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Common Fixed Point Theorems for Maps under a New Contractive Condition

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Abstract

In this paper fixed point and coincidence results are presented for two and three single-valued mappings. These results extend previous results given by Rhoades (2003) and Djoudi and Merghadi (2008).

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1. Introduction and preliminaries

Let (X, d) be a metric space and let $f, g : X \longrightarrow X$ be two single valued mappings. f, g are said to be:
(a) Weakly commuting if for all $x \in X$

$$d(fgx, gfx) \leq d(fx, gx). \tag{1.1}$$

(b) Weakly compatible if for all $t \in X$ such that

$$ft = gt \text{ then } fgt = gft. \tag{1.2}$$

Clearly, every pair of weakly commuting mappings is weakly compatible. But the converse is not true.(see [4; Example 3])

Let f, g be self-mappings on X satisfying the following condition

$$g(X) \subset f(X) \quad . \tag{1.3}$$

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Let now x_0 be an arbitrary point of X and generate inductively the sequence $\{y_n\}_{n=0}^{\infty}$ as follow,

$$g(x_n) = f(x_{n+1}) = y_n, \qquad n = 0, 1, 2, ...$$

$O(y_k, r)$ is called the rth orbit of y_k and defines as follow:

$$O(y_k, r) := \{y_k, y_{k+1}, ..., y_{k+r}\}, \quad k = 0, 1, 2, ...$$

Also we define,

$$O(y_0, \infty) := \{y_0, y_1, ..., y_n, ...\}.$$

For any set A, $\delta(A)$ will denote the diameter of A. Furthermore, we put for every $x, y \in X$,

$$M(x, y) := \max\{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}.$$

and

$$N(x, y) := \max\{d(fx, fy), d(fx, gx), d(fy, gy), \frac{1}{2}[d(fx, gy) + d(fy, gx)]\}.$$

We introduce the notation Φ for all nondecreasing and continuous from the right mapping $\varphi : \mathbb{R}_+ \to [0, +\infty]$ with $\varphi(t) < t$ for every $t > 0$.
Also we introduce the notation Ψ for all nondecreasing continuous mapping $F : \mathbb{R}_+ \to [0, +\infty]$ with $F^{-1}(0) = \{0\}$.

In [10], S. Sessa generalized an elegant result due to G. Jungck [7] and proved the following theorem.

Theorem 1.1. *Let f be a continuous self-mapping of X and $g : X \to X$ verifying the conditions:*

(a) $d(fgx, gfx) \leq d(fx, gx)$,
(b) $g(X) \subset f(X)$,
(c) $d(gx, gy) \leq \phi(M(x, y))$.

If there exists $x_0 \in X$ such that $\delta(O(y_0, \infty)) < \infty$, then f and g have a unique common fixed point.

In [9], trying to extend a theorem of Branciari [1] and theorem of Ćirić [2], B.E. Rhoades established two fixed point theorems satisfying a contractive inequality of integral type. In particular he proved the following theorem.

Theorem 1.2. *Let (X, d) be a complete metric space, $k \in [0, 1)$, $g : X \to X$ and $f = I : X \to X$ be the identity mapping of X. Suppose that for all $x, y \in X$, the condition*

$$\int_0^{d(gx,gy)} \varphi(t)\ dt \leq k \int_0^{M(x,y)} \varphi(t)\ dt, \tag{1.4}$$

is valid where,
(i) $\varphi : \mathbb{R}_+ \to [0, \infty]$ is a Lebesgue-integrable mapping which is summable, nonnegative, and satisfies $\int_0^{\varepsilon} \varphi(t)\, dt > 0$ for each $\varepsilon > 0$.
If there is a point $x \in X$ with bounded orbit, then g has a unique fixed point in X.

A. Djoudi and F. merghadi [4] (2008) proved the following two theorems in particular, they extended theorem 2 for maps which are not necessary continuous, that extended [5] and [6].

Theorem 1.3. *Let (X,d) be a complete metric space and let $f,g : X \to X$ be to mappings verifying conditions* (1.2), (1.3). *Suppose that*

$$\int_0^{d(gx,gy)} \varphi(t)\ dt \le \phi\left(\int_0^{M(x,y)} \varphi(t)\ dt\right), \tag{1.5}$$

for all $x,y \in X$, where $\phi \in \Phi$ and φ is a functions having the property (i). Assume that $f(X)$ is a closed subset of X, and that there exists $x_0 \in X$, such that $\delta(O(y_0,\infty)) < \infty$. Then f and g have a unique common fixed point.

