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Random differential inequalities and comparison principles for nonlinear hybrid random differential equations

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Abstract

In this paper, some basic results concerning strict, nonstrict inequalities, local existence theorem and differential inequalities have been proved for an IVP of first order hybrid random differential equations with the linear perturbation of second type. A comparison theorem is proved and applied to prove the uniqueness of random solution for the considered perturbed random differential equation. Finally an existence of extremal random solution is obtained in between the given upper and lower random solutions.

Keywords: Random differential inequalities; existence theorem; comparison principle; extremal solutions.
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1. Introduction

Let $\mathbb{R}$ be the real line. Given a bounded interval $J = [t_0, t_0 + a)$ in $\mathbb{R}$ for some fixed $t_0, a \in \mathbb{R}$ with $a > 0$, let $C(J, \mathbb{R})$ denote the space of continuous real-valued functions defined on J. Let $(\Omega, \mathcal{A}, \mu)$ be a complete probability measure space and given a measurable function $x : \Omega \to C(J, \mathbb{R})$, consider the initial value problems random hybrid differential equation (in short HRDE) satisfying for $\omega \in \Omega$ a.e.,

$$\left.\begin{aligned} \frac{d}{dt}\big[x(t,\omega) - f(t, x(t,\omega), \omega)\big] &= g(t, x(t,\omega), \omega), \\ x(t_0, \omega) &= x_0(\omega) \in \mathbb{R}, \end{aligned}\right\} \tag{1.1}$$

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for all $t \in J$, where $f, g : \Omega \to C(J \times \mathbb{R}, \mathbb{R})$ and $x_0 : \Omega \to \mathbb{R}$ is measurable.

By a *random solution* of the HRDE (1.1) we mean a measurable function $x : \Omega \to C(J, \mathbb{R})$ satisfying for $\omega \in \Omega$,

(i) the function $t \mapsto x - f(t, x, \omega)$ is continuous for each $x \in \mathbb{R}$, and

(ii) x satisfies the equations in (1.1).

The importance of the investigations of nonlinear hybrid random differential equations lies in the fact that they include several dynamic systems as special cases [8, 23, 24]. The consideration of hybrid differential equations is implicit in the works of Krasnoselskii [20] and extensively treated in the several papers on hybrid differential equations with different perturbations. See Krasnoselskii [20], Dhage [7] and the references therein. This class of hybrid random differential equations includes the perturbations of original random differential equations in different ways [9, 10, 11, 12, 13, 14].

In this paper, we initiate the basic theory of hybrid random differential equations of mixed perturbations of second type involving two nonlinearities and prove the basic result such as differential inequalities, existence theorem and maximal and minimal random solutions etc. We claim that the results of this chapter are basic and important contribution to the theory of nonlinear ordinary random differential equations.

2. Strict and Nonstrict Random Inequalities

We need frequently the following hypothesis in what follows.

(A_0) The function $x \mapsto x - f(t, x, \omega)$ is increasing in $\mathbb{R}$ for all $t \in J$ and $\omega \in \Omega$.

We begin by proving the basic results dealing with hybrid random differential inequalities.

Theorem 2.1. *Assume that the hypothesis (A_0) holds. Suppose that there exist measurable functions $y, z : \Omega \to C(J, \mathbb{R})$ satisfying for $\omega \in \Omega$ a.e.,*

$$\frac{d}{dt}\big[y(t, \omega) - f(t, y(t, \omega), \omega)\big] \le g(t, y(t, \omega), \omega), \tag{2.1}$$

and

$$\frac{d}{dt}\big[z(t, \omega) - f(t, z(t, \omega), \omega)\big] \ge g(t, z(t, \omega), \tag{2.2}$$

for all $t \in J$, If one of the inequalities (2.1) and (2.2) *is strict and*

$$y(t_0, \omega) < z(t_0, \omega), \ \ \omega \in \Omega \quad a.e. \tag{2.3}$$

then

$$y(t, \omega) < z(t, \omega), \ \ \omega \in \Omega \ a.e. \tag{2.4}$$

for all $t \in J$.

