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Approximately n-order linear differential equations

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Abstract

We prove the generalized Hyers–Ulam stability of n-th order linear differential equation of the form $y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_{n-1}(x)y'+p_n(x)y = f(x)$, with condition that there exists a non–zero solution of corresponding homogeneous equation. Our main results extend and improve the corresponding results obtained by many authors.

Keywords: Generalized Hyers–Ulam stability; Linear differential equation; homogeneous equation.

2010 MSC: Primary 47H10; Secondary 47H09.

1. Introduction and preliminaries

The stability problem of functional equations started with the question concerning stability of group homomorphisms proposed by S.M. Ulam [14] during a talk before a Mathematical Colloquium at the University of Wisconsin, Madison. In 1941, D. H. Hyers [5] gave a partial solution of $Ulam's$ problem for the case of approximate additive mappings in the context of Banach spaces. In 1978, Th. M. Rassias [12] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences $\|f(x+y)-f(x)-f(y)\| \le \epsilon(\|x\|^p+\|y\|^p), (\epsilon > 0, p \in [0,1))$. This phenomenon of stability that was introduced by Th. M. Rassias [12] is called the Hyers–Ulam–Rassias stability (or the generalized Hyers–Ulam stability).

Let X be a normed space over a scalar field $\mathbb{K}$ and let I be an open interval. Assume that for any function $f : I \to X$ satisfying the differential inequality

$$\|a_n(t)y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+...+a_1(t)y'(t)+a_0(t)y(t)+h(t)\| \le \epsilon$$

for all $t \in I$ and for some $\epsilon \ge 0$, there exists a function $f_0 : I \to X$ satisfying

$$a_n(t)y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+...+a_1(t)y'(t)+a_0(t)y(t)+h(t) = 0$$

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and

$$\|f(t) - f_0(t)\| \leq K(\epsilon)$$

for all $t \in I$, here $K(t)$ is an expression for ϵ with $lim_{\epsilon \to 0} K(\epsilon) = 0$. Then, we say that the above differential equation has the Hyers–Ulam stability.

If the above statement is also true when we replace ϵ and $K(\epsilon)$ by $\varphi(t)$ and $\phi(t)$, where $\varphi, \phi : I \to [0, \infty)$ are functions not depending on f and f_0 explicitly, then we say that the corresponding differential equation has the Hyers–Ulam–Rassias stability (or the generalized Hyers–Ulam stability).

The Hyers–Ulam stability of differential equation $y' = y$ was first investigated by Alsina and Ger [2]. This result has been generalized by Takahasi et al. [13] for the Banach space valued differential equation $y' = \lambda y$. In [10], Miura et al. proved the Hyers–Ulam–Rassias stability of linear differential of first order, $y' + g(t)y(t) = 0$, where $g(t)$ is a continuous function, while the author [6] proved the Hyers–Ulam–Rassias stability of linear differential equation of the form $c(t)y'(t) = y(t)$. Soon-Mo Jung [7] proved the Hyers–Ulam–Rassias stability of linear differential equation of first order of the form $y'(t)+g(t)y(t)+h(t) = 0$. We refer the interested readers for more information on such problems to the papers [1, 3, 4, 8, 9, 11] and [15].

In this paper, we investigate the generalized Hyers–Ulam stability of differential equations of the form

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x). \tag{1.1}$$

From now on, we assume that X is a complex Banach space and $I = (a, b)$ is an arbitrary interval, where $a, b \in \mathbb{R} \bigcup \{\pm\infty\}$ are arbitrarily given with $a < b$, and $y_1 : I \to X$ is a non–zero solution of corresponding homogeneous equation of (1.1), which

$$y_1^{(n)} + p_1(x)y_1^{(n-1)} + \cdots + p_{n-1}(x)y_!' + p_n(x)y_1 = 0. \tag{1.2}$$

2. Main results

Using the induction method, we are going to investigate the stability of n-th order linear differential equations. For the sake of convenience, all the integrals and derivations will be viewed as existing.

Theorem 2.1. *Assume that $p_1, p_2, \cdots, p_n : I \to \mathbb{C}$ and $f : I \to X$ are continuous functions and $y_1 : I \to X$ is a non–zero n–times continuously differentiable function satisfies the differential equation* (1.2). *If an n–times continuously differentiable function $y : I \to X$ satisfies*

$$\|y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y - f(x)\| \leq \varphi(x) \tag{2.1}$$

for all $x \in I$, where $\varphi : I \to (0, \infty)$ is a continuous function, then differential equation (1.1) *has the generalized Hyers–Ulam stability.*

Proof .

