Common Fixed Point Theorems for Maps under a New Contractive Condition

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\begin{abstract}
In this paper fixed point and coincidence results are presented for two and three single-valued mappings. These results extend previous results given by Rhoades (2003) and Djoudi and Merghadi (2008).

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\end{abstract}

\section{Introduction and preliminaries}

Let \((X, d)\) be a metric space and let \(f, g : X \rightarrow X\) be two single valued mappings. \(f, g\) are said to be:

\begin{enumerate}[(a)]
\item Weakly commuting if for all \(x \in X\)
\[ d(fgx,gfx) \leq d(fx,gx). \]  \tag{1.1}
\item Weakly compatible if for all \(t \in X\) such that \(ft = gt\) then \(f gt = g ft\).
\[ ft = gt \text{ then } f gt = g ft. \]  \tag{1.2}
\end{enumerate}

Clearly, every pair of weakly commuting mappings is weakly compatible. But the converse is not true.(see [4; Example 3])

Let \(f, g\) be self-mappings on \(X\) satisfying the following condition
\[ g(X) \subset f(X). \]  \tag{1.3}
Let now $x_0$ be an arbitrary point of $X$ and generate inductively the sequence $\{y_n\}_{n=0}^{\infty}$ as follow,
\[ g(x_n) = f(x_{n+1}) = y_n, \quad n = 0, 1, 2, \ldots \]

$O(y_k, r)$ is called the $r$th orbit of $y_k$ and defines as follow:
\[ O(y_k, r) := \{y_k, y_{k+1}, \ldots, y_{k+r}\}, \quad k = 0, 1, 2, \ldots \]

Also we define,
\[ O(y_0, \infty) := \{y_0, y_1, \ldots, y_n, \ldots\}. \]

For any set $A$, $\delta(A)$ will denote the diameter of $A$. Furthermore, we put for every $x, y \in X$,
\[ M(x, y) := \max\{d(fx, fy), d(fx, gx), d(fy, gy), d(fy, gx)\}. \]

and
\[ N(x, y) := \max\{d(fx, fy), d(fx, gx), d(fy, gy), \frac{1}{2}[d(fx, gy) + d(fy, gx)]\}. \]

We introduce the notation $\Phi$ for all nondecreasing and continuous from the right mapping $\varphi : \mathbb{R}_+ \to [0, +\infty]$ with $\varphi(t) < t$ for every $t > 0$.

Also we introduce the notation $\Psi$ for all nondecreasing continuous mapping $F : \mathbb{R}_+ \to [0, +\infty]$ with $F^{-1}(0) = \{0\}$.

In [10], S. Sessa generalized an elegant result due to G. Jungck [7] and proved the following theorem.

**Theorem 1.1.** Let $f$ be a continuous self-mapping of $X$ and $g : X \to X$ verifying the conditions:

(a) $d(fgx,gfx) \leq d(fx,gx)$,
(b) $g(X) \subset f(X)$,
(c) $d(gx,gy) \leq \varphi(M(x,y))$.

If there exists $x_0 \in X$ such that $\delta(O(y_0, \infty)) < \infty$, then $f$ and $g$ have a unique common fixed point.

In [9], trying to extend a theorem of Branciari [1] and theorem of Ćirić [2], B.E. Rhoades established two fixed point theorems satisfying a contractive inequality of integral type. In particular he proved the following theorem.

**Theorem 1.2.** Let $(X, d)$ be a complete metric space, $k \in [0, 1)$, $g : X \to X$ and $f = I : X \to X$ be the identity mapping of $X$. Suppose that for all $x, y \in X$, the condition
\[ \int_0^{d(gx,gy)} \varphi(t) \, dt \leq k \int_0^{M(x,y)} \varphi(t) \, dt, \tag{1.4} \]

is valid where,
(i) $\varphi : \mathbb{R}_+ \to [0, \infty]$ is a Lebesgue-integrable mapping which is summable, nonnegative, and satisfies
\[ \int_0^\varepsilon \varphi(t) \, dt > 0 \quad \text{for each} \quad \varepsilon > 0. \]

If there is a point $x \in X$ with bounded orbit, then $g$ has a unique fixed point in $X$.