Proof . Denote

$$\Theta = \{\omega \in \Omega \mid \text{ The inequality (2.1) is true } \},$$

$$\Delta = \{\omega \in \Omega \mid \text{ The inequality (2.2) is true } \},$$

and

$$\Gamma = \{\omega \in \Omega \mid \text{ The inequality (2.3) is true } \}.$$

Let

$$\Lambda = \Theta \cap \Gamma \cap \Delta.$$

Then,

$$\Lambda^c = (\Theta \cap \Gamma \cap \Delta)^c = \Theta^c \cup \Gamma^c \cup \Delta^c$$

and so that

$$\mu(\Lambda^c) = \mu(\Theta^c \cup \Gamma^c \cup \Delta^c) = 0.$$

Let $\omega \in \Lambda$ be arbitrary fixed element and suppose that the inequality (2.4) is false. Then the set P defined by

$$P = \{t \in J \mid y(t,\omega) \geq z(t,\omega)\} \tag{2.5}$$

is non-empty.

Denote $t_1 = \inf P$. Without loss of generality, we may assume that

$$y(t_1,\omega) = z(t_1,\omega) \text{ and } y(t,\omega) < z(t,\omega)$$

for all $t < t_1$.

Assume that

$$\frac{d}{dt}\big[z(t,\omega) - f(t, z(t,\omega), , \omega)\big] > g(t, z(t,\omega), \omega)$$

for $t \in J$.

Denote

$$Y(t,\omega) = y(t,\omega) - f(t, y(t,\omega), \omega)$$

and

$$Z(t,\omega) = z(t,\omega) - f(t, z(t,\omega), \omega)$$

for $t \in J$.

As hypothesis (A_0) holds, it follows from (2.5) that

$$Y(t_1,\omega) = Z(t_1,\omega) \quad \text{and} \quad Y(t,\omega) < Z(t,\omega) \tag{2.6}$$

for all $t_0 \leq t < t_1$. The above relation (2.6) further yields

$$\frac{Y(t_1+h,\omega) - Y(t_1,\omega)}{h} > \frac{Z(t_1+h,\omega) - Z(t_1,\omega)}{h}$$

for small $h < 0$. Taking the limit as $h \to 0$, we obtain

$$Y'(t_1,\omega) \geq Z'(t_1,\omega). \tag{2.7}$$

Hence, from (2.6) and (2.7), we get

$$g(t_1, y(t_1,\omega), \omega) \geq Y'(t_1,\omega) \geq Z'(t_1,\omega) > g(t_1, z(t_1,\omega), \omega)$$

for all $\omega \in \Lambda$. This is a contradiction and hence

$$y(t,\omega) < z(t,\omega)$$

for all $\omega \in \Lambda$. As a result, (2.4) is true and the proof of the theorem is complete. □

The next result is about the nonstrict inequality for the HRDE (1.1) on J which requires a one-sided Lipschitz condition.

Theorem 2.2. *Assume that the hypotheses of Theorem* 2.1 *hold. Suppose also that there exists a real number* $L > 0$ *satisfying for* $\omega \in \Omega$ *a.e.,*

$$\begin{aligned} g(t,&y(t,\omega),\omega) - g(t,z(t,\omega),\omega) \\ &\leq L \sup_{t_0 \leq s \leq t} \big[(y(s,\omega) - f(s,y(s,\omega),\omega)) - (z(s,\omega) - f(s,z(s,\omega),\omega))\big], \end{aligned} \tag{2.8}$$

whenever $y(s,\omega) \geq z(s,\omega)$, $t_0 \leq s \leq t$. *Then,*

$$y(t_0,\omega) \leq z(t_0,\omega), \ \ \omega \in \Omega \ a.e., \tag{2.9}$$

implies

$$y(t,\omega) \leq z(t,\omega), \ \ \omega \in \Omega \ a.e. \tag{2.10}$$

for all $t \in J$.