For $k = 1$, see [7]. We assume that the linear differential equation of order k, with condition that there exists a non–zero solution of corresponding homogeneous equation, satisfying the generalized Hyers–Ulam stability for $1 \leq k < n$. We will show that the linear differential equation of order n, with condition that there exists a non–zero solution of corresponding homogeneous equation, satisfies the generalized Hyers–Ulam stability.

Let $k = n$ and let

$$v(x) = \frac{y(x)}{y_1(x)} \tag{2.2}$$

for all $x \in I$. It follows from (1.2),(2.1) and (2.2) that

$$\begin{aligned}
&\|y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_{n-1}(x)y'(x) + p_n(x)y(x) - f(x)\| \\
&= \| \left(y_1(x)v(x)\right)^{(n)} + p_1(x)\left(y_1(x)v(x)\right)^{(n-1)} + \ldots + p_n(x)\left(y_1(x)v(x)\right) - f(x)\| \\
&= \|(v(x))^{(n)}y_1(x) + (v(x))^{(n-1)}\left(n(y_1(x))' + p_1(x)y_1(x)\right) \\
&+ (v(x))^{(n-2)}\left(n(y_1(x))'' + (n-1)p_1(x)(y_1(x))' + p_2(x)y_1(x)\right) \\
&+ (v(x))^{(n-3)}\big(n(y_1(x))^{(3)} + (n-1)p_1(x)(y_1(x))'' + (n-2)p_2(x)(y_1(x))' + p_3(x)y_1(x)\big) \\
&+ (v(x))'\big(n(y_1(x))^{(n-1)} + (n-1)p_1(x)(y_1(x))^{(n-2)} + \cdots + p_1(x)y_1(x)\big) \\
&+ \big((y_1(x))^{(n)} + p_1(x)(y_1(x))^{(n-1)} + p_2(x)(y_1(x))^{(n-2)} + \cdots + p_n(x)y_1(x)\big) - f(x)\| \\
&\le \varphi(x).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\|(v(x))^{(n)} + (v(x))^{(n-1)}\left(n\frac{(y_1(x))'}{y_1(x)} + p_1(x)\right) \\
&+ (v(x))^{(n-2)}\left(n\frac{(y_1(x))''}{y_1(x)} + (n-1)p_1(x)\frac{(y_1(x))'}{y_1(x)} + p_2(x)\right) \\
&+ (v(x))^{(n-3)}\left(n\frac{(y_1(x))^{(3)}}{y_1(x)} + (n-1)p_1(x)\frac{(y_1(x))''}{y_1(x)} + (n-2)p_2(x)\frac{(y_1(x))'}{y_1(x)} + p_3(x)\right) + \cdots \\
&+ (v(x))'\left(n\frac{(y_1(x))^{(n-1)}}{y_1(x)} + (n-1)p_1(x)\frac{(y_1(x))^{(n-2)}}{y_1(x)} + \cdots + p_1(x)\right) - \frac{f(x)}{y_1(x)}\| \le \frac{\varphi(x)}{\|y_1(x)\|}.
\end{aligned} \tag{2.3}$$

We suppose that

$$(v(x))' = w(x) \tag{2.4}$$

for all $x \in I$. It follows from (2.3) and (2.4) that

$$\begin{aligned}
&\|(w(x))^{(n-1)} + (w(x))^{(n-2)}\left(n\frac{(y_1(x))'}{y_1(x)} + p_1(x)\right) \\
&+ (w(x))^{(n-3)}\left(n\frac{(y_1(x))''}{y_1(x)} + (n-1)p_1(x)\frac{(y_1(x))'}{(y_1(x))} + p_2(x)\right) \\
&+ (w(x))^{(n-4)}\left(n\frac{(y_1(x))^{(3)}}{(y_1(x)} + (n-1)p_1(x)\frac{(y_1(x))''}{y_1(x)} + (n-2)p_2(x)\frac{(y_1(x))'}{(y_1(x)} + p_3(x)\right) + \cdots \\
&+ w(x)\left(n\frac{(y_1(x))^{(n-1)}}{(y_1(x)} + (n-1)p_1(x)\frac{(y_1(x))^{(n-2)}}{(y_1(x)} + \cdots + p_1(x)\right) - f(x)\| \le \frac{\varphi(x)}{\|y_1(x)\|}
\end{aligned} \tag{2.5}$$

we define a $n-1$ order differential equation of the form

$$\begin{aligned}
&(y(x))^{(n-1)} + (y(x))^{(n-2)}\left(n\frac{(y_1(x))'}{y_1(x)} + p_1(x)\right) \\
&+ (y(x))^{(n-3)}\left(n\frac{(y_1(x))''}{y_1(x)} + (n-1)p_1(x)\frac{(y_1(x))'}{(y_1(x))} + p_2(x)\right) \\
&+ (y(x))^{(n-4)}\left(n\frac{(y_1(x))^{(3)}}{(y_1(x)} + (n-1)p_1(x)\frac{(y_1(x))''}{y_1(x)} + (n-2)p_2(x)\frac{(y_1(x))'}{(y_1(x)} + p_3(x)\right) + \cdots \\
&+ (y(x))\left(n\frac{(y_1(x))^{(n-1)}}{(y_1(x)} + (n-1)p_1(x)\frac{(y_1(x))^{(n-2)}}{(y_1(x)} + \cdots + p_1(x)\right) = \frac{f(x)}{y_1(x)}.
\end{aligned} \tag{2.6}$$

