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Some common fixed point theorems for Gregus type mappings

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Abstract

In this paper, sufficient conditions for the existence of common fixed points for a compatible pair of self maps of Gregus type in the framework of convex metric spaces have been obtained. Also, established the existence of common fixed points for a pair of compatible mappings of type (B) and consequently for compatible mappings of type (A). The proved results generalize and extend some of the well known results of the literature.

Keywords: Common fixed point; convex set; compatible maps; affine map.
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1. Introduction and preliminaries

Fixed point theory has gained impetus, due to its wide range of applicability, to resolve diverse problems emanating from the theory of nonlinear differential equations, theory of nonlinear integral equations, game theory, mathematical economics, control theory, and so forth. For example, in theoretical economics, such as general equilibrium theory, a situation arises where one needs to know whether the solution to a system of equations necessarily exists; or, more specifically, under what conditions will a solution necessarily exist. The mathematical analysis of this question usually relies on fixed point theorems. Hence finding necessary and sufficient conditions for the existence of fixed points is an interesting aspect.

In 1986, Fisher and Sessa [6], obtained the following common fixed point theorem by generalising a theorem of Gregus [8].

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Theorem 1.1. Let $T, I : K \to K$ be two weakly commuting mappings on a closed convex subset K of a Banach space X satisfying

$$\|Tx - Ty\| \le a\ \|Ix - Iy\| + (1-a)\ \max\{\|Ix - Tx\|, \|Iy - Ty\|\} \tag{1.1}$$

for all $x, y \in K$, where $0 < a < 1$. If I is linear, nonexpansive in K such that $T(K) \subseteq I(K)$, then T and I have a unique common fixed point in K.

If I is an identity map, we have an immediate generalization of the Gregus fixed point theorem. Mukherjee and Verma [14] generalized Theorem 1.1 by replacing the linearity of I with a more general condition that I is affine, while Jungck [10] generalised it further by replacing commutativity and nonexpansiveness assumptions with compatibility and continuity respectively. Thereafter, many results which are closely related to Gregus's Theorem have appeared in literature (see e.g. [1], [2], [3], [4], [5], [17]). The purpose of this paper is to find sufficient conditions for the existence of common fixed points for a compatible pair of self maps of Gregus type when the underlying spaces are convex metric spaces. Also, established the existence of common fixed points for a pair of compatible mappings of type (B) and consequently for compatible mappings of type (A). Our technique, which is originally due to Gregus [8], has been used by many authors. Our results extend and generalize some of the results of Ćirić [2], [3], [4], Diviccaro, Fisher and Sessa [5], Fisher and Sessa [6], Gregus [8], Jungck [10], Jungck and Sessa [12], Mukherjee and Verma [14], Olaleru [17], Sahab, Khan and Sessa [20], Singh [21], Smoluk [22], Subrahmanyam [23] and of few others.

For a metric space (X, d), a continuous mapping $W : X \times X \times [0,1] \to X$ is said to be (s.t.b.) a **convex structure** on X if for all $x, y \in X$ and $\lambda \in [0,1]$,

$$d(u, W(x, y, \lambda)) \le \lambda d(u, x) + (1-\lambda) d(u, y)$$

holds for all $u \in X$. The metric space (X, d) together with a convex structure is called a **convex metric space** [24].

A subset K of a convex metric space (X, d) is s.t.b. a **convex set** [24] if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0,1]$.

A normed linear space and each of its convex subsets are simple examples of convex metric spaces with W given by $W(x, y, \lambda) = \lambda x + (1-\lambda) y$ for $x, y \in X$ and $0 \le \lambda \le 1$.

Example 1.2. Let $X = \{(x_1, x_2) \in R^2 : x_1, x_2 > 0\}$. For $x = (x_1, x_2)$, $y = (y_1, y_2)$ in X, and $\alpha \in [0,1]$, define a mapping $W : X \times X \times [0,1] \to X$ by $W(x, y, \alpha) = (\alpha x_1 + (1-\alpha) y_1, \frac{\alpha x_1 x_2 + (1-\alpha) y_1 y_2}{\alpha x_1 + (1-\alpha) y_1})$, and a metric $d : X \times X \to [0, \infty)$ by $d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|$. Then (X, d) is a convex metric space but not a normed linear space.

For more examples of convex metric spaces which are not normed linear spaces, we refer to [7], [24].

For a non-empty subset M of a metric space (X, d) and $x \in X$, an element $y \in M$ is s.t.b. a **best approximant** to x or a **best M-approximant** to x if $d(x, y) = d(x, M) \equiv \inf\{d(x, y) : y \in M\}$. The set of all such $y \in M$ is denoted by $P_M(x)$.

