f(2ax,2ay,2az,2w)-a^4f(2ax,2ay,2z,2w)\| \leq |\frac{1}{2}|\psi_3(2ax,2ay,2z,0,2w) \quad (2.11)$$

for all $x,y,z,w \in X$. Letting $w_2 = 0$ and replacing x,y,z,w_1 by $2ax,2ay,2az,2w$ in (2.7), we get

$$\|f(2ax,2ay,2az,2aw)-a^3f(2ax,2ay,2az,2w)\| \leq |\frac{1}{2}|\psi_4(2ax,2ay,2az,2w,0) \quad (2.12)$$

for all $x,y,z,w \in X$. Combining (2.9), (2.8), (2.11) and (2.12), we lead to

$$\|f(2ax,2ay,2az,2aw)-a^{10}f(2x,2y,2z,2w)\|$$
$$\leq |\tfrac{1}{2}|max\{|a^9|\psi_1(2x,0,2y,2z,2w),|a^7|\psi_2(2ax,2y,0,2z,2w), \quad (2.13)$$
$$|a^3|\psi_3(2ax,2ay,2z,0,2w),\psi_4(2ax,2ay,2az,2w,0)\}$$

for all $x,y,z,w \in X$. Replacing x,y,z and w by $\frac{x}{2}$, $\frac{y}{2}$, $\frac{z}{2}$ and $\frac{w}{2}$ in (2.13), we have

$$\|f(ax,ay,az,aw)-a^{10}f(x,y,z,w)\|$$
$$\leq |\tfrac{1}{2}|max\{|a^9|\psi_1(x,0,y,z,w),|a^7|\psi_2(ax,y,0,z,w), \quad (2.14)$$
$$|a^3|\psi_3(ax,ay,z,0,w),\psi_4(ax,ay,az,w,0)\}$$

for all $x,y,z,w \in X$. It follows from (2.14) that

$$\|\tfrac{1}{a^{10}}f(ax,ay,az,aw)-f(x,y,z,w)\|$$
$$\leq |\tfrac{1}{2}|max\{|a^{-1}|\psi_1(x,0,y,z,w),|a^{-3}|\psi_2(ax,y,0,z,w), \quad (2.15)$$
$$|a^{-7}|\psi_3(ax,ay,z,0,w),|a^{-10}|\psi_4(ax,ay,az,w,0)\}$$

$$\|a^{10}f(\tfrac{x}{a},\tfrac{y}{a},\tfrac{z}{a},\tfrac{w}{a})-f(x,y,z,w)\|$$
$$\leq |\tfrac{1}{2}|max\{|a^9|\psi_1(\tfrac{x}{a},0,\tfrac{y}{a},\tfrac{z}{a},\tfrac{w}{a}),|a^7|\psi_2(x,\tfrac{y}{a},0,\tfrac{z}{a},\tfrac{w}{a}), \tag{2.16}$$
$$|a^3|\psi_3(x,y,\tfrac{z}{a},0,\tfrac{w}{a}),\psi_4(x,y,z,\tfrac{w}{a},0)\}$$

for all $x,y,z,w\in X$. From the (2.15) and (2.16), we have

$$\|\frac{1}{a^{10\beta}}f(a^\beta x,a^\beta y,a^\beta z,a^\beta w)-f(x,y,z,w)\|\leq \Psi(x,y,z,w) \tag{2.17}$$

for all $x,y,z,w\in X$.

Consider

$$\Omega:=\{u|u:X\times X\times X\times X\to Y,\quad u(x,0,z,w)=u(x,y,0,w)=0,\forall x,y,z,w\in X\},$$

and let us introduce a generalized metric on Ω as follows:

$$d(u,v)=\inf\{\eta\in\mathbb{R}^+:\|u(x,y,z,w)-v(x,y,z,w)\|\leq\eta\Psi(x,y,z,w),\forall x,y,z,w\in X\},$$

where, as usual, $\inf\emptyset=+\infty$. The proof of the fact that (Ω,d) is a complete generalized metric space can be found in [4]. Now we consider the mapping $\Lambda:\Omega\to\Omega$ defined by

$$\Lambda u(x,y,z,w):=a^{-10\beta}u(a^\beta x,a^\beta y,a^\beta z,a^\beta w)$$

for all $u\in\Omega$ and $x,y,z,w\in X$. Let $\varepsilon>0$ and $f,g\in\Omega$ be such that $d(f,g)<\varepsilon$. Hence

