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Multilinear forms which are products of linear forms

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Abstract

The conditions under which, multilinear forms (the symmetric case and the non symmetric case), can be written as a product of linear forms, are considered. Also we generalize a result due to S. Kurepa for 2^n-functionals in a group G.

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1. Introduction and preliminaries

We define polynomials in infinite dimension spaces using multilinear mappings. Let E and F be two vector spaces on $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. We shall call the mapping $L : E^n \to F$ n-linear form if the mapping $x_i \mapsto L(x_1, \ldots, x_i, \ldots, x_n)$, $i = 1, 2, \ldots, n$, is linear. Also we shall call the $L : E^n \to F$ symmetric if

$$L(x_1, x_2, \ldots, x_n) = L(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \,,$$

$\forall\ (x_1, x_2, \ldots, x_n) \in E^n$ and every permutation of the first n natural numbers. If $L : E^n \to F$ is a n-linear form we put:

$$S\left(L\right)\left(x_1, x_2, \ldots, x_n\right) := \frac{1}{n!} \sum_{\sigma \in S_n} L\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\right)$$

where S_n is the set of all permutations of the first n natural numbers. Obviously $S(L) : E^n \to F$ is a symmetric n-linear form. We put

$$\widehat{L}\left(x\right) := L\left(x, x, \ldots, x\right) \ \forall\ x \in E$$

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and we define the mapping $P : E \to F$ as a homogeneous polynomial of n-degree if there exists an n-linear form $L : E^n \to F$ such that $P = \widehat{L}$, i.e.

$$P\left(x\right) = \widehat{L}\left(x\right) = L\left(x, x, \ldots, x\right) .$$

Generaly there is no a bijection between the n-linear forms and the homogeneous polynomials of n-degree. Though there exists a bijection between the symmetric n-linear forms and the homogeneous polynomials of n-degree.

The proof of this claim is based on the following Lemma where we use the polarization formulas.

Lemma 1.1. *If $L : E^n \to F$ is a symmetric n-linear form and $P : E \to F$ a homogeneous polynomial of n-degree with $P = \widehat{L}$, then:*

$$L(x_1, x_2, \ldots, x_n) = \frac{1}{2^n n!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n P\left(\sum_{k=1}^{n} \varepsilon_k x_k\right) ,$$

where the sum is over all $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{-1, 1\}$. If we use the Rademacher functions instead of ε_i the last formula takes the form:

$$L(x_1, x_2, \ldots, x_n) = \frac{1}{n!} \int_0^1 r_1(t) r_2(t) \cdots r_n(t) P\left(\sum_{k=1}^{n} r_k(t) x_k\right) dt$$

where $r_k(t) = sgn \sin 2^k \pi t$ is the k-Rademacher function for $1 \le k \le n$.

2. The symmetric case

Lemma 2.1. *Let V be a vector space and $F : V^m \to \mathbf{K}$, $\mathbf{K} = \mathbf{C}$ or $\mathbf{R}$, a symmetric m-linear form, $F \not\equiv 0$. If for some $x_0 \in V$, with $\widehat{F}(x_0) \neq 0$, we have*

$$F(x_0^{m-1}, x)^m = \widehat{F}(x_0)^{m-1} \cdot \widehat{F}(x) , \quad x \in V , \tag{2.1}$$

then

$$F(x_1, \ldots, x_m) = \frac{1}{\widehat{F}(x_0)^{m-1}} \cdot F(x_0^{m-1}, x_1) \cdots F(x_0^{m-1}, x_m) , \qquad x_1, \ldots, x_m \in V .$$