Theorem 1.4 (see [4; Theorem 11]). *Let f and g be two self-mappings of complete metric space (X,d) verifying conditions* (1.1), (1.3) *and* (1.5). *Assume that f is a continuous function on X and that there exists $x_0 \in X$, such that $\delta(O(y_0,\infty)) < \infty$. Then f and g have a unique common fixed point.*

Zang and song [12] proved the following theorem that extended Theorem 1.3 where $\phi(t) \equiv t$.

Theorem 1.5. *Let (X,d) be a complete metric space, and $T,S : X \to X$ two mapping such that for all $x,y \in X$*

$$d(Tx,Sy) \le N(x,y) - \varphi(N(x,y))$$

where $\varphi : \mathbb{R}_+ \to [0,\infty)$ is a lower semi-continuous function with $\varphi(t) > 0$ for $t \in (0,\infty)$ and $\varphi(0) = 0$ and

$$N(x,y) := \max\{d(x,y), d(x,Tx), d(y,Sy), \frac{1}{2}[d(y,Tx) + d(x,Sy)]\}.$$

Then there exists a unique point $u \in X$ such that $u = Tu = Su$.

In the proof of our main results, we will use the following lemma and refer to [11] for its proof.

Lemma 1.6. *Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a nondecreasing function and continuous from right. Then for all $t > 0$, $\varphi(t) < t$ if and only if $\lim_k \phi^k(t) = 0$, where ϕ^k denotes the k-times repeated composition of ϕ with itself.*

Rouhani and Moradi [3] (2010) extended Theorem 1.5 for multi-valued maps.
In section 2, we extend Djoudi-Merghadi's Theorem [Theorem 1.3 and Theorem 1.4].
In section 3, by same method in [12] we extend another type of Theorem 1.4 and Theorem 1.4 without assuming $\delta(O(y_0,\infty)) < +\infty$. These results extend extend the Rhoades Theorem (Theorem 1.2).

2. Extension of Djoudi-Merghadi's Theorem

The following theorem extends Djoudi and Merghadi's Theorem (Theorem 1.3).

Theorem 2.1. *Let f and g be two self-mappings of complete metric space (X,d) fulfilling conditions* (1.1), (1.3). *Suppose that,*

$$F(d(gx,gy)) \le \phi(F(M(x,y))), \tag{2.1}$$

for all $x,y \in X$, where $\phi \in \Phi$ and $F \in \Psi$. Assume that $f(X)$ is a closed subset of X and there exists $x_0 \in X$, such that $\delta(O(y_0,\infty)) < \infty$. Then f and g have a unique common fixed point.

Proof . We may assume that $\delta(O(y_k, r)) > 0$ for all $k \geq 0$ and $r \geq 1$, since, if there exist $k \geq 0$ and $r \geq 1$ such that $\delta(O(y_k, r)) = 0$, we immediately get, $y_k = y_{k+1} = y$ that is $f(x_{k+1}) = g(x_{k+1}) = y$, then from (1.2), $fy = gy$.
Hence

$$\begin{aligned} M(y, x_{k+1}) &= \max\{d(fy, fx_{k+1}), d(fy, gy), d(fx_{k+1}, gx_{k+1}), \\ &\quad d(fy, gx_{k+1}, d(fx_{k+1}, gy)\} \\ &= \max\{d(fy, y), d(gy, y)\} = d(gy, y). \end{aligned}$$

Therefore from (2.1)

$$\begin{aligned} F(d(gy, y)) = F(d(gy, gx_{k+1})) &\leq \phi(F(M(y, x_{k+1}))) \\ &= \phi(F(d(gy, y))). \end{aligned}$$

Since $\phi(t) < t$ for all $t > 0$, $F(d(gy, y)) = 0$ and since $F^{-1}(0) = \{0\}$, $d(gy, y) = 0$. Hence, $gy = y$ and so $fy = gy = y$.
So we may assume that $\delta(O(y_k, r)) > 0$ for all $k \geq 0$ and $r \geq 1$.
We break the argument into four steps.