For a convex subset M of a convex metric space (X, d), a mapping $g : M \to X$ is s.t.b. **affine** if for all $x, y \in M$, $g(W(x, y, \lambda)) = W(gx, gy, \lambda)$ for all $\lambda \in [0,1]$. g is s.t.b. **affine with respect to** $p \in M$ if $g(W(x, p, \lambda)) = W(gx, gp, \lambda)$ for all $x \in M$ and $\lambda \in [0,1]$.

Let M a nonempty subset of a metric space (X, d), a point $x \in M$ is a **common fixed (coincidence) point** of S and T if $x = Sx = Tx (Sx = Tx)$. The set of fixed points (respectively, coincidence points) of S and T is denoted by $F(S, T)$ (respectively, $C(S, T)$). The mappings $T, S : M \to M$ are s.t.b.

i) **commuting** on M if $STx = TSx$ for all $x \in M$;

ii) **weakly commuting** on M if $d(TSx, STx) \leq d(Tx, Sx)$ for all $x \in M$.

iii) **compatible**[9] if $\lim d(TSx_n, STx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim Tx_n = \lim Sx_n = t$ for some t in M.

v) **reciprocal continuous** [18] if $\lim_{n\to\infty} TSx_n = Tt$ and $\lim_{n\to\infty} STx_n = St$ whenever $\{x_n\}$ is a sequence in M such that
$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in M$.

vi) **compatible mappings of type (A)** [11], if

$$\lim_{n\to\infty} d(TSx_n, SSx_n) = 0$$

and

$$\lim_{n\to\infty} d(STx_n, TTx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in M such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in M$.

vii) **compatible mappings of type (B)** [19], if

$$\lim_{n\to\infty} d(STx_n, TTx_n) \leq \frac{1}{2}[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, SSx_n)]$$

and

$$\lim_{n\to\infty} d(TSx_n, SSx_n) \leq \frac{1}{2}[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, TTx_n)],$$

whenever $\{x_n\}$ is a sequence in M such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in M$.

Every weakly commuting pair of maps is compatible but the converse is not true (see [9]). Weakly commuting maps are of compatible of type (A) but the converse is not true (see [11]). However, compatible maps and compatible maps of type (A) are independent (see [11], [13]). Every compatible mappings of type (A) are compatible mappings of type (B) but the converse need not be true (see [19]). Clearly, every continuous pair of self maps is reciprocal continuous, but its converse need not to be true (see [18]).

2. Main Results

We begin the section with a following result which generalizes and extends the corresponding results of [5], [6] and [17].

Proposition 2.1. *Let M be a closed convex subset of a complete convex metric space (X, d), and f, $T : M \to M$ self mappings. Suppose that f, T satisfies*

$$\begin{aligned} d(Tx, Ty) \leq\ & a\ d(fx, fy) + (1-a)\ \max\{d(fx, Tx), d(fy, Ty), b\ d(fx, Ty), \\ & c\ d(fy, Tx)\} \end{aligned} \tag{2.1}$$

for all $x, y \in M$, where $0 < a < 1$, $0 \leq b < 1$, and $0 \leq c \leq \frac{1}{2}$. Further, if f and T are compatible on M, f is continuous then $Tw = fw$ for some $w \in M$ if and only if $A = \cap\{\overline{TK_n} : n \in \mathbb{N}\} \neq \emptyset$, where $K_n = \{x \in M : d(fx, Tx) \leq \frac{1}{n}\}$.

Proof . Suppose that $Tw = fw$ for some $w \in M$. Then $w \in K_n$ for all n and thus $Tw \in TK_n \subseteq \overline{TK_n}$ for all n. Hence $Tw \in A$ so that A is nonempty.

Assume that A is nonempty. If $w \in A$ for each n, then there is a $y_n \in TK_n$ such that $d(w, y_n) < \frac{1}{n}$. Hence for each n, there is an $x_n \in K_n$ such that $y_n = Tx_n$ and $d(w, Tx_n) < \frac{1}{n}$ for all n and so $Tx_n \to w$. Since $x_n \in K_n$, we have $d(fx_n, Tx_n) \leq \frac{1}{n}$. Thus $\lim fx_n = \lim Tx_n = w$. Since T and f are compatible mappings, we have

$$d(fTx_n, Tfx_n) \to 0.$$

Since f is continuous, we have

$$ffx_n, Tfx_n, fTx_n \to fw.$$

Now, consider

$$\begin{aligned} d(Tw, Tfx_n) \leq\ & a\ d(fw, ffx_n) + (1-a)\ \max\{d(fw, Tw), d(ffx_n, Tfx_n), \\ & b\ d(fw, Tfx_n), c\ d(ffx_n, Tw)\} \end{aligned}$$