Proof . Since $F \not\equiv 0$, from the polarization formula there exits x_0 such that $\widehat{F}(x_0) \neq 0$.
If r_n is the n^{th} Rademacher function, using Eq.(2.1) we obtain:

$$\begin{aligned} & \int_0^1 r_1(t) \cdots r_m(t) \cdot \left[F\left(x_0^{m-1}, \sum_{k=1}^{m} r_k(t) x_k\right)\right]^m dt \\ = & \int_0^1 r_1(t) \cdots r_m(t) \cdot \widehat{F}(x_0)^{m-1} \widehat{F}\left(\sum_{k=1}^{m} r_k(t) x_k\right) dt \\ = & \; m! \widehat{F}(x_0)^{m-1} \cdot F(x_1, \ldots, x_m) . \end{aligned}$$

Thus

$$
\begin{aligned}
& m!\widehat{F}(x_0)^{m-1} \cdot F(x_1,\dots,x_m) \\
= & \int_0^1 r_1(t)\cdots r_m(t)\cdot \left[F(x_0^{m-1}, r_1(t)x_1+\cdots+r_m(t)x_m)\right]^m dt \\
= & \int_0^1 r_1(t)\cdots r_m(t)\cdot \left[r_1(t)F(x_0^{m-1},x_1)+\cdots+r_m(t)F(x_0^{m-1},x_m)\right]^m dt \\
= & \int_0^1 r_1(t)\cdots r_m(t) \sum_{n_1+\ldots+n_m=m} \frac{m!}{n_1!\cdots n_m!}\, r_1(t)^{n_1}\cdots r_m(t)^{n_m}\cdot F(x_0^{m-1},x_1)^{n_1}\cdots F(x_0^{m-1},x_m)^{n_m} dt \\
= & m! F(x_0^{m-1},x_1)\cdots F(x_0^{m-1},x_m)\,,
\end{aligned}
$$

hence

$$\widehat{F}(x_0)^{m-1}\cdot F(x_1,\dots,x_m) = F(x_0^{m-1},x_1)\cdots F(x_0^{m-1},x_m)\,.$$

□

Proposition 2.2. *Let V be a vector space and let $F : V^m \to \mathbf{K}$ be a symmetric m-linear form, $F \not\equiv 0$. Then*

$$F(x_0^{m-1},x)^m = \widehat{F}(x_0)^{m-1}\cdot \widehat{F}(x)\ ,\ x\in V\,, \tag{2.2}$$

for some $x_0 \in V$, with $\widehat{F}(x_0)\neq 0$, if and only if

$$F(x_1,\dots,x_m) = c\cdot L(x_1)\cdots L(x_m),\quad x_1,\dots,x_m\in V\,, \tag{2.3}$$

for some constant $c\neq 0$, where $L : V\to \mathbf{K}$ is a linear form.

Proof . It is clear that Eq. (2.3) implies Eq.(2.2). We assume now that Eq.(2.2) holds true. Then, from the previous Lemma, we get:

$$F(x_1,\dots,x_m) = \frac{1}{\widehat{F}(x_0)^{m-1}}\cdot F(x_0^{m-1},x_1)\cdots F(x_0^{m-1},x_m)\,.$$

Thus

$$F(x_1,\dots,x_m) = c\cdot L(x_1)\cdots L(x_m),\qquad c=\frac{1}{\widehat{F}(x_0)^{m-1}}\,.$$

and $L : V\to \mathbf{K}$ is a linear map which is defined as:

$$L(x) = F(x_0^{m-1},x)\,.$$

□

Equivalenthy, the above Proposition can be stated as.

Corollary 2.3. *Let $\widehat{F} : V \to \mathbf{K}$ be a homogeneous polynomial of degree m, $\widehat{F} \not\equiv 0$, where V is a vector space and let $F : V^m \to \mathbf{K}$ be the symmetric m-linear form which corresponds to the polynomial $\widehat{F}$. Then*

$$\widehat{F}(x) = c \cdot L(x)^m, \;\; x \in V,$$

for some constant $c \neq 0$, where $L : V \to \mathbf{K}$ is a linear form, if and only if

$$F(x_0^{m-1}, x)^m = \widehat{F}(x_0)^{m-1} \cdot \widehat{F}(x)\;, \;\; x \in V,$$

for some $x_0 \in V$, with $\widehat{F}(x_0) \neq 0$.