Proof . Define the sets Θ and Δ as in the proof of Theorem 2.1. Denote

$$\Pi = \{\omega \in \Omega \mid \text{ The inequality (2.8) is true } \}$$

and

$$\digamma = \{= \{\omega \in \Omega \mid \text{ The inequality (2.9) is true } \}.$$

Let $\Lambda = \Theta \cap \Delta \cap \Pi \cap \digamma$. Then, we have

$$\mu(\Lambda^c) = \mu((\Gamma \cap \Delta \cap \Pi \cap \digamma)^c) = \mu(\Gamma^c \cup \Delta^c \cup \Pi^c \cup \digamma^c) = 0.$$

Let $\omega \in \Lambda$ be arbitrary fixed element. Let $\epsilon > 0$ and let a real number $L > 0$ be given. Set

$$z_\epsilon(t,\omega) - f(t,z_\epsilon(t,\omega),\omega) = z(t,\omega) - f(t,x(t,\omega),\omega) + \epsilon e^{2L(t-t_0)} \tag{2.11}$$

so that

$$z_\epsilon(t,\omega) - f(t,z_\epsilon(t,\omega),\omega) > z(t,\omega) - f(t,x(t,\omega),\omega).$$

Define

$$Z_\epsilon(t,\omega) = z_\epsilon(t,\omega) - f(t,z_\epsilon(t,\omega),\omega)$$

and

$$Z(t,\omega) = z(t,\omega) - f(t,z(t,\omega),\omega)$$

for $t \in J$.

Now using the one-sided Lipschitz condition (2.8), we obtain

$$\begin{aligned} g(t,z_\epsilon(t,\omega),\omega) &- g(t,z(t,\omega),\omega) \\ &\leq L \sup_{t_0 \leq s \leq t} [Z_\epsilon(s,\omega) - Z(s,\omega)] \\ &= L\epsilon e^{2L(t-t_0)}. \end{aligned}$$

Now,

$$\begin{aligned} Z'_\epsilon(t,\omega) &= Z'(t,\omega) + 2L\epsilon e^{2L(t-t_0)} \\ &\geq g(t,z(t,\omega),\omega) + 2L\epsilon e^{2L(t-t_0)} \\ &\geq g(t,z_\epsilon(t,\omega)) + 2L\epsilon e^{2L(t-t_0)} - L\epsilon e^{2L(t-t_0)} \\ &= (g(t,z_\epsilon(t,\omega),\omega) + L\epsilon e^{2L(t-t_0)} \\ &> g(t,z_\epsilon(t,\omega),\omega) \end{aligned}$$

for all $t \in J$ and $\omega \in \Lambda$. Also, we have

$$Z_\epsilon(t_0,\omega) > Z(t_0,\omega) \geq Y(t_0,\omega)$$

for all $\omega \in \Lambda$.

Now we apply Theorem 2.1 with $z = z_\epsilon$ to yield

$$Y(t,\omega) < Z_\epsilon(t,\omega), \;\; \omega \in \Omega \text{ a.e.}$$

for all $t \in J$. On taking $\epsilon \to 0$ in the above inequality, we get

$$Y(t,\omega) \leq Z(t,\omega), \;\; \omega \in \Omega \text{ a.e.}$$

which further in view of hypothesis (A_0) implies that (2.10) holds on J. This completes the proof. □

Remark 2.3. The conclusion of Theorems 2.1 and 2.2 also remains true if we replace the derivative in the inequalities (2.1) and (2.2) by Dini-derivative D_- of the function

$$x(t,\omega) - f(t,x(t,\omega),\omega)$$

on the bounded interval J.