A. Djoudi and F. Merghadi [4] (2008) proved the following two theorems in particular, they extended theorem 2 for maps which are not necessary continuous, that extended [5] and [6].
Theorem 1.3. Let \((X, d)\) be a complete metric space and let \(f, g : X \rightarrow X\) be to mappings verifying conditions (1.2), (1.3). Suppose that
\[
\int_0^\infty \varphi(t) \, dt \leq \phi \left( \int_0^\infty \varphi(t) \, dt \right),
\]
for all \(x, y \in X\), where \(\phi \in \Phi\) and \(\varphi\) is a functions having the property (i). Assume that \(f(X)\) is a closed subset of \(X\), and that there exists \(x_0 \in X\), such that \(\delta(O(y_0, \infty)) < \infty\). Then \(f\) and \(g\) have a unique common fixed point.

Theorem 1.4 (see [4; Theorem 11]). Let \(f\) and \(g\) be two self-mappings of complete metric space \((X, d)\) verifying conditions (1.1), (1.3) and (1.5). Assume that \(f\) is a continuous function on \(X\) and that there exists \(x_0 \in X\), such that \(\delta(O(y_0, \infty)) < \infty\). Then \(f\) and \(g\) have a unique common fixed point.

Zang and Song [12] proved the following theorem that extended Theorem 1.3 where \(\phi(t) \equiv t\).

Theorem 1.5. Let \((X, d)\) be a complete metric space, and \(T, S : X \rightarrow X\) two mapping such that for all \(x, y \in X\)
\[
d(Tx, Sy) \leq N(x, y) - \varphi(N(x, y))
\]
where \(\varphi : \mathbb{R}_+ \rightarrow [0, \infty)\) is a lower semi-continuous function with \(\varphi(t) > 0\) for \(t \in (0, \infty)\) and \(\varphi(0) = 0\) and
\[
N(x, y) := \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}[d(y, Tx) + d(x, Sy)]\}.
\]
Then there exists a unique point \(u \in X\) such that \(u = Tu = Su\).

In the proof of our main results, we will use the following lemma and refer to [11] for its proof.

Lemma 1.6. Let \(\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) is a nondecreasing function and continuous from right. Then for all \(t > 0\), \(\varphi(t) < t\) if and only if \(\lim_{t \to 0} \phi^k(t) = 0\), where \(\phi^k\) denotes the \(k\)-times repeated composition of \(\phi\) with itself.


In section 2, we extend Djoudi-Merghadi’s Theorem [Theorem 1.3 and Theorem 1.4].
In section 3, by same method in [12] we extend another type of Theorem 1.4 and Theorem 1.4 without assuming \(\delta(O(y_0, \infty)) < +\infty\). These results extend extend the Rhoades Theorem (Theorem 1.2).

2. Extension of Djoudi-Merghadi’s Theorem

The following theorem extends Djoudi and Merghadi’s Theorem (Theorem 1.3).

Theorem 2.1. Let \(f\) and \(g\) be two self-mappings of complete metric space \((X, d)\) fulfilling conditions (1.1), (1.3). Suppose that,
\[
F(d(gx, gy)) \leq \phi(F(M(x, y))),
\]
for all \(x, y \in X\), where \(\phi \in \Phi\) and \(F \in \Psi\). Assume that \(f(X)\) is a closed subset of \(X\) and there exists \(x_0 \in X\), such that \(\delta(O(y_0, \infty)) < \infty\). Then \(f\) and \(g\) have a unique common fixed point.
Proof. We may assume that $\delta(O(y_k, r)) > 0$ for all $k \geq 0$ and $r \geq 1$, since, if there exist $k \geq 0$ and $r \geq 1$ such that $\delta(O(y_k, r)) = 0$, we immediately get, $y_k = y_{k+1} = y$ that is $f(x_{k+1}) = g(x_{k+1}) = y$, then from (1.2), $fy = gy$.

Hence

$$M(y, x_{k+1}) = \max \{d(fy, fx_{k+1}), d(fy, gy), d(fx_{k+1}, gx_{k+1}), \}
\{d(fy, gx_{k+1}, d(fx_{k+1}, gy))\}
\leq \max \{d(fy, y), d(gy, y)\} = d(gy, y).$$

Therefore from (2.1)

$$F(d(gy, y)) = F(d(gy, gx_{k+1})) \leq \phi(F(M(y, x_{k+1})))
= \phi(F(d(gy, y))).$$

Since $\phi(t) < t$ for all $t > 0$, $F(d(gy, y)) = 0$ and since $F^{-1}(0) = \{0\}$, $d(gy, y) = 0$. Hence, $gy = y$ and so $fy = gy = y$.