Step 1.$\{y_n\}$ is Cauchy.
Proof . From the definition of $\delta(O(y_k, r))$, there exist m, n satisfying $k \leq n < m \leq k + r$ such that $\delta(O(y_k, r)) = d(y_n, y_m)$. So

$$\begin{aligned} F(\delta(O(y_k, r))) = F(d(y_n, y_m)) &= F(d(gx_n, gx_m)) \\ &\leq \phi(F(x_n, x_m)), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} M(x_n, x_m) &= \max\{d(y_{n-1}, y_{m-1}), d(y_{n-1}, y_n), d(y_{m-1}, y_m), \\ &\quad d(y_{n-1}, y_m), d(y_{m-1}, y_n)\} \\ &\leq \delta(O(y_{k-1}, r + 1)). \end{aligned} \tag{2.3}$$

From (2.2), (2.3) and using induction we conclude that

$$\begin{aligned} F(\delta(O(y_k, r))) &\leq \phi(F(\delta(O(y_{k-1}, r + 1)))) \\ &\leq \phi^2(F(\delta(O(y_{k-2}, r + 2)))) \\ &\leq \ldots \\ &\leq \phi^k(F(\delta(O(y_0, r + k)))) \end{aligned} \tag{2.4}$$

For every m, n integer with $m > n$, $d(y_m, y_n) \leq \delta(O(y_n, m))$. So from (2.4) and $\delta(O(y_0, \infty) < \infty$

$$\begin{aligned} F(d(y_m, y_n)) &\leq \phi(F(\delta(O(y_n, m)))) \\ &\leq \phi^n(F(\delta(O(y_0, n + m)))) \\ &\leq \phi^n(F(\delta(O(y_0, +\infty)))) \end{aligned} \tag{2.5}$$

Using Lemma 1.6, $\lim_{n\to\infty} \phi^n(t) = 0$ and hence, from (2.5), $\lim_{n,m\to\infty} F(d(y_n, y_m)) = 0$. Since $F \in \Psi$, $\lim_{n,m\to\infty} d(y_n, y_m) = 0$. Therefore $\{y_n\}$ is Chauchy. □

Step 2. $gz = fz$ for some $z \in X$.
Proof . By Step 1 and completeness of X, there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$. That is

$$z = \lim_{n\to\infty} y_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_{n+1}.$$

Since $f(X)$ is closed, there exists a point $u \in X$ such that $z = f(u)$. Using (2.1),

$$F(d(gu, gx_n)) \leq \phi(F(M(u, x_n)), \tag{2.6}$$

where

$$\begin{aligned} M(u, x_n) &= \max\{d(fu, fx_n), d(fu, gu), d(fx_n, gx_n), \\ &\quad d(fu, gx_n), d(fx_n, gu)\} \\ &= \max\{d(z, y_{n-1}), d(z, gu), d(y_{n-1}, y_n), \\ &\quad d(z, y_n), d(y_{n-1}, gu)\}, \end{aligned}$$

and this shows that $\lim_{n\to\infty} M(u, x_n) = d(z, gu)$.
Hence from (2.6)

$$F(d(gu, z)) \leq \phi(F(M(z, gu)),$$

and so $d(gu, z) = 0$. Therefore $gu = z$. From $fu = gu = z$ and (1.2) we obtain $fz = gz$. □

Step 3. f and g have a common fixed point.
Proof . Using (1.2) and $fz = gz = t$, we get $ft = gt$. From (1.6) we conclude that

$$F(d(gt, gz)) \leq \phi(F(M(t, z)), \tag{2.7}$$

where

$$\begin{aligned} M(t, z) &= \max\{d(ft, fz), d(ft, gt), d(fz, gz), \\ &\quad d(ft, gz), d(fz, gt)\} \\ &= d(gt, gz), \end{aligned}$$

and this shows that $d(gt, gz) = 0$. So $gt = gz$. Hence $g(t) = t$ and so $f(t) = g(t) = t$. Therefor f and g have a common fixed point. Unicity of the common fixed point follows from (2.1) and this completes the proof. □ □

Remark 2.2. *One can cheek without great difficulty that Theorem 2.2 is still true if we have "$g(X)$ is closed" instead of "$f(X)$ is closed". Moreover, the theorem also remains valid if we have g or f is surjective instead of "$g(X)$ is closed".*

Remark 2.3. *We derive Theorem 8 of A. Djoudi and F. Merghadi [4] (Theorem 1.4) if we let, in Theorem 2.2, $F(t) = \int_0^t \phi(s)ds$.*

The following theorem extends Theorem 1.4 ([5; Theorem 11]).