3. The Existence Result

In this section, we prove an existence result for the HRDE (1.1) on a closed and bounded interval $J = [t_0, t_0 + a]$ under mixed Lipschitz and compactness conditions on the nonlinearities involved in it. We place the HRDE (1.1) in the function space $C(J,\mathbb{R})$ of continuous real-valued functions defined on J. Define a supremum norm $\|\cdot\|$ in $C(J,\mathbb{R})$ defined by

$$\|x\| = \sup_{t\in J} |x(t)|.$$

Clearly $C(J,\mathbb{R})$ is a separable Banach space with respect to the above supremum norm. We prove the existence of random solutions for the HRDE (1.1) via the following hybrid random fixed point theorem in the Banach spaces due to Dhage [4]. Before stating the fixed point theorem, we give some preliminaries and definitions that will be used in what follows.

Definition 3.1. A mapping $T : \Omega \times E \to E$ is called a random mapping $T(\cdot, x) : \Omega \to E$ is measurable. A random mapping $T : \Omega \times E \to E$ is said to be continuous if and only if there exists a set $\mathcal{N}_T \subset \Omega$ such that the mapping $T(\omega,\cdot) : E \to E$ is continuous for each $\omega \in \mathcal{N}_T$ and $\mu(\mathcal{N}_T^c) = 0$, where $\mathcal{N}_T^c$ is a complement of $\mathcal{N}_T$ in Ω.

Definition 3.2. A mapping $\psi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is called a **dominating function** or, in short, **$\mathcal{D}$-function** if $\psi(\omega, r)$ is an upper semi-continuous and nondecreasing function in r satisfying $\psi(\omega,0) = 0$ for all $\omega \in \Omega$. A random mapping $Q : \Omega \times E \to E$ is called **$\mathcal{D}$-Lipschitz** if there is a $\mathcal{D}$-function $\psi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying for each $\omega \in \Omega$,

$$\|Q(\omega)\phi - Q(\omega)\xi\| \leq \psi_\omega(\|\phi - \xi\|) \tag{3.1}$$

for all $\phi, \xi \in E$. The function $\psi_\omega(r) = \psi(\omega, r)$ is called a $\mathcal{D}$-function of Q on E. If $\psi(r,\omega) = k(\omega)\,r$, $k > 0$, then Q is called **Lipschitz** with the Lipschitz constant $k(\omega)$. In particular, if $k(\omega) < 1$ for all $\omega \in \Omega$, then Q is called a **contraction** on X with the contraction constant $k(\omega)$. Further, if $\psi_\omega(r) < r$,$r > 0$ for all $\omega \in \Omega$, then Q is called **nonlinear random $\mathcal{D}$-contraction** and the function ψ_ω is called random $\mathcal{D}$-function of Q on X.

The details of different types of random contractions appear in the monographs of Dhage [4] and Granas and Dugundji [16]. There do exist random $\mathcal{D}$-functions and the commonly used $\mathcal{D}$-functions are $\psi_\omega(r) = k(\omega)\, r$ and $\psi_\omega(r) = \dfrac{r}{1+r}$, etc. These $\mathcal{D}$-functions have been widely used in the theory of nonlinear differential and integral equations for proving the existence results via fixed point methods.

Another notion that we need in the sequel is the following definition.

Definition 3.3. A random operator $Q : \Omega \times E \to E$ is called compact if $Q(\omega)(E)$ is a relatively compact subset of E for each $\omega \in \Omega$. Q is called totally bounded if for any bounded subset S of E, $Q(\omega)(S)$ is a relatively compact subset of E for each $\omega \in \Omega$. If Q is continuous and totally bounded, then it is called completely continuous random operator on E.

Theorem 3.4. *Suppose that S is a closed, convex and bounded subset of the separable Banach space E and let $A : \Omega \times E \to E$ and $B : \Omega \times S \to E$ be two continuous random operators satisfying for $\omega \in \Omega$ a.e.,*

(a) $A(\omega)$ is a nonlinear $\mathcal{D}$-contraction,

(b) $B(\omega)$ is compact and continuous, and

(c) $x = A(\omega)x + B(\omega)y$ for all $y \in S \Longrightarrow x \in S$.