So we may assume that $\delta(O(y_k, r)) > 0$ for all $k \geq 0$ and $r \geq 1$.

We break the argument into four steps.

Step 1. $\{y_n\}$ is Cauchy.

Proof. From the definition of $\delta(O(y_k, r))$, there exist $m, n$ satisfying $k \leq n < m \leq k + r$ such that $\delta(O(y_k, r)) = d(y_n, y_m)$. So

$$F(\delta(O(y_k, r))) = F(d(y_n, y_m)) = F(d(gx_n, gx_m))
\leq \phi(F(x_n, x_m)), \quad (2.2)$$

where

$$M(x_n, x_m) = \max \{d(y_{n-1}, y_{m-1}), d(y_{n-1}, y_n), d(y_{m-1}, y_m),
\{d(y_{n-1}, y_m), d(y_{m-1}, y_n)\}\}
\leq \delta(O(y_{k-1}, r + 1)). \quad (2.3)$$

From (2.2), (2.3) and using induction we conclude that

$$F(\delta(O(y_k, r))) \leq \phi(F(\delta(O(y_{k-1}, r + 1))))
\leq \phi^2(F(\delta(O(y_{k-2}, r + 2))))
\leq \ldots
\leq \phi^k(F(\delta(O(y_0, r + k)))) \quad (2.4)$$

For every $m, n$ integer with $m > n$, $d(y_m, y_n) \leq \delta(O(y_n, m))$. So from (2.4) and $\delta(O(y_0, \infty) < \infty$

$$F(d(y_m, y_n)) \leq \phi(F(\delta(O(y_m, m))))
\leq \phi^n(F(\delta(O(y_0, n + m)))) \quad (2.5)$$

Using Lemma 1.6, $\lim_{n \to \infty} \phi^n(t) = 0$ and hence, from (2.5), $\lim_{n,m \to \infty} F(d(y_n, y_m)) = 0$. Since $F \in \Psi$, $\lim_{n,m \to \infty} d(y_n, y_m) = 0$. Therefore $\{y_n\}$ is Cauchy. \(\square\)
Step 2. $g z = f z$ for some $z \in X$.

**Proof.** By Step 1 and completeness of $X$, there exists $z \in X$ such that $\lim_{n \to \infty} y_n = z$. That is

$$z = \lim_{n \to \infty} y_n = \lim_{n \to \infty} g x_n = \lim_{n \to \infty} f x_{n+1}.$$ 

Since $f(X)$ is closed, there exists a point $u \in X$ such that $z = f(u)$. Using (2.1),

$$F(d(gu, gx_n)) \leq \phi(F(M(u, x_n))),$$

where

$$M(u, x_n) = \max\{d(fu, fx_n), d(fu, gu), d(fx_n, gx_n),$$

$$d(fu, gx_n), d(fx_n, gu)\}$$

$$= \max\{d(z, y_{n-1}), d(z, gu), d(y_{n-1}, y_n),$$

$$d(z, y_n), d(y_{n-1}, gu)\},$$

and this shows that $\lim_{n \to \infty} M(u, x_n) = d(z, gu)$. Hence from (2.6)

$$F(d(gu, z)) \leq \phi(F(M(z, gu))),$$

and so $d(gu, z) = 0$. Therefore $gu = z$. From $fu = gu = z$ and (1.2) we obtain $fz = gz$. \qed

Step 3. $f$ and $g$ have a common fixed point.

**Proof.** Using (1.2) and $fz = gz = t$, we get $ft = gt$. From (1.6) we conclude that

$$F(d(gt, gz)) \leq \phi(F(M(t, z))),$$

where

$$M(t, z) = \max\{d(ft, fz), d(ft, gt), d(fz, gz),$$

$$d(ft, fz), d(fz, gt)\}$$

$$= d(gt, gz),$$

and this shows that $d(gt, gz) = 0$. So $gt = gz$. Hence $g(t) = t$ and so $f(t) = g(t) = t$. Therefore $f$ and $g$ have a common fixed point. Unicity of the common fixed point follows from (2.1) and this completes the proof. \qed

**Remark 2.2.** One can check without great difficulty that Theorem 2.2 is still true if we have "$g(X)$ is closed" instead of "$f(X)$ is closed". Moreover, the theorem also remains valid if we have $g$ or $f$ is surjective instead of "$g(X)$ is closed".