Theorem 2.4. *Let f and g be two self-mappings of complete metric space (X, d) verifying conditions (1.1), (1.3) and (2.1), for ϕ and F that introduced in Theorem 2.1 and let f be a continuous function of X, such that $\delta(O(y_0, \infty)) < \infty$ for some $y_0 \in X$. Then f and g have a unique common fixed point.*

Proof . Unicity of the common fixed point follows from (2.1). Following the proof of Theorem 2.1 we may conclude that $\{y_n\}$ is a Cauchy sequence converging to some z in X and

$$z = \lim_{n\to\infty} y_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_{n+1}.$$

Since f is continuous fy_n converges to fz. Furthermore, condition (1.1) triangular inequality imply

$$\begin{aligned} d(gy_n, fz) &\leq d(gy_n, fy_{n+1}) + d(fy_{n+1}, fz) \\ &= d(gfx_{n+1}, fgx_{n+1}) + d(fy_{n+1}, fz) \\ &\leq d(fx_{n+1}, gx_{n+1}) + d(fy_{n+1}, fz) \\ &= d(y_n, y_{n+1}) + d(fy_{n+1}, fz). \end{aligned} \tag{2.8}$$

Letting $n \to \infty$ in the above inequality, we conclude that $\lim_{n\to\infty} gy_n = fz$.

$$M(y_n, z) = \max\{d(fy_n, fz), d(fy_n, gy_n), d(fz, gz), d(fy_n, gz), d(fz_n, gy_n)\},$$

converges from the right to $d(fz, gz)$. Consequently, we obtain from (2.1),

$$F(d(gy_n, gz)) \leq \phi(F(M(y_n, z)), \tag{2.9}$$

and by taking the limit of (2.9), as n $\to \infty$, gives

$$F(d(fz, gz)) \leq \phi(F(M(fz, z)).$$

So $F(d(fz, gz)) = 0$. Hence, $d(fz, gz) = 0$ and therefore $fz = gz$. Thus $fgz = gfz = ggz$. Now we show that gz is a common fixed point for f and g. Using (2.9),

$$\begin{aligned} F(d(ggz, gz)) &\leq \phi(F(M(gz, z)) \\ &= \phi(F(d(ggz, z)) \end{aligned}$$

and hence $d(ggz, gz) = 0$. Therefore, $ggz = gz$. Thus $fgz = ggz = gz$. This shows that gz is common fixed point for f and g. □

Remark 2.5. *By define* $F(t) = \int_0^t \phi(s)ds$ *we can conclude Theorem* 1.4. *Also by define* $\phi(t) = t$ *we can extend Ćirić theorem* [2].

3. Extension of Rhoade's Theorem

By the same method in Zang and Song [12] we have two extension of Rhoads Theorem. In particular $M(x, y)$ in Theorem 2.1 and Theorem 2.4 is replaced by $N(x, y)$ without assuming $\delta(O(y_0, \infty)) < \infty$.

Theorem 3.1. *Let* f, g_1 *and* g_2 *be three self-mappings of complete metric space* (X, d) *verifying the conditions :*

(a) $\forall t \in X$ *if* $ft = g_i t$ *Then* $fg_i t = g_i ft$ $(i = 1, 2)$,
(b) $g_1(X) \subseteq f(X)$ *and* $g_2(X) \subseteq f(X)$,
(c) $F(d(g_1 x, g_2 y) \leq \phi(F(N(x, y)))$.

where $\phi, F \in \Psi$, $\phi(t) < t$ *for all* $t > 0$ *and where*

$$N(x,y) = \max\{d(fx,fy), d(fx,g_1x), d(fy,g_2y), \frac{1}{2}[d(fx,g_2y) + d(fy,g_1x)]\}.$$

Assume that $f(X)$ *is a closed subset of* X. *Then* f, g_1, g_2 *have a unique common fixed point.*