Then the random equation $A(\omega)x + B(\omega)x = x$, $\omega \in \Omega$ a.e., has a random solution in S.

Proof . Let

$$\mathcal{N}_A = \{\omega \in \Omega \mid A(\omega, \cdot) \text{ is continuous on } E\},$$

$$\mathcal{N}_B = \{\omega \in \Omega \mid B(\omega, \cdot) \text{ is continuous on } E\},$$

$$\Gamma = \{\omega \in \Omega \mid A(\omega) \text{ is a nonlinear } \mathcal{D}\text{-contraction on } E\},$$

$$\Delta = \{\omega \in \Omega \mid B(\omega) \text{ is compact on } S\}$$

and

$$\Lambda = \{\omega \in \Omega \mid x = A(\omega)x + B(\omega)y \text{ for all } y \in S \Longrightarrow x \in S\}.$$

Take $\Sigma = \mathcal{N}_A \cap \mathcal{N}_B \cap \Gamma \cap \Delta \cap \Lambda$. Then,

$$\Sigma^c = (\mathcal{N}_A \cap \mathcal{N}_B \Gamma \cap \Delta \cap \Lambda)^c = \mathcal{N}_A^c \cup \mathcal{N}_B^c \cup \Gamma^c \cup \Delta^c \cup \Lambda^c.$$

Hence,

$$\mu(\Sigma^c) = \mu(\mathcal{N}_A^c \cup \mathcal{N}_B^c \cup \Gamma^c \cup \Delta^c \cup \Lambda^c) = 0.$$

Let $\omega \in \Sigma$ be fixed element. Then by a classical fixed point theorem of Dhage [6], the operator equation $A(\omega)x + B(\omega)x = x$ has a solution in S. Define a multi-valued map $F : \Sigma \to S$ by

$$F(\omega) = \big\{x \in S \mid A(\omega)x + B(\omega)x = x\big\}. \tag{3.2}$$

Clearly, $F(\omega)$ is non-empty and compact subset of S for each $\omega \in \Sigma$. See Dhage [6] and the references therein. We shall show that F is measurable on Σ. Now, for any closed subset C of S, denote

$$L(C) = \bigcap_{n=1}^{\infty} \bigcup_{x_i \in C_n} \left\{\omega \in \Sigma \,\middle|\, \|x_i - (A(\omega)x_i + B(\omega)x_i)\| < \frac{1}{n}\right\} \tag{3.3}$$

where, $C_n = \{x \in S \mid d(x, C) < \frac{1}{n}\}$ and $d(x, C) = \inf\{d(x, c) \mid c \in C\}$.

Obviously, $L(C)$ is measurable. Now proceeding with the arguments similar to that given in the proof of Theorem 2.1 of Itoh [19], it is proved that $F^{-1}(C) = L(C)$. Hence F is measurable on Σ. Since $F(\omega)$ is compact, it has closed values for each $\omega \in \Sigma$. Now an application of a selection theorem of Kuratowskii and Ryll-Nardzewskii [21] yields that F has a measurable selection ξ on Σ. This further implies that $A(\omega)\xi(\omega) + B(\omega)\xi(\omega) = \xi(\omega)$ for each $\omega \in \Sigma$. As a result, we have $A(\omega)\xi(\omega) + B(\omega)\xi(\omega) = \xi(\omega)$, $\omega \in \Omega$ a.e. and the proof of the theorem is complete. □

Before stating the needed hypotheses, we give a useful definition.

Definition 3.5. A mapping $f : J \times \mathbb{R} \times \Omega \to \mathbb{R}$ is said to be Carathéodory if

(i) the map $(t, x) \mapsto f(t, x, \omega)$ is jointly continuous for $\omega \in \Omega$, and

(ii) the map $\omega \mapsto f(t, x, \omega)$ is measurable for each $t \in J$ and $x \in \mathbb{R}$.