$$\begin{aligned}\|\Lambda f(x,y,z,w)-\Lambda g(x,y,z,w)\| &= \|a^{-10\beta}f(a^\beta x,a^\beta y,a^\beta z,a^\beta w)-a^{-10\beta}g(a^\beta x,a^\beta y,a^\beta z,a^\beta w)\|\\ &= |a^{-10\beta}|\|f(a^\beta x,a^\beta y,a^\beta z,a^\beta w)-g(a^\beta x,a^\beta y,a^\beta z,a^\beta w)\| \qquad (2.18)\\ &\leq |a^{-10\beta}|\Psi(a^\beta x,a^\beta y,a^\beta z,a^\beta w)\leq L\varepsilon\Psi(x,y,z,w)\end{aligned}$$

for all $x,y,z,w\in X$, that is, if $d(f,g)<\varepsilon$, we have $d(\Lambda f,\Lambda g)\leq L\varepsilon$. This means that $d(\Lambda f,\Lambda g)\leq Ld(f,g)$ for all $f,g\in\Omega$. This means that, Λ is a strictly contractive self-mapping on Ω with the Lipschitz constant L. It follows from (2.17) that $d(\Lambda f,f)\leq 1$. Due to Theorem 1.2, there exists a unique mapping $D:X\times X\times X\times X\to Y$ such that D is a fixed point of Λ, i.e., $D(a^\beta x,a^\beta y,a^\beta z,a^\beta w)=a^{-10\beta}D(x,y,z,w)$ for all $x,y,z,w\in X$. Also, $d(\Lambda^n f,D)\to 0$ as $n\to\infty$, which implies the equality

$$\lim_{n\to\infty}a^{-10\beta n}f(a^{\beta n}x,a^{\beta n}y,a^{\beta n}z,a^{\beta n}w)=D(x,y,z,w)$$

for all $x,y,z,w\in X$. By Theorem 1.2, we have

$$d(f,D)\leq\frac{1}{1-L}d(f,\Lambda f)\leq\frac{1}{1-L}.$$

This implies that inequality (2.4).

On the other hand by (2.3), (2.4), (2.5), (2.6) and (2.7), we have

$$\begin{aligned}
&\|D(ax_1+bx_2,y,z,w)+D(ax_1-bx_2,y,z,w)-2aD(x_1,y,z,w)\| \\
&=\lim_{n\to\infty}|a^{-10\beta n}|\|f(a^{\beta n}ax_1+a^{\beta n}bx_2,a^{\beta n}y,a^{\beta n}z,a^{\beta n}w) \\
&+f(a^{\beta n}ax_1-a^{\beta n}bx_2,a^{\beta n}y,a^{\beta n}z,a^{\beta n}w)-2af(a^{\beta n}x_1,a^{\beta n}y,a^{\beta n}z,a^{\beta n}w)\| \\
&\le\lim_{n\to\infty}|a^{-10\beta n}|\psi_1(a^{\beta n}x_1,a^{\beta n}x_2,a^{\beta n}y,a^{\beta n}z,a^{\beta n}w)=0,
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
&\|D(x,ay_1+by_2,z,w)+D(x,ay_1-by_2,z,w)-2a^2D(x,y_1,z,w)-2b^2D(x,y_2,z,w)\| \\
&=\lim_{n\to\infty}|a^{-10\beta n}|\|f(a^{\beta n}x,a^{\beta n}ay_1+a^{\beta n}by_2,a^{\beta n}z,a^{\beta n}w) \\
&+f(a^{\beta n}x,a^{\beta n}ay_1-a^{\beta n}by_2,a^{\beta n}z,a^{\beta n}w)-2a^2f(a^{\beta n}x,a^{\beta n}y_1,a^{\beta n}z,a^{\beta n}w) \\
&-2b^2f(a^{\beta n}x,a^{\beta n}y_2,a^{\beta n}z,a^{\beta n}w)\| \\
&\le\lim_{n\to\infty}|a^{-10\beta n}|\psi_2(a^{\beta n}x,a^{\beta n}y_1,a^{\beta n}y_2,a^{\beta n}z,a^{\beta n}w)=0,
\end{aligned} \tag{2.20}$$