Taking $n \to \infty$, we have

$$\begin{aligned} d(Tw, fw) &\leq (1-a)\ \max\{d(fw, Tw), 0, 0, 0\} \\ &= (1-a)d(fw, Tw) \end{aligned}$$

a contradiction. Thus $Tw = fw$. □

The following result, extends and generalizes the corresponding results of [2], [3], [4], [5], [6], [10], [14] and [17].

Theorem 2.2. *Let C be a closed convex subset of a complete convex metric space (X, d), and f, $T : C \to C$ self mappings and satisfies condition* (2.1)*. If f and T are compatible on C, $T(C) \subseteq f(C)$ and f is affine and continuous, then T and f have a unique common fixed point in C.*

Proof . Let $x = x_\circ$ be an arbitrary point of C. Let x_1, x_2, x_3 be points in C such that $fx_1 = Tx_0$, $fx_2 = Tx_1$, $fx_3 = Tx_2$, so that $Tx_{r-1} = fx_r$, for $r = 1, 2, 3$, as $T(C) \subseteq f(C)$. Consider

$$\begin{aligned} d(Tx_r, fx_r) &= d(Tx_r, Tx_{r-1}) \\ &\leq a\ d(fx_r, fx_{r-1}) + (1-a)\ \max\{d(fx_r, Tx_r), d(fx_{r-1}, Tx_{r-1}), \\ &\quad b\ d(fx_r, Tx_{r-1}), c\ d(fx_{r-1}, Tx_r)\} \\ &\leq a\ d(Tx_{r-1}, fx_{r-1}) + (1-a)\ \max\{d(fx_r, Tx_r), d(fx_{r-1}, Tx_{r-1}), \\ &\quad b\ d(fx_r, fx_r), c\ [d(fx_{r-1}, Tx_{r-1}) + d(Tx_{r-1}, Tx_r)]\} \\ &= a\ d(Tx_{r-1}, fx_{r-1}) + (1-a)\ \max\{d(fx_r, Tx_r), d(fx_{r-1}, Tx_{r-1}), \\ &\quad c\ [d(fx_{r-1}, Tx_{r-1}) + d(fx_r, Tx_r)]\}. \end{aligned}$$

If $d(Tx_{r-1}, fx_{r-1}) < d(Tx_r, fx_r)$, then we have

$$\begin{aligned} d(Tx_r, fx_r) &< a\ d(Tx_r, fx_r) + (1-a)\ \max\{d(fx_r, Tx_r), 2c\ \{d(fx_r, Tx_r)\}\} \\ &= d(Tx_r, fx_r), \end{aligned}$$

a contradiction. Thus, we have

$$d(Tx_r, fx_r) \leq d(Tx_{r-1}, fx_{r-1}) \leq d(Tx_0, fx_0).$$

So, it follows that

$$\begin{aligned} d(Tx_2, fx_1) &= d(Tx_2, Tx_0) \\ &\leq a\ d(fx_2, fx_0) + (1-a)\ \max\{d(fx_2, Tx_2), d(fx_0, Tx_0), \\ &\quad b\ d(fx_2, Tx_0), c\ d(fx_0, Tx_2)\} \\ &\leq a\ [d(fx_2, fx_1) + d(fx_1, fx_0)] + (1-a)\ \max\{d(fx_2, Tx_2), d(fx_0, Tx_0), \\ &\quad b\ d(fx_2, Tx_0), c\ [d(Tx_0, fx_0) + d(Tx_2, Tx_0)]\} \\ &\leq a\ [d(Tx_0, fx_0) + d(Tx_0, fx_0)] + (1-a)\ \max\{d(fx_0, Tx_0), d(fx_0, Tx_0), \\ &\quad b\ d(Tx_0, fx_0), c\ [d(Tx_0, fx_0) + d(Tx_2, Tx_0)]\}. \end{aligned}$$