In the case, where the vector space V is of finite dimension, say $V = \mathbf{K}^n$, it is known that the m-homogeneous polynomial $\widehat{F} : \mathbf{K}^n \to \mathbf{K}$ can be written in the form

$$\widehat{F}(x) = \sum_{j=1}^{N} \alpha_j L_j(x)^m\,,$$

where $N = (m+1)^{n-1}$, $\alpha_j \in \mathbb{K}$ and $L_j : \mathbf{K}^n \to \mathbf{K}$ are linear forms, $j = 1, \ldots, N$. For a relatively easy proof of this known result we refer to work (see [3]). Hence, in the case of vector spaces of finite dimension, the previous result gives us the sufficent and necessary condition that an m-homogeneous polynomial can be written as an m^{th} power of a linear form.

In the case of Hermitian forms we have an analogous result.

Proposition 2.4. *Let V be a complex vector space and let $F : V \times V \to \mathbf{C}$ be a Hermitian form, $F \not\equiv 0$. Then*

$$|F(x_0, x)|^2 = \widehat{F}(x_0) \cdot \widehat{F}(x)\;,\; x \in V\,, \tag{2.4}$$

for some $x_0 \in V$, with $\widehat{F}(x_0) \neq 0$, if and only if

$$F(x, y) = c \cdot L(x) \cdot \overline{L(y)} \tag{2.5}$$

for some constant $c \neq 0$, where $L : V \to \mathbf{C}$ is a linear form.

Proof . It is clear that Eq.(2.5) implies Eq.(2.4).We suppose that Eq.(2.4) is true. We have

$$\begin{aligned}
&\tfrac{1}{(2\pi)^2} \int_0^{2\pi}\!\!\int_0^{2\pi} e^{-i\vartheta_1} \cdot e^{i\vartheta_2} \left|F(x_0, xe^{i\vartheta_1} + ye^{i\vartheta_2})\right|^2 d\vartheta_1 d\vartheta_2 \\
= &\tfrac{1}{(2\pi)^2} \int_0^{2\pi}\!\!\int_0^{2\pi} e^{-i\vartheta_1} \cdot e^{i\vartheta_2} \cdot F\left(x_0, xe^{i\vartheta_1} + ye^{i\vartheta_2}\right) \cdot F\left(xe^{i\vartheta_1} + ye^{i\vartheta_2}, x_0\right) d\vartheta_1 d\vartheta_2 \\
= &F(x, x_0) \cdot F(x_0, y)\,.
\end{aligned}$$

Thus

$$\begin{aligned}
F(x, x_0) \cdot F(x_0, y) &= \tfrac{1}{(2\pi)^2} \int_0^{2\pi}\!\!\int_0^{2\pi} e^{-i\vartheta_1} \cdot e^{i\vartheta_2} \cdot \widehat{F}\left(x_0\right) \cdot \widehat{F}(xe^{i\vartheta_1} + ye^{i\vartheta_2}) \Big) d\vartheta_1 d\vartheta_2 \\
&= \tfrac{\widehat{F}(x_0)}{(2\pi)^2} \cdot \int_0^{2\pi}\!\!\int_0^{2\pi} e^{-i\vartheta_1} \cdot e^{i\vartheta_2} \cdot F\left(xe^{i\vartheta_1} + ye^{i\vartheta_2}, xe^{i\vartheta_1} + ye^{i\vartheta_2}\right) d\vartheta_1 d\vartheta_2 \\
&= \widehat{F}(x_0) \cdot F(x, y)\,.
\end{aligned}$$

So we have proved that

$$F(x,y) = c \cdot L(x) \cdot \overline{L(y)}\,,$$

where $c = \frac{1}{\widehat{F}(x_0)}$ and $L(x) = F(x, x_0)$ is a linear form. □