Proof . If for some $x, y \in X$, $d(g_1x, g_2y) \geq N(x,y)$ then

$$F(d(g_1x, g_2y)) \geq F(N(x,y)) \geq \phi(F(N(x,y)).$$

Since $\phi(t) < t$ for all $t > 0$, $F(N(x,y)) = 0$. So $N(x,y) = 0$. Hence $f(x) = f(y) = g_1(x) = g_2(y) = t$. Using (a) we conclude that $g_1t = ft = g_2t$. From (c),

$$F(d(g_1x, g_2t)) \leq \phi(F(N(x,t)).$$

where

$$\begin{aligned} N(x,t) &= \max\{d(fx,ft), d(fx,g_1x), d(ft,g_2t), \\ &\quad \frac{1}{2}[d(fx,g_2t) + d(ft,g_1x)]\} \\ &= \max\{d(g_1x,g_2t), \frac{1}{2}[d(g_1x,g_2t) + d(g_2t,g_1x)]\} \\ &= d(g_1x,g_2t), \end{aligned}$$

and this shows that $F(d(g_1x, g_2t)) = 0$. So $d(g_1x, g_2t) = 0$ and hence $t = g_1x = g_2t$. Therefore, $g_1t = ft = g_2t = t$. So we may assume that for all $x, y \in X$, $d(g_1x, g_2y) < N(x,y)$. Let $x_0 \in X$. Using (b) there exist $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ such that $y_0 = g_1(x_0) = f(x_1)$, $y_1 = g_2(x_1) = f(x_2)$, ... $y_{2n} = g_1(x_{2n}) = f(x_{2n+1})$, $y_{2n+1} = g_2(x_{2n+1}) = f(x_{2n+2})$, ...
We break the proof into four steps.

Step 1. $\lim_{n,m\to\infty} d(y_n, y_m) = 0$.
Proof . For all $n \in \mathbb{N}$

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(g_1x_{2n}, g_2x_{2n+1}) < N(x_{2n}, x_{2n+1}) \\ &= \max\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ &\quad \frac{1}{2}[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})]\} \\ &\leq \max\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ &\quad \frac{1}{2}[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]\} \\ &= \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\}. \end{aligned}$$

So

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &< N(x_{2n}, x_{2n+1}) \\ &= \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} \quad (3.1) \\ &= d(y_{2n-1}, y_{2n}). \end{aligned}$$

Similarly,

$$d(y_{2n+2}, y_{2n+1}) < N(x_{2n+2}, x_{2n+1}) = d(y_{2n+1}, y_{2n}). \quad (3.2)$$

Therefore by (3.1) and (3.2) we conclude that

$$d(y_{k+1}, y_k) < d(y_k, y_{k-1}),$$

for all $k \in \mathbb{N}$. Therefore the sequence $\{d(y_{n+1}, y_n)\}$ is decreasing and bounded belove. So there exists $r \geq 0$ such that $\lim_{n\to\infty} d(y_{n+1}, y_n) = r$. We need to show that $r = 0$. Using (3.1), (3.2) and condition (c) we have

$$F(d(y_{n+1}, y_n)) \leq \phi(F(N(x_n, x_{n+1}))) = \phi(F(d(y_n, y_{n-1}))), \tag{3.3}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (3.3), $F(r) \leq \phi(F(r))$. So $F(r) = 0$ and hence $r = 0$. $\square$

Step 2. $\{y_n\}$ is a bounded sequence.
Proof . If $\{y_n\}$ is a unbounded, then by Step 1, $\{y_{2n}\}$ and $\{y_{2n-1}\}$ are unbounded. We choose the sequence $\{n(k)\}_{k=1}^{\infty}$ such that $n(1) = 1$, $n(2) > n(1)$ is even and minimal in sense such that $d(y_{n(2)}, y_{n(1)}) > 1$ and similarly $n(3) > n(2)$ is odd and minimal in sense such that $d(y_{n(3)}, y_{n(2)} > 1$, ..., $n(2k) > n(2k-1)$ is even and minimal in sense such that $d(y_{n(2k)}), y_{n(2k-1)} > 1$ and $n(2k+1) > n(2k)$ is odd and minimal in sense such that $d(y_{n(2k+1)}, y_{n(2k)} > 1$. Obviously $n(k) \geq k$ for every $k \in \mathbb{N}$. By Step 1 there exists $N_0 \in \mathbb{N}$ such that for all $k \geq N_0$, $d(y_{k+1}, y_k) < \frac{1}{4}$. So for $k \geq N_0$ we have $n(k+1) - n(k) \geq 2$ and