Hence, we have

$$d(Tx_2, fx_1) = d(Tx_2, Tx_0) \leq \frac{1+3a}{1+a} d(Tx_0, fx_0).$$

Let $z = W(x_2, x_3, \frac{1}{2})$. Since C is convex and f is affine, $fz = fW(x_2, x_3, \frac{1}{2}) = W(fx_2, fx_3, \frac{1}{2}) = W(Tx_1, Tx_2, \frac{1}{2})$. Therefore,

$$\begin{aligned} d(fz, fx_1) &= d(W(Tx_1, Tx_2, \frac{1}{2}), Tx_0) \\ &\leq \frac{1}{2} d(Tx_1, Tx_0) + \frac{1}{2} d(Tx_2, Tx_0) \\ &\leq \frac{1}{2}[d(Tx_1, fx_1) + \frac{1+3a}{1+a} d(Tx_0, fx_0)] \\ &\leq \frac{1}{2}[d(Tx_0, fx_0) + \frac{1+3a}{1+a} d(Tx_0, fx_0)] \\ &= \frac{1+2a}{1+a} d(Tx_0, fx_0), \end{aligned}$$

$$d(fz, fx_2) = d(W(Tx_1, Tx_2, \frac{1}{2}), Tx_1) \leq \frac{1}{2} d(Tx_1, Tx_1) + \frac{1}{2} d(Tx_2, Tx_1) \leq \frac{1}{2} d(Tx_0, fx_0)$$

and

$$d(fz, fx_3) = d(W(Tx_1, Tx_2, \frac{1}{2}), Tx_2) \leq \frac{1}{2} d(Tx_1, Tx_2) + \frac{1}{2} d(Tx_2, Tx_2) \leq \frac{1}{2} d(Tx_0, fx_0).$$

Consider

$$\begin{aligned} & d(Tz, fz) \\ = \ & d\left(Tz, W\left(Tx_1, Tx_2, \frac{1}{2}\right)\right) \\ \leq \ & \frac{1}{2} d(Tz, Tx_1) + \frac{1}{2} d(Tz, Tx_2) \\ \leq \ & \frac{1}{2}[a\ d(fz, fx_1) + (1-a)\ \max\{d(fz, Tz), d(fx_1, Tx_1), b\ d(fz, Tx_1), \\ & c\ d(fx_1, Tz)\}] + \frac{1}{2}[a\ d(fz, fx_2) + (1-a)\ \max\{d(fz, Tz), d(fx_2, Tx_2), \\ & b\ d(fz, Tx_2), c\ d(fx_2, Tz)\}] \end{aligned}$$

and then

$$\begin{aligned}
& d(Tz, fz) \\
\leq\ & \frac{1}{2}[a\ d(fz, fx_1) + (1-a)\ \max\{d(fz, Tz), d(fx_2, Tx_2), b\ d(fz, Tx_1), \\
& c\ [d(fx_1, fz) + d(fz, Tz)]\}] + \frac{1}{2}[a\ d(fz, fx_2) + (1-a)\ \max\{d(fz, Tz), \\
& d(fx_2, Tx_2), b\ d(fz, Tx_2), c\ [d(fx_2, fz) + d(fz, Tz)]\}] \\
=\ & \frac{1}{2}[a\ d(fz, fx_1) + (1-a)\ \max\{d(fz, Tz), d(fx_2, Tx_2), b\ d(fz, fx_2), \\
& c\ [d(fx_1, fz) + d(fz, Tz)]\}] + \frac{1}{2}[a\ d(fz, fx_2) + (1-a)\ \max\{d(fz, Tz), \\
& d(fx_2, Tx_2), b\ d(fz, fx_3), c\ [d(fx_2, fz) + d(fz, Tz)]\}] \\
\leq\ & \frac{1}{2}\Big[\frac{a(2a+1)}{a+1} d(Tx_0, fx_0) + (1-a)\ \max\Big\{d(fz, Tz), d(Tx_0, fx_0), \frac{b}{2} d(Tx_0, fx_0), \\
& c\ \left[\frac{2a+1}{a+1} d(Tx_0, fx_0) + d(fz, Tz)\right]\Big\}\Big] + \frac{1}{2}[\frac{a}{2} d(Tx_0, fx_0) + (1-a)\ \max\{d(fz, Tz), \\
& d(Tx_0, fx_0), \frac{b}{2} d(Tx_0, fx_0), c\ [\frac{1}{2} d(Tx_0, fx_0) + d(fz, Tz)]\}] \\
\leq\ & \frac{5a^2+3a}{4(a+1)} d(Tx_0, fx_0) + (1-a)\ \max\Big\{d(fz, Tz), d(Tx_0, fx_0), \\
& c\ \left[\frac{2a+1}{a+1} d(Tx_0, fx_0) + d(fz, Tz)\right]\Big\}
\end{aligned}$$