Furthermore, a function $f(t, x, \omega)$ is called $\mathcal{C}$-Carathéodory if

(iii) f is Carathéodory and there exists a continuous function $h : J \to \mathbb{R}$ such that

$$|f(t, x, \omega)| \leq h(t)$$

for all $x \in \mathbb{R}$ and $\omega \in \Omega$.

Definition 3.6. A function $f : J \times \mathbb{R} \times \Omega \to \mathbb{R}$ is said to be nonlinear $\mathcal{D}$-Lipschitz if there exist measurable functions $L, K : \Omega \to \mathbb{R}_+$ satisfying for $\omega \in \Omega$,

$$|f(t, x, \omega) - f(t, y, \omega)| \leq \frac{L(\omega)\,|x - y|}{K(\omega) + |x - y|},$$

for all $t \in J$ and $x, y \in \mathbb{R}$. Moreover, $L(\omega) \leq K(\omega)$ for $\omega \in \Omega$.

The following lemma is useful in the sequel.

Lemma 3.7. *Assume that the function f is Carathéodory on $J \times \mathbb{R} \times \Omega$ and hypothesis (A_0) holds. Then for any continuous function $h : J \to \mathbb{R}$, the measurable function $x : \Omega \to C(J, \mathbb{R})$ is a random solution of the HRDE satisfying for each $\omega \in \Omega$,*

$$\left.\begin{aligned} \frac{d}{dt}\Big[x(t, \omega) - f(t, x(t, \omega), \omega)\Big] &= h(t), \quad t \in J \\ x(0, \omega) &= x_0(\omega) \in \mathbb{R} \end{aligned}\right\} \tag{3.4}$$

if and only if x satisfies the hybrid random integral equation (HRIE)

$$x(t, \omega) = x_0(\omega) - f(t_0, x_0(\omega), \omega) + f(t, x(t, \omega), \omega) + \int_{t_0}^{t} h(s)\, ds, \quad t \in J. \tag{3.5}$$

Proof . Let $h \in C(J, \mathbb{R})$. Assume first that x is a solution of the HRDE (3.4). By definition, $t \mapsto x(t, \omega) - f(t, x(t, \omega), \omega)$ is continuous on J, and so, differentiable there, whence $\frac{d}{dt}\big[x(t, \omega) - f(t, x(t, \omega), \omega)\big]$ is integrable on J. Applying integration to (3.4) from t_0 to t, we obtain the HIE (3.5) on J.

Conversely, assume that x satisfies the HIE (3.5). Then by direct differentiation we obtain the first equation in (3.4). Again, substituting $t = t_0$ in (3.5) yields

$$x(t_0,\omega) - f(t_0, x(t_0,\omega),\omega) = x_0(\omega) - f(t_0, x_0(\omega),\omega).$$

Since the mapping $x \mapsto x - f(t,x,\omega)$ is increasing in $\mathbb{R}$ for all $t \in J$ and $\omega \in \Omega$, the mapping $x \mapsto x - f(t_0,x,\omega)$ is injective in $\mathbb{R}$, whence $x(t_0,\omega) = x_0(\omega)$. Hence the proof of the lemma is complete. □

We need the following hypotheses in what follows.

(A_1) f is Carathéodory and nonlinear $\mathcal{D}$-Lipschitz on $J \times \mathbb{R} \times \Omega$ for $\omega \in \Omega$ a.e.

(A_2) g is $\mathcal{C}$-Carathéodory on $J \times \mathbb{R} \times \Omega$ for $\omega \in \Omega$ a.e.

(A_3) There exist constants $F_0 > 0$ and $F_1 > 0$ such that

$$|f(t_0, 0, \omega| \le F_0 \ \text{ and } \ |x_0(\omega) - f(t_0, x_0(\omega), \omega)| \le F_1$$

for $\omega \in \Omega$ a.e.

Now we are in a position to prove the following existence theorem for the HRDE (1.1) on J.