$$\begin{aligned} 1 &< d(y_{n(k+1)}, y_{n(k)}) \\ &\leq d(y_{n(k+1)}, y_{n(k+1)-2}) + d(y_{n(k+1)-2}, y_{n(k)}) \\ &\leq d(y_{n(k+1)}, y_{n(k+1)-2}) + 1. \end{aligned}$$

Hence $\lim_{k\to\infty} d(y_{n(k+1)}, y_{n(k)}) = 1$. Also

$$\begin{aligned} 1 &< d(y_{n(k+1)}, y_{n(k)}) \\ &\leq d(y_{n(k+1)}, y_{n(k+1)+1}) + d(y_{n(k+1)+1}, y_{n(k)+1}) + d(y_{n(k)+1}, y_{n(k)}) \\ &\leq d(y_{n(k+1)}, y_{n(k+1)+1}) + d(y_{n(k+1)+1}, y_{n(k+1)}) + d(y_{n(k+1)}, y_{n(k)}) \\ &+ d(y_{n(k)}, y_{n(k)+1}) + d(y_{n(k)+1}, y_{n(k)}) \\ &= 2d(y_{n(k+1)}, y_{n(k+1)+1}) + d(y_{n(k+1)}, y_{n(k)}) + 2d(y_{n(k)+1}, y_{n(k)}), \end{aligned}$$

and this shows that $\lim_{k\to\infty} d(y_{n(k+1)+1}, y_{n(k)+1}) = 1$.
Using (c),

$$F(d(y_{n(k+1)+1}, y_{n(k)+1})) \leq \phi(F(N(x_{n(k+1)+1}, x_{n(k)+1}))), \tag{3.4}$$

where

$$\begin{aligned} d(y_{n(k+1)}, y_{n(k)}) &\leq N(x_{n(k+1)+1}, x_{n(k)+1}) \\ &= \max\{d(y_{n(k+1)}, y_{n(k)}), d(y_{n(k+1)}, y_{n(k+1)+1}), d(y_{n(k)}, y_{n(k)+1}), \\ &\quad \frac{1}{2}[d(y_{n(k+1)}, y_{n(k)+1}) + d(y_{n(k)}, y_{n(k+1)+1}]\} \\ &\leq \max\{d(y_{n(k+1)}, y_{n(k)}), d(y_{n(k+1)}, y_{n(k+1)+1}), d(y_{n(k)}, y_{n(k)+1}), \\ &\quad \frac{1}{2}[2d(y_{n(k+1)}, y_{n(k)}) + d(y_{n(k)}, y_{n(k)+1}) + d(y_{n(k+1)}, y_{n(k+1)+1})]\}, \end{aligned}$$

and this shows that $\lim_{k\to\infty} N(x_{n(k+1)+1}, x_{n(k)+1}) = 1$. Since (3.4) holds and $F \in \Psi$, $F(1) \le \phi(F(1))$. So $F(1) = 0$ and this is a contradiction. □

Step 3. $\{y_n\}$ is Cauchy.
Proof . Let $C_n = \sup\{d(y_i, y_j) : i, j \ge n\}$. Since $\{y_n\}$ is bounded, $C_n < +\infty$ for all $n \in \mathbb{N}$. Obviously $\{C_n\}$ is decreasing. So there exists $C \ge 0$ such that $\lim_{n\to\infty} C_n = C$. We need to show that $C = 0$.
For every $k \in \mathbb{N}$, there exists $n(k), m(k) \in \mathbb{N}$ such that $m(k) > n(k) \ge k$ and

$$C_k - \frac{1}{k} \le d(y_{m(k)}, y_{n(k)}) \le C_k. \tag{3.5}$$

By (3.5), we conclude that

$$\lim_{k\to\infty} d(y_{m(k)}, y_{n(k)}) = C. \tag{3.6}$$

From Step 1 and (3.6), we have

$$\begin{aligned}\lim_{k\to\infty} d(y_{m(k)+1}, y_{n(k)+1}) &= \lim_{k\to\infty} d(y_{m(k)+1}, y_{n(k)}) \\ &= \lim_{k\to\infty} d(y_{m(k)}, y_{n(k)+1}) \\ &= \lim_{k\to\infty} d(y_{m(k)}, y_{n(k)}) = C.\end{aligned}$$