Now the following three possible cases may arise.
Case 1. If $d(fz, Tz)$ is maximum, then we have

$$\begin{aligned}
d(Tz, fz) &\leq \frac{5a^2+3a}{4(a+1)} d(Tx_0, fx_0) + (1-a)\ d(fz, Tz) \\
&\leq \frac{5a+3}{4(a+1)} d(Tx_0, fx_0)
\end{aligned}$$

Case 2. If $d(Tx_0, fx_0)$ is maximum, then we have

$$\begin{aligned}
d(Tz, fz) &\leq \frac{5a^2+3a}{4(a+1)} d(Tx_0, fx_0) + (1-a)\ d(Tx_0, fx_0) \\
&\leq \frac{a^2+3a+4}{4(a+1)} d(Tx_0, fx_0)
\end{aligned}$$

Case 3. If $c\ [\frac{2a+1}{a+1} d(Tx_0, fx_0) + d(fz, Tz)]$ is maximum, then we have

$$\begin{aligned}
d(Tz, fz) &\leq \frac{5a^2+3a}{4(a+1)} d(Tx_0, fx_0) + (1-a)\ c\ [\frac{2a+1}{a+1} d(Tx_0, fx_0) + d(fz, Tz)] \\
&\leq \frac{5a^2+3a+4c-8a^2c+4ac}{4(a+1)(1-c+ac)} d(Tx_0, fx_0)
\end{aligned}$$

and so from the above cases we have

$$d(Tz, fz) \leq \lambda \; d(Tx_0, fx_0)$$

where

$$\lambda = \max\{\frac{5a+3}{4(a+1)}, \frac{a^2+3a+4}{4(a+1)}, \frac{5a^2+3a+4c-8a^2c+4ac}{4(a+1)(1-c+ac)}\} < 1.$$

We therefore have

$$\inf\{d(Tz, fz) : z = W(x_2, x_3, \frac{1}{2}\} \leq \lambda \; \inf\{d(Tx, fx) : x \in C\}$$

and since

$$\inf\{d(Tz, fz) : z = W(x_2, x_3, \frac{1}{2}\} \geq \inf\{d(Tx, fx) : x \in C\},$$

it follows that $\inf\{d(Tx, fx) : x \in C\} = 0$. Then the sets defined by $K_n = \{x \in C : d(Tx, fx) \leq \frac{1}{n}\}$, for $n = 1, 2, \ldots$ must be nonempty and $K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \supseteq \ldots$. Thus $cl(TK_n)$ is nonempty for $n = 1, 2 \ldots$ and $cl(TK_1) \supseteq cl(TK_2) \supseteq \ldots \supseteq cl(TK_n) \supseteq \ldots$. Further, for all $x, y \in K_n$,

$$\begin{aligned} d(Tx, Ty) &\leq a \; d(fx, fy) + (1-a) \; \max\{d(fx, Tx), d(fy, Ty), b \; d(fx, Ty), \\ &\quad c \; d(fy, Tx)\} \end{aligned}$$

Case 1. If $d(fx, Tx)$ or $d(fy, Ty)$ is maximum, then we have

$$\begin{aligned} d(Tx, Ty) &\leq a \; d(fx, fy) + (1-a) \; \frac{1}{n} \\ &\leq a\{d(fx, Tx) + d(Tx, Ty) + d(Ty, fy)\} + \frac{1-a}{n} \\ &= \frac{a+1}{n} + ad(Tx, Ty), \end{aligned}$$

which implies that $d(Tx, Ty) \leq \frac{a+1}{(1-a)n}$.

Case 2. If $b \; d(fx, Ty)$ is maximum, then we have

$$\begin{aligned} d(Tx, Ty) &\leq a \; d(fx, fy) + (1-a) \; b \; d(fx, Ty) \\ &\leq a \; d(fx, fy) + (1-a)b \; \{d(fx, fy) + d(fy, Ty)\} \\ &\leq (a + (1-a)b) \; d(fx, fy) + (1-a)b \; \frac{1}{n} \\ &\leq (a + (1-a)b)\{d(fx, Tx) + d(Tx, Ty) + d(Ty, fy)\} + \frac{(1-a)b}{n} \\ &\leq \frac{2a + 3(1-a)b}{n} + (a + (1-a)b) \; d(Tx, Ty), \end{aligned}$$

which implies that $d(Tx, Ty) < \frac{2a+3(1-a)b}{n(1-(a+(1-a)b))}$.