**Remark 2.3.** We derive Theorem 8 of A. Djoudi and F. Merghadi [4] (Theorem 1.4) if we let, in Theorem 2.2, $F(t) = \int_0^t \phi(s)ds$.

The following theorem extends Theorem 1.4 ([5; Theorem 11]).

**Theorem 2.4.** Let $f$ and $g$ be two self-mappings of complete metric space $(X, d)$ verifying conditions (1.1), (1.3) and (2.1), for $\phi$ and $F$ that introduced in Theorem 2.1 and let $f$ be a continuous function of $X$, such that $\delta(O(y_0, \infty)) < \infty$ for some $y_0 \in X$. Then $f$ and $g$ have a unique common fixed point.
Proof. Unicity of the common fixed point follows from (2.1). Following the proof of Theorem 2.1 we may conclude that \( \{y_n\} \) is a Cauchy sequence converging to some \( z \) in \( X \) and

\[
z = \lim_{n \to \infty} y_n = \lim_{n \to \infty} gx_n = \lim_{n \to \infty} fx_{n+1}.
\]

Since \( f \) is continuous \( fy_n \) converges to \( fz \). Furthermore, condition (1.1) triangular inequality imply

\[
\begin{align*}
d(gy_n, fz) & \leq d(gy_n, fy_{n+1}) + d(fy_{n+1}, fz) \\
& = d(gfx_{n+1}, gfx_{n+1}) + d(fy_{n+1}, fz) \\
& \leq d(fx_{n+1}, gx_{n+1}) + d(fy_{n+1}, fz) \\
& = d(y_n, y_{n+1}) + d(fy_{n+1}, fz).
\end{align*}
\]

(2.8)

Letting \( n \to \infty \) in the above inequality, we conclude that \( \lim_{n \to \infty} gy_n = fz \).

\[
M(y_n, z) = \max\{d(fy_n, fz), d(fy_n, gy_n), d(fz, gz), d(fy_n, gz), d(fz_n, gy_n)\},
\]

converges from the right to \( d(fz, gz) \). Consequently, we obtain from (2.1),

\[
F(d(gy_n, gz)) \leq \phi(F(M(y_n, z))),
\]

(2.9)

and by taking the limit of (2.9), as \( n \to \infty \), gives

\[
F(d(fz, gz)) \leq \phi(F(M(fz, z))).
\]

So \( F(d(fz, gz)) = 0 \). Hence, \( d(fz, gz) = 0 \) and therefore \( fz = gz \). Thus \( fgz = gfz = ggz \). Now we show that \( gz \) is a common fixed point for \( f \) and \( g \). Using (2.9),

\[
F(d(ggz, gz)) \leq \phi(F(M(gz, z))
\]

and hence \( d(ggz, gz) = 0 \). Therefore, \( ggz = gz \). Thus \( fgz = ggz = gz \). This shows that \( gz \) is common fixed point for \( f \) and \( g \). \( \square \)

Remark 2.5. By define \( F(t) = \int_0^t \phi(s)ds \) we can conclude Theorem 1.4. Also by define \( \phi(t) = t \) we can extend Ćirić theorem [2].

3. Extension of Rhoade’s Theorem

By the same method in Zang and Song [12] we have two extension of Rhoads Theorem. In particular \( M(x, y) \) in Theorem 2.1 and Theorem 2.4 is replaced by \( N(x, y) \) without assuming \( \delta(O(y_0, \infty)) < \infty \).

Theorem 3.1. Let \( f, g_1 \) and \( g_2 \) be three self-mappings of complete metric space \( (X, d) \) verifying the conditions :

(a) \( \forall t \in X \) if \( ft = g_it \) Then \( fg_it = g_itf \) \((i = 1, 2)\),

(b) \( g_1(X) \subseteq f(X) \) and \( g_2(X) \subseteq f(X) \),

(c) \( F(d(g_1x, g_2y)) \leq \phi(F(N(x, y))) \).

where $\phi, F \in \Psi$, $\phi(t) < t$ for all $t > 0$ and where

$$N(x, y) = \max \{d(fx, fy), d(fx, g_1x), d(fy, g_2y), \frac{1}{2}[d(fx, g_2y) + d(fy, g_1x)]\}.$$ 

Assume that $f(X)$ is a closed subset of $X$. Then, $g_1$, $g_2$ have a unique common fixed point.