So we can assume that for every $k \in \mathbb{N}$, $m(k)$ is odd and $n(k)$ is even. Hence,

$$F(d(y_{m(k)}, y_{n(k)})) \le \phi(F(M(x_{m(k)}, x_{n(k)}))), \tag{3.7}$$

where

$$\begin{aligned}N(x_{m(k)}, x_{n(k)}) = \max\{&d(y_{m(k)-1}, y_{n(k)-1}), d(y_{m(k)-1}, y_{m(k)}), d(y_{n(k)-1}, y_{n(k)}), \\ &\frac{1}{2}[d(y_{m(k)-1}, y_{n(k)}) + d(y_{n(k)-1}, y_{m(k)})]\}.\end{aligned}$$

This inequality shows that $\lim_{k\to\infty} N(x_{m(k)}, x_{n(k)}) = C$. and from (3.7) we get $F(C) \le \phi(F(C))$. So $C = 0$. □

Step 4. f, g_1 and g_2 have a common fixed point.
Proof . Since (X, d) is complete and $\{y_n\}$ is Cauchy there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Since $f(X)$ is closed, there exists a point $u \in X$ such that $z = f(u)$. For all $n \in \mathbb{N}$ For all $n \in \mathbb{N}$

$$F(d(g_1u, y_{2n+1})) = F(d(g_1u, g_2x_{2n+1})) \le \phi(F(N(u, x_{2n+1})))), \tag{3.8}$$

where

$$\begin{aligned}N(u, x_{2n+1}) &= \max\{d(fu, fx_{2n+1}), d(fu, g_1u), d(fx_{2n+1}, g_2x_{2n+1}), \\ &\quad \frac{1}{2}[d(fu, g_2x_{2n+1}) + d(fx_{2n+1}, g_1u)]\} \\ &= \max\{d(z, y_{2n}), d(z, g_1u), d(y_{2n}, y_{2n+1}), \\ &\quad \frac{1}{2}[d(z, y_{2n+1}) + d(y_{2n}, g_1u)]\},\end{aligned}$$

and this shows that $\lim_{n\to\infty} N(u, x_{2n+1}) = d(z, g_1u)$. Letting $n \to \infty$ in (3.8), we conclude that

$$F(d(g_1u, z)) \leq \phi(F(d(g_1u, z)))),$$

and so $F(d(g_1u, z)) = 0$. Hence $d(g_1u, z) = 0$ and therefore $g_1u = z$. Similarity, $g_2u = z$. Therefore $fu = g_1u = g_2u = z$. Using condition (a) we conclude that $g_1z = fz = g_2z$. Now we prove that z is a common fixed point for f, g_1 and g_2.
For all $n \in \mathbb{N}$,

$$F(d(g_1z, y_{2n+1})) = F(d(g_1z, g_2x_{2n+1})) \leq \phi(F(N(z, x_{2n+1})))), \tag{3.9}$$

where

$$\begin{aligned} N(z, x_{2n+1}) &= \max\{d(fz, fx_{2n+1}), d(fz, g_1z), d(fx_{2n+1}, g_2x_{2n+1}), \\ &\quad \frac{1}{2}[d(fz, g_2x_{2n+1}) + d(fx_{2n+1}, g_1z)]\} \\ &= \max\{d(g_1z, y_{2n}), 0, d(y_{2n}, y_{2n+1}), \\ &\quad \frac{1}{2}[d(g_1z, y_{2n+1}) + d(y_{2n}, g_1z)]\}, \end{aligned}$$

and this shows that $\lim_{n\to\infty} N(z, x_{2n+1}) = d(z, g_1z)$. Letting $n \to \infty$ in (3.9), we get

$$F(d(g_1z, z)) \leq \phi(F(d(g_1z, z)))),$$

This shows that $F(d(g_1z, z)) = 0$ and so $d(g_1z, z) = 0$. Hence $g_1z = z$. Therefore $f(z) = g_1(z) = g_2(z) = z$. Unicity of the common fixed point follows from (c). □ □

Theorem 3.2. *Let f, g_1 and g_2 be three self-mappings of complete metric space (X, d) verifying the conditions (b) and (c) of Theorem 3.1, where $\phi, F \in \Psi$ and where $\phi(t) < t$ for all $t > 0$. Assume that f is a continuous function of X. If for all $x \in X$,*

$$d(fg_ix, g_ifx) \leq d(fx, g_ix), \quad i = 1, 2. \tag{3.10}$$

Then f, g_1, g_2 have a unique common fixed point.