Case 3. If $c \; d(fy, Tx)$ is maximum, then we have

$$\begin{aligned} d(Tx, Ty) &\leq a \; d(fx, fy) + (1-a) \; c \; d(fy, Tx) \\ &\leq a \; d(fx, fy) + (1-a)c \; \{d(fx, fy) + d(fx, Tx)\} \\ &\leq (a + (1-a)c) \; d(fx, fy) + (1-a)c \; \frac{1}{n} \\ &\leq (a + (1-a)c)\{d(fx, Tx) + d(Tx, Ty) + d(Ty, fy)\} + \frac{(1-a)c}{n} \\ &\leq \frac{2a + 3(1-a)c}{n} + (a + (1-a)c) \; d(Tx, Ty), \end{aligned}$$

which implies that $d(Tx, Ty) < \frac{2a+3(1-a)c}{n(1-(a+(1-a)c))}$.

From the above three cases, we have

$$d(Tx, Ty) \leq \max\{\frac{a+1}{(1-a)n}, \frac{2a+3(1-a)b}{n(1-(a+(1-a)b))}, \frac{2a+3(1-a)c}{n(1-(a+(1-a)c))}\}.$$

Thus

$$\lim diam(TK_n) = \lim diam(cl(TK_n)) = 0,$$

i.e. $cl(TK_n)$ is a decreasing sequence of nonempty closed subsets of C whose sequence $\{diam(cl(TK_n))\}$ of the diameters converges to zero and by Cantor's Intersection Theorem, $A = \cap_{n=1}^{\infty}\{cl(TK_n) : n \in \mathbb{N}\}$ contains exactly one point w(say). Thus from Proposition 2.1, $fw = Tw$.

Now, consider

$$\begin{aligned} d(Tw, Tx_n) \leq\ & a\ d(fw, fx_n) + (1-a)\ \max\{d(fw, Tw), d(fx_n, Tx_n), b\ d(fw, Tx_n), \\ & c\ d(fx_n, Tw)\}. \end{aligned}$$

Taking $n \to \infty$, we have

$$\begin{aligned} d(Tw, w) &\leq a\ d(fw, w) + (1-a)\ \max\{d(fw, Tw), d(w, w), b\ d(fw, w), c\ d(w, Tw)\} \\ &= [a + (1-a)b]\ d(Tw, w) \\ &< d(Tw, w),\ (\text{since } [a + (1-a)b] < 1) \end{aligned}$$

a contradiction. Thus $Tw = w$, so that $Tw = w = fw$.

Now we prove the uniqueness. Suppose that v and w are common fixed points of T and f i.e., there exists $v \in C$ such that $Tv = v = fv$. Then

$$\begin{aligned} d(v, w) &= d(Tv, Tw) \\ &\leq a\ d(fv, fw) + (1-a)\ \max\{d(fv, Tv), d(fw, Tw), b\ d(fv, Tw), c\ d(fw, Tv)\} \\ &= [a + (1-a)b]\ d(w, v) \\ &< d(v, w). \end{aligned}$$

This gives that $v = w$. □

Corollary 2.3. Let C be a closed convex subset of a complete convex metric space (X, d), and f, $T : C \to C$ self mappings and satisfies

$$d(Tx, Ty) \leq a\ d(fx, fy) + (1-a)\ \max\{d(fx, Tx), d(fy, Ty)\}$$

for all $x, y \in C$, where $0 < a < 1$. If $T(C) \subseteq f(C)$ and f is affine and continuous, then T and f have a unique common fixed point in C.

Corollary 2.4. Let C be a closed convex subset of a complete convex metric space (X, d), and $T : C \to C$ self mappings and satisfies

$$d(Tx, Ty) \leq a\ d(x, y) + (1-a)\ \max\{d(x, Tx), d(y, Ty)\}$$

for all $x, y \in C$, where $0 < a < 1$. Then T has a unique fixed point in C.

Example 2.5. Let $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. Define self maps $T, f : X \to X$ by $Tx = \frac{2+x}{3}$ and $fx = \frac{3x-1}{2}$, $x \in X$. Clearly, f is continuous and affine, but neither nonexpansive nor linear. Here T and f are compatible mappings on X. Now for any $x, y \in X$, $d(Tx, Ty) = |\frac{x-y}{3}| = \frac{2}{9}d(fx, fy)$ and $\{1\}$ is a unique common fixed point of T and f.

By using Proposition 2.1 and Theorem 2.2, we have a following result which extends and generalizes the corresponding results of [6], [10], [14] and [17].