**Proof.** If for some $x, y \in X$, $d(g_1x, g_2y) \geq N(x, y)$ then

$$F(d(g_1x, g_2y)) \geq F(N(x, y)) \geq \phi(F(N(x, y))).$$

Since $\phi(t) < t$ for all $t > 0$, $F(N(x, y)) = 0$. So $N(x, y) = 0$. Hence $f(x) = f(y) = g_1(x) = g_2(y) = t$. Using (a) we conclude that $g_1t = ft = g_2t$. From (c),

$$F(d(g_1x, g_2y)) \leq \phi(F(N(x, t))).$$

where

$$N(x, t) = \max \{d(fx, ft), d(fx, g_1x), d(ft, g_2t), \frac{1}{2}[d(fx, g_2t) + d(ft, g_1x)]\}$$

and this shows that $F(d(g_1x, g_2t)) = 0$. So $d(g_1x, g_2t) = 0$ and hence $t = g_1x = g_2t$. Therefore, $g_1t = ft = g_2t = t$. So we may assume that for all $x, y \in X$, $d(g_1x, g_2y) < N(x, y)$. Let $x_0 \in X$.

Using (b) there exist $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ such that $y_0 = g_1(x_0) = f(x_1)$, $y_1 = g_2(x_1) = f(x_2)$, ... $y_{2n} = g_1(x_{2n})$, $y_{2n+1} = g_2(x_{2n+1}) = f(x_{2n+2})$, ...

We break the proof into four steps.

**Step 1.** $\lim_{n,m \to \infty} d(y_n, y_m) = 0$.

**Proof.** For all $n \in \mathbb{N}$

$$d(y_{2n}, y_{2n+1}) = d(g_1x_{2n}, g_2x_{2n+1}) < N(x_{2n}, x_{2n+1})$$

$$= \max \{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \frac{1}{2}[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})]\}$$

$$\leq \max \{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \frac{1}{2}[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]\}$$

$$= \max \{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\}.$$ 

So

$$d(y_{2n}, y_{2n+1}) < N(x_{2n}, x_{2n+1})$$

$$= \max \{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\}$$

$$= d(y_{2n-1}, y_{2n}). \quad (3.1)$$

Similarly,

$$d(y_{2n+2}, y_{2n+1}) < N(x_{2n+2}, x_{2n+1}) = d(y_{2n+1}, y_{2n}). \quad (3.2)$$
Therefore by (3.1) and (3.2) we conclude that
\[ d(y_{k+1}, y_k) < d(y_k, y_{k-1}), \]
for all \( k \in \mathbb{N} \). Therefore the sequence \( \{d(y_{n+1}, y_n)\} \) is decreasing and bounded below. So there exists \( r \geq 0 \) such that \( \lim_{n \to \infty} d(y_{n+1}, y_n) = r \). We need to show that \( r = 0 \). Using (3.1), (3.2) and condition (c) we have
\[ F(d(y_{n+1}, y_n)) \leq \phi(F(N(x_n, x_{n+1}))) = \phi(F(d(y_n, y_{n-1}))), \quad (3.3) \]
for all \( n \in \mathbb{N} \). Letting \( n \to \infty \) in (3.3), \( F(r) \leq \phi(F(r)) \). So \( F(r) = 0 \) and hence \( r = 0 \). \( \square \)