$$\begin{aligned}
&\|D(x,y,az_1+bz_2,w)+D(x,y,az_1-bz_2,w)-a^2b^2(D(x,y,z_1+z_2,w) \\
&+D(x,y,z_1-z_2,w))-2a^2(a^2-b^2)D(x,y,z_1,w)+2b^2(a^2-b^2)D(x,y,z_2,w)\| \\
&=\lim_{n\to\infty}|a^{-10\beta n}|\|f(a^{\beta n}x,a^{\beta n}y,a^{\beta n}az_1+a^{\beta n}bz_2,a^{\beta n}w) \\
&+f(a^{\beta n}x,a^{\beta n}y,a^{\beta n}az_1-a^{\beta n}bz_2,a^{\beta n}w)-a^2b^2(f(a^{\beta n}x,a^{\beta n}y,a^{\beta n}z_1+a^{\beta n}z_2,a^{\beta n}w) \\
&+f(a^{\beta n}x,a^{\beta n}y,a^{\beta n}z_1-a^{\beta n}z_2,a^{\beta n}w))-2a^2(a^2-b^2)f(a^{\beta n}x,a^{\beta n}y,a^{\beta n}z_1,a^{\beta n}w) \\
&+2b^2(a^2-b^2)f(a^{\beta n}x,a^{\beta n}y,a^{\beta n}z_2,a^{\beta n}w)\| \\
&\le\lim_{n\to\infty}|a^{-10\beta n}|\psi_3(a^{\beta n}x,a^{\beta n}y,a^{\beta n}z_1,a^{\beta n}z_2,a^{\beta n}w)=0,
\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}
&\|D(x,y,z,aw_1+bw_2)+D(x,y,z,aw_1-bw_2)-ab^2(D(x,y,z,w_1+w_2) \\
&+D(x,y,z,w_1-w_2))-2a(a^2-b^2)D(x,y,z,w_1)\| \\
&=\lim_{n\to\infty}|a^{-10\beta n}|\|f(a^{\beta n}x,a^{\beta n}y,a^{\beta n}z,a^{\beta n}aw_1+a^{\beta n}bw_2) \\
&+f(a^{\beta n}x,a^{\beta n}y,a^{\beta n}z,a^{\beta n}aw_1-a^{\beta n}bw_2)-ab^2(f(a^{\beta n}x,a^{\beta n}y,a^{\beta n}z,a^{\beta n}w_1+a^{\beta n}w_2) \\
&+f(a^{\beta n}x,a^{\beta n}y,a^{\beta n}z,a^{\beta n}w_1-a^{\beta n}w_2))-2a(a^2-b^2)f(a^{\beta n}x,a^{\beta n}y,a^{\beta n}z,a^{\beta n}w_1)\| \\
&\le\lim_{n\to\infty}|a^{-10\beta n}|\psi_4(a^{\beta n}x,a^{\beta n}y,a^{\beta n}z,a^{\beta n}w_1,a^{\beta n}w_2)=0
\end{aligned} \tag{2.22}$$

for all $x, y, z, w, x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2 \in X$. It follows from (2.19), (2.20), (2.21) and (2.22) that D satisfies (1.4), that is, D is dectic mapping. Since D is the unique fixed point of Λ in the set $\Delta = \{g \in \Omega : d(f, g) < \infty\}$, D is the unique mapping satisfying (1.4). □

Remark 2.2. *Let X be a normed space and let Y be a Banach space in Theorem 2.1. Using the fixed point method, one can show that there exists a unique dectic mapping $D : X \times X \times X \times X \to Y$ satisfying (1.4) and*

$$\|f(x, y, z, w) - D(x, y, z, w)\| \le \frac{1}{1-L}\widehat{\Psi}(x, y, z, w) \tag{2.23}$$

for all $x, y, z, w \in X$ and

$$\begin{aligned} \widehat{\Psi}(x, y, z, w) \ &:= |\tfrac{1}{2}|\{|a^{-5\beta+4}|\psi_1(a^{\frac{\beta-1}{2}}x, 0, a^{\frac{\beta-1}{2}}y, a^{\frac{\beta-1}{2}}z, a^{\frac{\beta-1}{2}}w) \\ &+|a^{-5\beta+2}|\psi_2(a^{\frac{\beta+1}{2}}x, a^{\frac{\beta-1}{2}}y, 0, a^{\frac{\beta-1}{2}}z, a^{\frac{\beta-1}{2}}w) \\ &+|a^{-5\beta-2}|\psi_3(a^{\frac{\beta+1}{2}}x, a^{\frac{\beta+1}{2}}y, a^{\frac{\beta-1}{2}}z, 0, a^{\frac{\beta-1}{2}}w) \\ &+|a^{-5\beta-5}|\psi_4(a^{\frac{\beta+1}{2}}x, a^{\frac{\beta+1}{2}}y, a^{\frac{\beta+1}{2}}z, a^{\frac{\beta-1}{2}}w, 0)\} \end{aligned} \tag{2.24}$$

for all $x, y, z, w \in X$.