Theorem 2.6. Let C be a closed convex subset of a complete convex metric space (X, d), and f, $T : C \to C$ self mappings and satisfies condition (2.1). If f and T are compatible on C, $T(C) \subseteq f(C)$ and f is affine and continuous, then T and f have a unique common fixed point in C if and only if

$$A = \cap\{\overline{TK_n} : n \in \mathbb{N}\} \neq \emptyset,$$

where

$$K_n = \{x \in C : d(fx, Tx) \leq \frac{1}{n}\}.$$

In the following result, we prove a common fixed point theorem for a compatible pair of self maps which are reciprocal continuous.

Theorem 2.7. *Let C be a closed convex subset of a complete convex metric space (X, d), and f, $T : C \to C$ self mappings and satisfies condition (2.1). If f and T are compatible and reciprocal continuous on C, $T(C) \subseteq f(C)$ and f is affine, then T and f have a unique fixed point in C if and only if*

$$A = \cap\{\overline{TK_n} : n \in \mathbb{N}\} \neq \emptyset,$$

where

$$K_n = \{x \in C : d(fx, Tx) \leq \frac{1}{n}\}.$$

Proof . If w is a common fixed point of T and f, then $A \neq \emptyset$ follows from Proposition 2.1. Conversely, assume that A is nonempty. If $w \in A$ for each n, then there is a $y_n \in TK_n$ such that $d(w, y_n) < \frac{1}{n}$. Hence for each n, there is an $x_n \in K_n$ such that $y_n = Tx_n$ and $d(w, Tx_n) < \frac{1}{n}$ for all n and so $Tx_n \to w$. Since $x_n \in K_n$, we have $d(fx_n, Tx_n) \leq \frac{1}{n}$. Thus $\lim fx_n = \lim Tx_n = w$. Since T and f are reciprocally continuous mappings, $\lim Tfx_n = Tw$ and $\lim fTx_n = Tw$. Now since T and f are compatible mappings, $Tw = \lim Tfx_n = \lim fTx_n = fw$. Now proceeding as in Theorem 2.2, we can prove that w is a common fixed point of T and f. □

Example 2.8. Let $X = \mathbb{R}$ with usual metric $d(x, y) = |x - y|$. Define self maps $T, f : X \to X$ by

$$Tx = \begin{cases} \frac{1}{2} & , \ if \ x \leq 0 \ and \ x = \frac{5}{2} \\ \frac{1+x}{2} & , \ if \ x > 0 \ and \ x \neq \frac{5}{2} \end{cases}$$

and $fx = \frac{3x-1}{2}$, $x \in X$. Clearly, f is affine, but neither nonexpansive nor linear. Here T and f are reciprocal continuous and compatible mappings on X. Further, for any $x, y \in X$,

$$d(Tx, Ty) = |\frac{x - y}{2}| = \frac{1}{3} d(fx, fy)$$

and $\{1\}$ is a unique common fixed point of T and f.

The following result will be needed in the proof of our next theorem.

Proposition 2.9. *If M is a subset of a convex metric space (X, d), $u \in X$ and $y \in P_M(u)$, then the line segment $\{W(y, u, \lambda) : 0 < \lambda < 1\}$ and the set M are disjoint.*

Proof . Since $y \in P_M(u)$, consider

$$\begin{aligned} d(u, W(y, u, \lambda)) &\leq \lambda\, d(u, y) \\ &< d(u, M), \text{ for every } 0 < \lambda < 1. \end{aligned}$$

This implies that $W(y, u, \lambda) \notin M$ for any λ, $0 < \lambda < 1$. Therefore the line segment $\{W(y, u, \lambda) : 0 < \lambda < 1\}$ and the set M are disjoint. □

Theorem 2.10. *Let M be a subset of a complete convex metric space (X, d) and T, S are self mappings of M such that $u \in F(S) \cap F(T)$ for some $u \in X$ and $T(\partial M \cap M) \subseteq M$. Suppose that $P_M(u)$ is nonempty, closed and convex, S is affine, and continuous on $P_M(u)$ and $T(P_M(u)) \subseteq S(P_M(u))$. If (T, S) is compatible and satisfies*

$$d(Tx, Ty) \leq \begin{cases} d(Sx, Sy), & \text{if } y = u \\ a\ d(Sx, Sy) + (1-a)\ \max\{d(Sx, Tx), d(Sy, Ty), & \\ \quad b\ d(Sx, Ty), c\ d(Sy, Tx)\} & \text{if } y \in P_M(u), \end{cases} \tag{2.2}$$

then $P_M(u) \cap F(S) \cap F(T) \neq \emptyset$.