It follows from (1.1), (2.6) and replacing $y(x)$ by $y_1(x)$ that

$$
\begin{aligned}
&(y_1(x))\cdot((y_1(x))^{(n-1)}+(y_1(x))^{(n-2)}\left(n\frac{(y_1(x))'}{y_1(x)}+p_1(x)\right)\\
&+(y_1(x))^{(n-3)}\left(n\frac{(y_1(x))''}{y_1(x)}+(n-1)p_1(x)\frac{(y_1(x))'}{(y_1(x))}+p_2(x)\right)\\
&+(y_1(x))^{(n-4)}\left(n\frac{(y_1(x))^{(3)}}{(y_1(x)}+(n-1)p_1(x)\frac{(y_1(x))''}{y_1(x)}+(n-2)p_2(x)\frac{(y_1(x))'}{(y_1(x)}+p_3(x)\right)+\cdots\\
&+y_1(x))\left(n\frac{(y_1(x))^{(n-1)}}{(y_1(x)}+(n-1)p_1(x)\frac{(y_1(x))^{(n-2)}}{(y_1(x)}+\cdots+p_1(x)\right))\\
&=(y_1(x))^{(n)}+p_1(x)(y_1(x))^{(n-1)}+\cdots+p_n(x)(y_1(x))=0
\end{aligned}
$$

according to assumption, $y_1: I \to X$ is a non–zero function. Hence, it follows that

$$
\begin{aligned}
&(y_1(x))^{(n-1)}+(y_1(x))^{(n-2)}\left(n\frac{(y_1(x))'}{y_1(x)}+p_1(x)\right)\\
&+(y_1(x))^{(n-3)}\left(n\frac{(y_1(x))''}{y_1(x)}+(n-1)p_1(x)\frac{(y_1(x))'}{(y_1(x))}+p_2(x)\right)\\
&+(y_1(x))^{(n-4)}\left(n\frac{(y_1(x))^{(3)}}{(y_1(x)}+(n-1)p_1(x)\frac{(y_1(x))''}{y_1(x)}+(n-2)p_2(x)\frac{(y_1(x))'}{(y_1(x)}+p_3(x)\right)+\cdots\\
&+y_1(x))\left(n\frac{(y_1(x))^{(n-1)}}{(y_1(x)}+(n-1)p_1(x)\frac{(y_1(x))^{(n-2)}}{(y_1(x)}+\cdots+p_1(x)\right)=0.
\end{aligned}
\tag{2.7}
$$

So $y_1(x)$ is a non–zero solution of corresponding homogeneous equation of (2.6). Thus, it follows from assumption of induction and (2.5) that there exists $w_0(x): I \to X$ satisfying (2.6) and

$$\|w(x)-w_0(x)\| \leq \psi(x) \tag{2.8}$$

where $\psi: I \to (0,\infty)$ is a continuous function. For simplicity, we use the following notation:

$$z(x) := \left(\frac{y(x)}{(y_1(x)}\right) - \int_a^x w_0(t)dt$$

for each $x \in I$. By making use of this notation and by (2.8), we get

$$
\begin{aligned}
\|z(x)-z(l)\| &= \|\left(\frac{y(x)}{(y_1(x)}\right)-\int_a^x w_0(t)dt-\left(\frac{y(l)}{(y_1(l)}\right)-\int_a^l w_0(t)dt\|\\
&= \|\int_l^x dt\left(\left(\frac{y(t)}{(y_1(t)}\right)-\int_a^t w_0(u)du\right)\| = \|\int_l^x\left(\left(\frac{y(t)}{(y_1(t)}\right)'-w_0(t)\right)dt\|\\
&= \|\int_l^x((v(t)')-w_0(t))\,dt\| \leq \int_l^x \|w(t)-w_0(t)\|dt \leq \int_l^x \psi(t)dt
\end{aligned}
\tag{2.9}
$$