**Step 2.** \( \{y_n\} \) is a bounded sequence.

**Proof.** If \( \{y_n\} \) is an unbounded, then by Step 1, \( \{y_{2n}\} \) and \( \{y_{2n-1}\} \) are unbounded. We choose the sequence \( \{n(k)\}_{k=1}^{\infty} \) such that \( n(1) = 1 \), \( n(2) > n(1) \) is even and minimal in sense such that \( d(y_{n(2)}, y_{n(1)}) > 1 \) and similarly \( n(3) > n(2) \) is odd and minimal in sense such that \( d(y_{n(3)}, y_{n(2)}) > 1 \), ..., \( n(2k) > n(2k-1) \) is even and minimal in sense such that \( d(y_{n(2k)}, y_{n(2k-1)}) > 1 \) and \( n(2k+1) > n(2k) \) is odd and minimal in sense such that \( d(y_{n(2k+1)}, y_{n(2k)}) > 1 \). Obviously \( n(k) \geq k \) for every \( k \in \mathbb{N} \). By Step 1 there exists \( N_0 \in \mathbb{N} \) such that for all \( k \geq N_0 \), \( d(y_{k+1}, y_k) < \frac{1}{k} \). So for \( k \geq N_0 \) we have \( n(k+1) - n(k) \geq 2 \) and
\[
1 < d(y_{n(k+1)}, y_{n(k)}) \leq d(y_{n(k+1)}, y_{n(k)-2}) + d(y_{n(k)-2}, y_{n(k)}) \leq d(y_{n(k)+1}, y_{n(k)-2}) + 1.
\]
Hence \( \lim_{k \to \infty} d(y_{n(k+1)}, y_{n(k)}) = 1 \). Also
\[
1 < d(y_{n(k)+1}, y_{n(k)}) \leq d(y_{n(k)+1}, y_{n(k)+1}) + d(y_{n(k)+1}, y_{n(k)}) \leq d(y_{n(k)+1}, y_{n(k)+1}) + d(y_{n(k)+1}, y_{n(k)}) + d(y_{n(k)}, y_{n(k)+1}) + d(y_{n(k)}, y_{n(k)})
\]
\[
= 2d(y_{n(k)+1}, y_{n(k)+1}) + d(y_{n(k)+1}, y_{n(k)}) + 2d(y_{n(k)+1}, y_{n(k)}),
\]
and this shows that \( \lim_{k \to \infty} d(y_{n(k)+1}, y_{n(k)+1}) = 1 \).

Using (c),
\[ F(d(y_{n(k)+1}, y_{n(k)+1})) \leq \phi(F(N(x_{n(k)+1}, x_{n(k)+1}))), \quad (3.4) \]
where
\[
d(y_{n(k)+1}, y_{n(k)}) \leq N(x_{n(k)+1}, x_{n(k)+1}) = \max \left\{ d(y_{n(k)+1}, y_{n(k)}), d(y_{n(k)}, y_{n(k)+1}) \right\} \leq \frac{1}{2} \left[ d(y_{n(k)+1}, y_{n(k)+1}) + d(y_{n(k)}, y_{n(k)+1}) \right]
\]
\[
\leq \max \left\{ d(y_{n(k)+1}, y_{n(k)}), d(y_{n(k)}, y_{n(k)+1}) \right\} \leq \frac{1}{2} \left[ 2d(y_{n(k)+1}, y_{n(k)}) + d(y_{n(k)}, y_{n(k)+1}) + d(y_{n(k)+1}, y_{n(k)+1}) \right],
\]
and this shows that \( \lim_{k \to \infty} N(x_{n(k)+1}, x_{n(k)+1}) = 1 \). Since (3.4) holds and \( F \in \Psi \), \( F(1) \leq \phi(F(1)) \). So \( F(1) = 0 \) and this is a contradiction. \( \square \)

**Step 3.** \( \{y_n\} \) is Cauchy.  

**Proof.** Let \( C_n = \sup \{d(y_i, y_i) : i, j \geq n\} \). Since \( \{y_n\} \) is bounded, \( C_n < +\infty \) for all \( n \in \mathbb{N} \). Obviously \( \{C_n\} \) is decreasing. So there exists \( C \geq 0 \) such that \( \lim_{n \to \infty} C_n = C \). We need to show that \( C = 0 \).

For every \( k \in \mathbb{N} \), there exists \( n(k), m(k) \in \mathbb{N} \) such that \( m(k) > n(k) \geq k \) and

\[
C_k - \frac{1}{k} \leq d(y_{m(k)}, y_{n(k)}) \leq C_k. \tag{3.5}
\]

By (3.5), we conclude that

\[
\lim_{k \to \infty} d(y_{m(k)}, y_{n(k)}) = C. \tag{3.6}
\]

From Step 1 and (3.6), we have

\[
\lim_{k \to \infty} d(y_{m(k)+1}, y_{n(k)+1}) = \lim_{k \to \infty} d(y_{m(k)+1}, y_{n(k)}) = \lim_{k \to \infty} d(y_{m(k)}, y_{n(k)+1}) = \lim_{k \to \infty} d(y_{m(k)}, y_{n(k)}) = C.
\]