Theorem 2.3. *Let X be a normed space and let Y be a Banach space in Theorem 2.1. Using the direct method, one can show that there exists a unique dectic mapping $D : X \times X \times X \times X \to Y$ satisfying (1.4) and*

$$\begin{aligned} \|f(x, y, z, w) - D(x, y, z, w)\| \le |\tfrac{1}{2}|\Big(&|a^{-1}|\widehat{\psi_1}(x, 0, y, z, w) + |a^{-3}|\widehat{\psi}_2(x, y, 0, z, w) \\ &+ |a^{-7}|\widehat{\psi}_3(x, y, z, 0, w) + |a^{-10}|\widehat{\psi}_4(x, y, z, w, 0)\Big) \end{aligned} \tag{2.25}$$

for all $x, y, z, w \in X$, where we assume that

$$\widehat{\psi_1}(x, 0, y, z, w) := \sum_{i=\frac{1-\beta}{2}}^{\infty} a^{-10\beta i}\psi_1(a^{\beta i}x, 0, a^{\beta i}y, a^{\beta i}z, a^{\beta i}w) < \infty,$$

$$\widehat{\psi}_2(x, y, 0, z, w) := \sum_{i=\frac{1-\beta}{2}}^{\infty} a^{-10\beta i}\psi_2(a^{1+\beta i}x, a^{\beta i}y, 0, a^{\beta i}z, a^{\beta i}w) < \infty,$$

$$\widehat{\psi}_3(x, y, z, 0, w) := \sum_{i=\frac{1-\beta}{2}}^{\infty} a^{-10\beta i}\psi_3(a^{1+\beta i}x, a^{1+\beta i}y, a^{\beta i}z, 0, a^{\beta i}w) < \infty,$$

$$\widehat{\psi}_4(x, y, z, w, 0) := \sum_{i=\frac{1-\beta}{2}}^{\infty} a^{-10\beta i}\psi_4(a^{1+\beta i}x, a^{1+\beta i}y, a^{1+\beta i}z, a^{\beta i}w, 0) < \infty.$$

Corollary 2.4. *Let $\beta \in \{-1, 1\}$ be fixed and $\delta, \rho > 0$ be real numbers such that $10\beta > \rho\beta$, and let X be a normed space and Y a Banach space. If $f : X \times X \times X \times X \to Y$ is a mapping such that $f(x, 0, z, w) = f(x, y, 0, w) = 0$ for all $x, y, z, w \in X$, and*

$$\begin{cases} \|f(ax_1 + bx_2, y, z, w) + f(ax_1 - bx_2, y, z, w) - 2af(x_1, y, z, w)\| \\ \quad \le \delta(\|x_1\|^\rho + \|x_2\|^\rho + \|y\|^\rho + \|z\|^\rho + \|w\|^\rho), \\ \|f(x, ay_1 + by_2, z, w) + f(x, ay_1 - by_2, z, w) - 2a^2 f(x, y_1, z, w) - 2b^2 f(x, y_2, z, w)\| \\ \quad \le \delta(\|x\|^\rho + \|y_1\|^\rho + \|y_2\|^\rho + \|z\|^\rho + \|w\|^\rho), \\ \|f(x, y, az_1 + bz_2, w) + f(x, y, az_1 - bz_2, w) - a^2b^2(f(x, y, z_1 + z_2, w) + f(x, y, z_1 - z_2, w)) \\ \quad - 2a^2(a^2 - b^2)f(x, y, z_1, w) + 2b^2(a^2 - b^2)f(x, y, z_2, w)\| \\ \quad \le \delta(\|x\|^\rho + \|y\|^\rho + \|z_1\|^\rho + \|z_2\|^\rho + \|w\|^\rho), \\ \|f(x, y, z, aw_1 + bw_2) + f(x, y, z, aw_1 - bw_2) - ab^2(f(x, y, z, w_1 + w_2) + f(x, y, z, w_1 - w_2)), \\ \quad - 2a(a^2 - b^2)f(x, y, z, w_1)\| \le \delta(\|x\|^\rho + \|y\|^\rho + \|z\|^\rho + \|w_1\|^\rho + \|w_2\|^\rho), \end{cases}$$

for all $x, y, z, w, x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2 \in X$, then there exists a unique dectic mapping $D : X \times X \times X \times X \to Y$ satisfying (1.4) and a constant $M > 0$ such that

$$\|f(x, y, z, w) - D(x, y, z, w)\| \le M(\|x\|^\rho + \|y\|^\rho + \|z\|^\rho + \|w\|^\rho)$$

for all $x, y, z, w \in X$.