We consider now the case where $(G, +)$ is a group, $\mathbf{K}$ is a field and $F : G^m \to \mathbf{K}$ is a symmetric m-additive map. Since

$$F(x_1, \ldots, x+y, \ldots, x_m) = F(x_1, \ldots, x, \ldots, x_m) + F(x_1, \ldots, y, \ldots, x_m)\,,$$

we have

$$F(x_1, \ldots, 0, \ldots, x_m) = 0 \;\; and \;\; F(x_1, \ldots, -x, \ldots, x_m) = -F(x_1, \ldots, x, \ldots, x_m)\,,$$

We note also that if $\widehat{F}(x_0) = F(x_0, \ldots, x_0) \neq 0$, $\widehat{F}(x_0) \in \mathbf{K}$, then $\widehat{F}(x_0)^n \neq 0$ for every $n \in \mathbf{N}$.

If we denote by $char\mathbf{K}$ the characteristic number of the field $\mathbf{K}$, by repeating the proofs of Lemma and Proposition 2.2, we obtain:

Proposition 2.5. *For $m \in \mathbf{N}$, let G be a group and $\mathbf{K}$ a field with $char\mathbf{K} = 0$ or $char\mathbf{K} > m$. A map $F : G^m \to \mathbf{K}$, $F \not\equiv 0$, is symmetric m-additive and satisfies*

$$F(x_0^{m-1}, x)^m = \widehat{F}(x_0)^{m-1} \cdot \widehat{F}(x) \tag{2.6}$$

for some $x_0 \in G$, with $\widehat{F}(x_0) \neq 0$, if an only if there exits a constant $c \in \mathbf{K} - \{0\}$ and an additive map $A : G \to \mathbf{K}$, such that

$$F(x_1, \ldots, x_m) = c \cdot A(x_1) \cdots A(x_m)\,. \tag{2.7}$$

Remark 2.6. *We notice that:*

1. *Proposition 2.5 is Theorem 1 in (see [1]) and it is due to Ebanks. Let us mention that Ebanks proved Theorem 1 with the less powerfull condition that G is a groupoid. Though his proof is much more complicated than the proof of Proposition 2.5. Also the hypothesis*

   $$F(x_1, \ldots, x_m)^m = \widehat{F}(x_1) \cdots \widehat{F}(x_m),\; x_1, \ldots, x_m \in G\,,$$

   it is used in the proof of Theorem 1 in (see [1]), which is more powerfull than the hypothesis Eq.(2.6) of Proposition 2.5.
2. *In addition Ebanks (page 183 in (see [1])) gives an application for "quartic functionals" which really is a generalization of a result due to S. Kurepa.*
   We say that a mapping $q : G \to \mathbf{K}$ is a quartic functional, if q satisfies the following functional equation:

$$\begin{aligned} & q(x_1+x_2+x_3+x_4) + q(x_1-x_2+x_3+x_4) + q(x_1+x_2-x_3+x_4) + \\ & q(x_1+x_2+x_3-x_4) + q(x_1-x_2-x_3+x_4) + q(x_1-x_2+x_3-x_4) + \\ & q(x_1+x_2-x_3-x_4) + q(x_1-x_2-x_3-x_4) \\ = \;& 8 \cdot q(x_1) + 8 \cdot q(x_2) + 8 \cdot q(x_3) + 8 \cdot q(x_4)\,, \quad x_1, x_2, x_3, x_4 \in G\,. \end{aligned}$$

If $\mathrm{char}\mathbf{K} = 0$ *or* $\mathrm{char}\mathbf{K} > 4$, *we define* $F : G^4 \to \mathbf{K}$ *as follows*:

$$\begin{aligned} 2^3 \cdot 4! F(x_1, x_2, x_3, x_4) &= q(x_1 + x_2 + x_3 + x_4) - q(x_1 - x_2 + x_3 + x_4) - \\ &\quad q(x_1 + x_2 - x_3 + x_4) - q(x_1 + x_2 + x_3 - x_4) + \\ &\quad q(x_1 - x_2 - x_3 + x_4) + q(x_1 - x_2 + x_3 - x_4) + \\ &\quad q(x_1 + x_2 - x_3 - x_4) - q(x_1 - x_2 - x_3 - x_4)\,. \end{aligned}$$