Proof . Following the proof of Theorem 3.1 we may conclude that $\{y_n\}$ is Cauchy sequence converging to some z in X and

$$z = \lim_{n\to\infty} y_n = \lim_{n\to\infty} g_1x_{2n} = \lim_{n\to\infty} g_2x_{2n+1} = \lim_{n\to\infty} fx_n.$$

Since f is continuous, fy_n converges fz. Using (3.10) and triangular inequality for all $n \in \mathbb{N}$

$$\begin{aligned} d(g_1y_{2n+1}, fz) &\leq d(g_1y_{2n+1}, fy_{2n+2}) + d(fy_{2n+2}, fz) \\ &= d(g_1fx_{2n+2}, fg_1x_{2n+2}) + d(fy_{2n+2}, fz) \\ &\leq d(fx_{2n+2}, g_1x_{2n+2}) + d(fy_{2n+2}, fz) \\ &= d(y_{2n+1}, y_{2n+2}) + d(fy_{2n+2}, fz). \end{aligned}$$

This shows that $\lim_{n\to\infty} d(g_1y_{2n+1}, fz) = 0$.
From (c)

$$F(d(g_1y_{2n+1}, g_2z)) \leq \phi(F(N(y_{2n+1}, z)), \tag{3.11}$$

where

$$\begin{aligned} N(y_{2n+1}, z) &= \max \{d(fy_{2n+1}, fz), d(fy_{2n+1}, g_1 y_{2n+1}), d(fz, g_2 z), \\ &\quad \frac{1}{2}[d(fy_{2n+1}, g_2 z) + d(fz, g_1 y_{2n+1})]\}, \end{aligned}$$

and this shows that $\lim_{n\to\infty} N(y_{2n+1}, z) = d(fz, g_2 z)$. Hence, from (3.11) we conclude that $F(d(fz, g_2 z)) \leq \phi(F(d(fz, g_2 z)))$. So $F(d(fz, g_2 z)) = 0$ and hence $d(fz, g_2 z) = 0$. Therefore $fz = g_2 z$. Similarly $fz = g_1 z$. Therefore $fz = g_1 z = g_2 z = t$.

Now we prove that t is common fixed point for f, g_1 and g_2. Using $fz = g_1 z = g_2 z = t$ and condition (3.10), we have $ft = g_1 t = g_2 t$. So from (c)

$$F(d(g_1 t, t)) = F(d(g_1 g_1 t, g_2 t)) \leq \phi(F(N(g_1 z, z)),$$

where

$$\begin{aligned} N(g_1 z, z) &= \max \{d(ft, fz), d(ft, g_1 t), d(fz, g_2 z), \\ &\quad \frac{1}{2}[d(ft, g_2 z) + d(fz, g_1 t)]\}, \\ &= d(ft, fz) = d(g_1 t, t). \end{aligned}$$

Hence $F(d(g_1 t, t)) \leq \phi(F(d(g_1 t, t)))$ and so $F(d(g_1 t, t)) = 0$. Therefore $d(g_1 t, t) = 0$ and this shows that $g_1 t = t$. Hence $ft = g_1 t = g_2 t = t$. By (c) we conclude that the common fixed point is unique and this completes the proof. □

4. Conclusion

We have extended Djoudi-Merghadi's Theorem by replacing $\int_0^t \phi(s)ds$ with F where $F \in \Psi$. We have also extended Rhoades theorem by assuming f, g_1 and g_2 verifying the condition (c) in Theorem 3.1 and 3.2, and found the common fixed point of f, g_1 and g_2. For the future directions of research we can consider three maps f, g_1 and g_2 in Theorem 2.1 and Theorem 2.4.

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