for all $l, x \in I$. Since $\psi(x)$ is integrable on I, we may select $l_0 \in I$, for any given $\epsilon > 0$, such that $l, x \geq l_0$ implies that $\|z(x)-z(l)\| < \epsilon$. That is, $\{z(l)\}_{l\in I}$ is a Cauchy net. By completeness of X, there exists an $x_0 \in X$ such that $z(l)$ converges to x_0 as $l \to b$. It follows from (2.9) and the above argument that for any $x \in I$,

$$
\begin{aligned}
\|y(x)-y_1(x)\left(x_0+\textstyle\int_a^x w_0(t)dt\right)\| &= \|y_1(x)\left(z(x)+\textstyle\int_a^x w_0(t)dt\right)-y_1(x)\left(x_0+\textstyle\int_a^x w_0(t)dt\right)\|\\
&= \|y_1(x)\left(z(x)-x_0\right)\| \leq \|y_1(x)\|.\left(\|z(x)-z(l)\|+\|z(l)-x_0\|\right)
\end{aligned}
$$

$$\leq \|y_1(x)\|.\left(|\int_l^x \psi(t)dt| + \|z(l) - x_0\|\right)\| \to \|y_1(x)\|.|\int_x^b \psi(t)dt| \tag{2.10}$$

as $l \to b$. Moreover, $y_1(x)\left(x_0 + \int_a^x w_0(t)dt\right)$ is a solution of (1.1). Now, we prove the uniqueness property of x_0. Assume that $x_1 \in X$ satisfies the inequality (2.10). Then, we have

$\|y(x) - y_1(x)\left(x_0 + \int_a^x w_0(t)dt\right) - y(x) + y_1(x)\left(x_1 + \int_a^x w_0(t)dt\right)\|$

$\leq \|y_1(x)\|.\|x_0 - x_1\| \leq 2.\|y_1(x)\|.|\int_x^b \psi(t)dt| \to 0$

as $s \to b$. It follows that $x_1 = x_0$. □

Remark 2.2. If we replace $\mathbb{C}$ by $\mathbb{R}$ in the proof of Theorem 2.1 and we assume that $p_1, p_2, \cdots, p_n$ are real–valued continuous functions, then we can see that Theorem 2.1 is true for a real Banach space X. Hence, all n-th order linear differential equations have the generalized Hyers–Ulam stability with condition that there exist a solution of corresponding homogeneous equation or the general solution in the ordinary differential equations.

Remark 2.3. Linear differential equations of n-th order with constant coefficients have the generalized Hyers–Ulam stability with condition that there exist a solution of corresponding homogeneous equation in $\mathbb{C}$. That is, all linear differential equations with constant coefficients that are solved in ordinary differential equations satisfy the generalized Hyers–Ulam stability.

References

[1] M.R. Abdollahpour and A. Najati, *Stability of linear differential equations of third order*, Appl. Math. Lett., In Press, DOI: 10.1016/j.aml.2011.04.043.

[2] C. Alsina and R. Ger, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl. 2 (1998) 373–380.

[3] D.S. Cimpean and D. Popa, *On the stability of the linear differential equation of higher order with constant coefficients*, Appl. Math. Comput. 217 (2010) 4141–4146.

[4] D.S. Cimpean and D. Popa, *Hyers–Ulam stability of Euler's equation*, Appl. Math. Lett. 24 (2011) 1539–1543.

[5] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. 27 (1941) 222–224.

[6] S.-M. Jung, *stability of linear differential of first order*, Appl. Math. Lett. 17(2004) 1135–1140.

[7] S.-M. Jung, *Hyers–Ulam stability of linear differential of first order, II*, Appl. Math. Lett. 19 (2006) 854–858.

[8] S.-M. Jung, *Hyers–Ulam stability of linear partial differential equations of first order*, Appl. Math. Lett. 22 (2009) 70–74.

[9] Y. Li and Y. Shen, *Hyers–Ulam stability of linear differential equations of second order*, Appl. Math. Lett. 23 (2010) 306–309.

[10] T. Miura, S. Miyajima and S.-E. Takahasi, *A characterization of Hyers–Ulam stability of first order linear differential operators*, J. Math. Anal. Appl. 286 (2003) 136–146.

[11] D. Popa and I. Rasa, *On the Hyers–Ulam stability of the linear differential equation*, J. Math. Anal. Appl. 381 (2011) 530–537.

[12] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72 (1978) 297–300.

[13] S.-E. Takahasi, T. Miura and S. Miyajima, *On the Hyers–Ulam stability of Banach space–valued differential equation* $y' = \lambda y$, Bull. Korean Math. Soc. 39 (2002) 309–315.

[14] S.M. Ulam, *Problems in Modern Mathematics, Chapter VI*, Science ed. Wiley, New York, 1940.

[15] G. Wang, M. Zhou and L. Sun, *Hyers–Ulam stability of linear differential equations of first order*, Appl. Math. Lett. 21 (2008) 1024–1028.