So we can assume that for every \( k \in \mathbb{N} \), \( m(k) \) is odd and \( n(k) \) is even. Hence,

\[
F(d(y_{m(k)}, y_{n(k)})) \leq \phi(F(M(x_{m(k)}, x_{n(k)}))), \tag{3.7}
\]

where

\[
N(x_{m(k)}, x_{n(k)}) = \max \{d(y_{m(k)-1}, y_{n(k)-1}), d(y_{m(k)-1}, y_{m(k)}), d(y_{n(k)-1}, y_{n(k)}), \frac{1}{2}[d(y_{m(k)-1}, y_{n(k)}) + d(y_{n(k)-1}, y_{m(k)})]\}.
\]

This inequality shows that \( \lim_{k \to \infty} N(x_{m(k)}, x_{n(k)}) = C \). and from (3.7) we get \( F(C) \leq \phi(F(C)) \). So \( C = 0 \). \( \square \)

**Step 4.** \( f_1 \) and \( g_2 \) have a common fixed point.  

**Proof.** Since \((X, d)\) is complete and \( \{y_n\} \) is Cauchy there exists \( z \in X \) such that \( \lim_{n \to \infty} y_n = z \). Since \( f(X) \) is closed, there exists a point \( u \in X \) such that \( z = f(u) \). For all \( n \in \mathbb{N} \) For all \( n \in \mathbb{N} \)

\[
F(d(g_1 u, y_{2n+1})) = F(d(g_1 u, g_2 x_{2n+1})) \leq \phi(F(N(u, x_{2n+1})))), \tag{3.8}
\]

where

\[
N(u, x_{2n+1}) = \max \{d(f u, f x_{2n+1}), d(f u, g_1 u), d(f x_{2n+1}, g_2 x_{2n+1}), \frac{1}{2}[d(f u, g_2 x_{2n+1}) + d(f x_{2n+1}, g_1 u)]\}
\begin{align*}
&= \max \{d(z, y_{2n}), d(z, g_1 u), d(y_{2n}, y_{2n+1}), \frac{1}{2}[d(z, y_{2n+1}) + d(y_{2n}, g_1 u)]\},
\end{align*}
\]
and this shows that \( \lim_{n \to \infty} N(u, x_{2n+1}) = d(z, g_1 u) \). Letting \( n \to \infty \) in (3.8), we conclude that
\[
F(d(g_1 u, z)) \leq \phi(F(d(g_1 u, z))),
\]
and so \( F(d(g_1 u, z)) = 0 \). Hence \( d(g_1 u, z) = 0 \) and therefore \( g_1 u = z \). Similarly, \( g_2 u = z \). Therefore \( f u = g_1 u = g_2 u = z \). Using condition (a) we conclude that \( g_1 z = f z = g_2 z \). Now we prove that \( z \) is a common fixed point for \( f, g_1 \) and \( g_2 \).
For all \( n \in \mathbb{N} \),
\[
F(d(g_1 z, y_{2n+1})) = F(d(g_1 z, g_2 x_{2n+1})) \leq \phi(F(N(z, x_{2n+1}))),
\]
(3.9)
where
\[
N(z, x_{2n+1}) = \max \{ d(fz, fx_{2n+1}), d(fz, g_1 z), d(fx_{2n+1}, g_2 x_{2n+1}),
\]
\[
\frac{1}{2} [d(fz, g_2 x_{2n+1}) + d(fx_{2n+1}, g_1 z)] \}
\]
= \max \{ d(g_1 z, y_{2n}), 0, d(y_{2n}, y_{2n+1}),
\]
\[
\frac{1}{2} [d(g_1 z, y_{2n+1}) + d(y_{2n}, g_1 z)] \},
\]
and this shows that \( \lim_{n \to \infty} N(z, x_{2n+1}) = d(z, g_1 z) \). Letting \( n \to \infty \) in (3.9), we get
\[
F(d(g_1 z, z)) \leq \phi(F(d(g_1 z, z))),
\]
This shows that \( F(d(g_1 z, z)) = 0 \) and so \( d(g_1 z, z) = 0 \). Hence \( g_1 z = z \). Therefore \( f(z) = g_1(z) = g_2(z) = z \). Unicity of the common fixed point follows from (c). □ □

**Theorem 3.2.** Let \( f, g_1 \) and \( g_2 \) be three self-mappings of complete metric space \( (X, d) \) verifying the conditions (b) and (c) of Theorem 3.1, where \( \phi, F \in \Psi \) and where \( \phi(t) < t \) for all \( t > 0 \). Assume that \( f \) is a continuous function of \( X \). If for all \( x \in X \),
\[
d(fg_1 x, g_1 f x) \leq d(fx, g_1 x), \quad i = 1, 2.
\]
(3.10)
Then \( f, g_1, g_2 \) have a unique common fixed point.