Proof . Let $x \in P_M(u)$. For any $\lambda \in (0, 1)$, we have

$$d(W(u, x, \lambda), u) \leq \lambda d(u, u) + (1-\lambda) d(x, u) = (1-\lambda) d(x, u) < dist(u, M).$$

It follows from Proposition 2.9 that the open line segment $\{W(u, x, \lambda) : 0 < \lambda < 1\}$ and the set M are disjoint. Thus x is not in the interior of M and so $x \in \partial M \cap M$. Since $T(\partial M \cap M) \subset M$, Tx must be in M. Also $Sx \in P_M(u)$, $u \in F(T) \cap F(S)$, and (T, S) satisfy (2.2), we have

$$d(Tx, u) = d(Tx, Tu) \leq d(Sx, Su) = d(Sx, u) \leq dist(u, M).$$

This implies that $Tx \in P_M(u)$. Moreover, $T(P_M(u)) \subseteq S(P_M(u))$. Hence the result follows from Theorem 2.2. □

Remark 2.11. Theorem 2.10 extends and generalizes the corresponding results of [12], [15], [16], [20], [21], [22] and [23].

We now prove the existence of common fixed points for a pair of compatible mappings of type (B), and obtain a result on the existence of common fixed point for a pair of compatible mappings of type (A). The following result will be used in the sequel which generalizes and extends the corresponding result of Pathak and Khan [19].

Proposition 2.12. *Let T and f be self-maps of a metric space (X, d) and are compatible mappings of type (B). Suppose that $\lim f x_n = \lim T x_n = t$, for some $t \in X$. If f is continuous at t, then $\lim TT x_n = ft$.*

Proof . Suppose that f is continuous at t. Since $\lim f x_n = \lim T x_n = t$ for some $t \in X$, we have $ffx_n, fTx_n \to ft$. Since f and T are compatible of type (B), we have

$$\begin{aligned} \lim d(ft, TTx_n) &= \lim d(fTx_n, TTx_n) \\ &\leq \frac{1}{2}[\lim d(fTx_n, ft) + \lim d(ft, ffx_n)] \\ &= d(ft, ft) \\ &= 0. \end{aligned}$$

Therefore, $\lim TTx_n = ft$. □

Proposition 2.1 remains true, if we replace compatible mappings by compatible mappings of type (B). The following result extends the corresponding result of [17].

Proposition 2.13. *Let M be a closed convex subset of a complete convex metric space (X, d), T and f are self-maps of M satisfying condition (2.1) and compatible mappings of type (B). If f is continuous then $Tw = fw$ for some $w \in X$ if and only if $A = \cap\{\overline{TK_n} : n \in \mathbb{N}\} \neq \emptyset$, where $K_n = \{x \in X : d(fx, Tx) \leq \frac{1}{n}\}$.*

Proof . Proceeding as in Proposition 2.1 and using Proposition 2.12, we get the result. □

Proceeding as Theorem 2.2 and using Proposition 2.13, we have the following result.

Theorem 2.14. *Let M be a closed convex subset of a complete convex metric space (X, d) and T and f are self-maps of M, satisfying (2.1) and are compatible mappings of type (B). If f is continuous and affine on M and $T(M) \subseteq f(M)$, then T and f have a unique common fixed point in M.*

Theorem 2.15. Let M be a closed convex subset of a complete convex metric space (X, d) and T and f are self-maps of M, satisfying (2.1) and are compatible mappings of type (B). If f is continuous and affine on M and $T(M) \subseteq f(M)$, then T and f have a unique common fixed point in M if and only if $A = \cap\{\overline{TK_n} : n \in \mathbb{N}\} \neq \emptyset$, where $K_n = \{x \in X : d(fx, Tx) \leq \frac{1}{n}\}$.

Since compatible mappings of type (A) implies compatible mappings of type (B), we have the following result.

Corollary 2.16. Let M be a closed convex subset of a complete convex metric space (X, d) and T and f are self-maps of M, satisfying (2.1) and are compatible mappings of type (A). If f is continuous and affine on M and $T(M) \subseteq f(M)$, then T and f have a unique common fixed point in M if and only if $A = \cap\{\overline{TK_n} : n \in \mathbb{N}\} \neq \emptyset$, where $K_n = \{x \in X : d(fx, Tx) \leq \frac{1}{n}\}$.

Remark 2.17. Theorems 2.14 and 2.15 generalize and extend the corresponding results of [8], [17] and [19].

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