Proof . Let $\psi_1, \psi_2, \psi_3, \psi_4 : X \times X \times X \times X \times X \to [0, \infty)$ be defined by

$$\psi_1(x_1, x_2, y, z, w) := \delta(\|x_1\|^\rho + \|x_2\|^\rho + \|y\|^\rho + \|z\|^\rho + \|w\|^\rho),$$

$$\psi_2(x, y_1, y_2, z, w) := \delta(\|x\|^\rho + \|y_1\|^\rho + \|y_2\|^\rho + \|z\|^\rho + \|w\|^\rho),$$

$$\psi_3(x, y, z_1, z_2, w) := \delta(\|x\|^\rho + \|y\|^\rho + \|z_1\|^\rho + \|z_2\|^\rho + \|w\|^\rho),$$

$$\psi_4(x, y, z, w_1, w_2) := \delta(\|x\|^\rho + \|y\|^\rho + \|z\|^\rho + \|w_1\|^\rho + \|w_2\|^\rho)$$

for all $x, y, z, w, x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2 \in X$. Then the corollary is followed from Theorem 2.3, where

$$M := \frac{\delta a^{5(1-\beta)}}{2\beta(a^{10} - |a|^\rho)} \max\{(a^9 + a^7|a|^\rho + a^3|a|^\rho + |a|^\rho), (a^9 + a^7 + a^3|a|^\rho + |a|^\rho), \\ (a^9 + a^7 + a^3 + |a|^\rho), (a^9 + a^7 + a^3 + 1)\}.$$

□

Approximation in non-Archimedean normed spaces is better than the approximation in (Archimedean) normed spaces. The following example shows that the previous corollary is not valid in non-Archimedean spaces.

Example 2.5. *Let* $X = Y = \mathbb{Q}_p$ *for prime number* $p > 3$ *and define* $f : X \times X \times X \times X \to Y$ *by* $f(x, y, z, w) = xyzw$*. Then for* $\delta = 1$*,* $\rho = 1$ *and* $x, y, z, w, x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2 \neq 0$ *with* $|x|_p < 1$*,* $|y|_p < 1$*,* $|z|_p < 1$*,* $|w|_p < 1$*, we have*

$$|f(2x_1 + x_2, y, z, w) + f(2x_1 - x_2, y, z, w) - 4f(x_1, y, z, w)|_p$$

$$= |0|_p = 0 \leq |x_1|_p + |x_2|_p + |y|_p + |z|_p + |w|_p,$$

$$|f(x, 2y_1 + y_2, z, w) + f(x, 2y_1 - y_2, z, w) - 8f(x, y_1, z, w) - 2f(x, y_2, z, w)|_p$$

$$= |xzw|_p| - 4y_1 - 2y_2|_p \leq \max\{| - 4y_1|_p, | - 2y_2|_p\}$$

$$\leq \max\{|y_1|_p, |y_2|_p\} \leq |x|_p + |y_1|_p + |y_2|_p + |z|_p + |w|_p,$$

$$|f(x, y, 2z_1 + z_2, w) + f(x, y, 2z_1 - z_2, w) - 4(f(x, y, z_1 + z_2, w) + f(x, y, z_1 - z_2, w))$$

$$- 2a^2(a^2 - b^2)f(x, y, z_1, w) + 2b^2(a^2 - b^2)f(x, y, z_2, w)|_p$$

$$= |xyw|_p| - 24z_1 + 6z_2|_p \leq \max\{| - 24z_1|_p, |6z_2|_p\}$$

$$\leq \max\{|z_1|_p, |z_2|_p\} \leq |x|_p + |y|_p + |z_1|_p + |z_2|_p + |w|_p,$$

and

$$|f(x, y, z, 2w_1 + w_2) + f(x, y, z, 2w_1 - w_2) - 2(f(x, y, z, w_1 + w_2) + f(x, y, z, w_1 - w_2))$$

$$- 12f(x, y, z, w_1)|_p = |xyz|_p| - 12w_1|_p \leq |w_1|_p$$

$$\leq |x|_p + |y|_p + |z|_p + |w_1|_p + |w_2|_p.$$

On the other hand for each natural number n*, we have*

$$|2^{-10\beta(n+1)}f(2^{\beta(n+1)}x, 2^{\beta(n+1)}y, 2^{\beta(n+1)}z, 2^{\beta(n+1)}w) - 2^{-10\beta n}f(2^{\beta n}x, 2^{\beta n}y, 2^{\beta n}z, 2^{\beta n}w)|_p \leq |xyzw|_p.$$

Hence, for each $x, y, z, w \neq 0$*, the sequence* $\{2^{-10\beta n}f(2^{\beta n}x, 2^{\beta n}y, 2^{\beta n}z, 2^{\beta n}w)\}$*is not convergent.*

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