It can be easily checked that F *is* 4*-additional, symmetric and that satisfies the relation*

$$\widehat{F}(x) = F(x, x, x, x) = q(x)\,.$$

Therefore we have:

Corollary 2.7. *Let* G *be a group,* $\mathbf{K}$ *is a field with* $\mathrm{char}\mathbf{K} = 0$ *or* $\mathrm{char}\mathbf{K} > 4$ *and* $q : G \to \mathbf{K}$ *is a quartic functional. Then there exits an additive map* $A : G \to \mathbf{K}$ *and a constant* $c \neq 0$ *for which*

$$q(x) = c \cdot A(x)^4, \quad x \in G\,,$$

if and only if q *satisfies the relation*

$$\begin{aligned} &[q(x_1 + x_2 + x_3 + x_4) - q(x_1 - x_2 + x_3 + x_4) - q(x_1 + x_2 - x_3 + x_4) - \\ &q(x_1 + x_2 + x_3 - x_4) + q(x_1 - x_2 - x_3 + x_4) + q(x_1 - x_2 + x_3 - x_4) + \\ &q(x_1 + x_2 - x_3 - x_4) - q(x_1 - x_2 - x_3 - x_4)]^4 \\ &= (2^3 \cdot 4!)^4 q(x_1) q(x_2) q(x_3) q(x_4)\quad , \end{aligned} \tag{2.8}$$

where $x_1, x_2, x_3, x_4 \in G$.

Remark 2.8. *Similarly we define a* 2^n*-functional* $q : G \to \mathbf{K}$. *Thus, we can generalize a S. Kurepa's result for* 2^n*-functionals in a group* G.

3. The non symmetric case

If V is a vector space, which condition is sufficient and necessary so that a twolinear form $F : V^2 \to \mathbf{K}$, $\mathbf{K} = \mathbf{C}$ or $\mathbf{R}$, can be written as a product of linear forms? i.e. under which condition F can be written as

$$F(x, y) = L_1(x) L_2(y)\,,$$

where $L_1, L_2 : V \to \mathbf{K}$ are linear functionals?

A nondegenerate twolinear functional $F : V^2 \to \mathbf{K}$ can not be written in the form

$$F(x, y) = L_1(x) L_2(y)\,,$$

where $L_1, L_2 : V \to \mathbf{K}$ are linear functionals, see Theorem 8 in work (see [2]). We recall that F is nondegenerate if $F(x, y) = 0$ for every $y \in V$ implies that $x = 0$. On the contrary we have the following result.

Proposition 3.1. *Let* V *a vector space on* $\mathbf{K}$, $\mathbf{K} = \mathbf{C}$ *or* $\mathbf{R}$. *A mapping* $F : V^2 \to \mathbf{K}$ *is twolinear and satisfies*

$$F(x, y) \cdot F(y, x) = \widehat{F}(x) \cdot \widehat{F}(y)\ , \quad x, y \in V\,, \tag{3.1}$$

if and only if there exits linear forms $L_1, L_2 : V \to \mathbf{K}$ *such that*

$$F(x, y) = L_1(x) \cdot L_2(y)\ , \quad x, y \in V\,. \tag{3.2}$$

The proof of Proposition 3.1 is similar to that of Theorem 2 in Ebanks's work (see [1]). Note that Ebanks's Theorem 2 is more general than Proposition 3.1.

(Some simpler proof for Proposition 3.1, than the one given in Theorem 2 (see [1]), could probably help us to prove an analogous result for n-linear forms, $n > 2$).

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