**Proof.** Following the proof of Theorem 3.1 we may conclude that \( \{y_n\} \) is Cauchy sequence converging to some \( z \) in \( X \) and
\[
z = \lim_{n \to \infty} y_n = \lim_{n \to \infty} g_1 x_{2n} = \lim_{n \to \infty} g_2 x_{2n+1} = \lim_{n \to \infty} f x_n.
\]
Since \( f \) is continuous, \( f y_n \) converges \( f z \). Using (3.10) and triangular inequality for all \( n \in \mathbb{N} \)
\[
d(g_1 y_{2n+1}, f z) \leq d(g_1 y_{2n+1}, f y_{2n+2}) + d(f y_{2n+2}, f z)
\]
\[
= d(g_1 f x_{2n+2}, g_1 x_{2n+2}) + d(f y_{2n+2}, f z)
\]
\[
\leq d(f x_{2n+2}, g_1 x_{2n+2}) + d(f y_{2n+2}, f z)
\]
\[
= d(y_{2n+2}, g_1 x_{2n+2}) + d(f y_{2n+2}, f z).
\]
This shows that \( \lim_{n \to \infty} d(g_1 y_{2n+1}, f z) = 0 \).
From (c)
\[
F(d(g_1 y_{2n+1}, g_2 z)) \leq \phi(F(N(y_{2n+1}, z))),
\]
(3.11)
where
\[ N(y_{2n+1}, z) = \max \{d(fy_{2n+1}, fz), d(fy_{2n+1}, g_1y_{2n+1}), d(fz, g_2z), \]
\[ \frac{1}{2}[d(fy_{2n+1}, g_2z) + d(fz, g_1y_{2n+1})] \}, \]
and this shows that \( \lim_{n \to \infty} N(y_{2n+1}, z) = d(fz, g_2z) \). Hence, from (3.11) we conclude that \( F(d(fz, g_2z)) \leq \phi(F(d(fz, g_2z))) \). So \( F(d(fz, g_2z)) = 0 \) and hence \( d(fz, g_2z) = 0 \). Therefore \( fz = g_2z \). Similarly \( fz = g_1z \). Therefore \( fz = g_1z = g_2z = t \).

Now we prove that \( t \) is common fixed point for \( f, g_1 \) and \( g_2 \). Using \( fz = g_1z = g_2z = t \) and condition (3.10), we have \( ft = g_1t = g_2t \). So from (c)
\[ F(d(g_1t, t)) = F(d(g_1g_1t, g_2t)) \leq \phi(F(N(g_1z, z))), \]
where
\[ N(g_1z, z) = \max \{d(ft, fz), d(ft, g_1t), d(fz, g_2z), \]
\[ \frac{1}{2}[d(ft, g_2z) + d(fz, g_1t)] \}, \]
\[ = d(ft, fz) = d(g_1t, t). \]
Hence \( F(d(g_1t, t)) \leq \phi(F(d(g_1t, t))) \) and so \( F(d(g_1t, t)) = 0 \). Therefore \( d(g_1t, t) = 0 \) and this shows that \( g_1t = t \). Hence \( ft = g_1t = g_2t = t \). By (c) we conclude that the common fixed point is unique and this completes the proof. □

4. Conclusion

We have extended Djoudi-Merghadi’s Theorem by replacing \( \int_0^t \phi(s)ds \) with \( F \) where \( F \in \Psi \). We have also extended Rhoades theorem by assuming \( f, g_1 \) and \( g_2 \) verifying the condition (c) in Theorem 3.1 and 3.2, and found the common fixed point of \( f, g_1 \) and \( g_2 \). For the future directions of research we can consider three maps \( f, g_1 \) and \( g_2 \) in Theorem 2.1 and Theorem 2.4.

References