Theorem 3.8. *Assume that the hypotheses (A_0) through (A_3) hold. Then the HRDE* (1.1) *has a random solution on J.*

Proof . Set $E = C(J,\mathbb{R})$ and define a subset S of E defined by

$$S = \{x : \Omega \to E \mid \|x(\omega)\| \le N \text{ for all } \omega \in \Omega\} \tag{3.6}$$

where,

$$N = F_1 + L + F_0 + a\|h\|.$$

Clearly S is a closed, convex and bounded subset of the Banach space E. Now, using the hypotheses (A_0) and (A_2) it can be shown by an application of Lemma 3.7 that the HRDE (1.1) is equivalent to the nonlinear HIE

$$\begin{aligned} x(t,\omega) = x_0(\omega) - f(t_0, x_0(\omega) + f(t, x(t,\omega),\omega) \\ + \int_{t_0}^{t} g(s, x(s,\omega),\omega)\,ds, \ \ \omega \in \Omega \text{ a.e..} \end{aligned} \tag{3.7}$$

Define two operators $A : \Omega \times E \to E$ and $B : \Omega \times S \to E$ by

$$A(\omega)x(t) = f(t, x(t,\omega),\omega), \ t \in J, \tag{3.8}$$

and

$$B(\omega)x(t) = x_0 - f(t_0, x_0(\omega),\omega) + \int_{t_0}^{t} g(s, x(s,\omega),\omega)\,ds, \ t \in J. \tag{3.9}$$

Then, the HIE (3.8) is transformed into an random operator equation as

$$A(\omega)x(t) + B(\omega)x(t) = x(t,\omega), \ \ \omega \in \Omega \text{ a.e.} \tag{3.10}$$

We shall show that the random operators $A(\omega)$ and $B(\omega)$ satisfy all the conditions of Theorem 3.4. First, we show that A is a nonlinear random $\mathcal{D}$-contraction on E with a $\mathcal{D}$-function ψ. Denote

$$\Phi = \{\omega \in \Omega \mid f \text{ is Carathéodory and nonlinear } \mathcal{D}\text{-Lipschitz on } J \times \mathbb{R} \times \Omega\},$$

$$\Psi = \{\omega \in \Omega \mid g \text{ is } \mathcal{C}\text{-Carathéodory on } J \times \mathbb{R} \times \Omega\}$$

and

$$\Upsilon = \{\omega \in \Omega \mid \text{ The hypothesis (A}_3\text{) is true }\}.$$

Denote $\Sigma = \Phi \cap \Psi \cap \Upsilon$. Then, we have

$$\Sigma^c = \big(\Phi \cap \Psi \cap \Upsilon\big)^c = \Phi^c \cup \Psi^c \cup \Upsilon^c \quad \text{and} \quad \mu\big(\Sigma^c\big) = \mu(\Phi^c \cup \Psi^c \cup \Upsilon^c) = 0.$$

Let $\omega \in \Sigma$ be fixed element and let $x, y \in E$. Then, by hypothesis (A_1),

$$\begin{aligned}|A(\omega)x(t) - A(\omega)y(t)| &= |f(t, x(t,\omega), \omega) - f(t, y(t,\omega), \omega)| \\ &\le \frac{L|x(t,\omega) - y(t,\omega)|}{M + |x(t,\omega) - y(t,\omega)|} \\ &\le \frac{L\|x(\omega) - y(\omega)\|}{M + \|x(\omega) - y(\omega)\|}\end{aligned}$$

for $\omega \in \Sigma$. Taking supremum over t, we obtain

$$\|A(\omega)x - A(\omega)y\| \le \frac{L\|x(\omega) - y(\omega)\|}{M + \|x(\omega) - y(\omega)\|},$$

for all $x, y \in E$. This shows that $A(\omega)$ is a continuous and nonlinear random $\mathcal{D}$-contraction on E with the $\mathcal{D}$-function ψ defined by $\psi(r) = \dfrac{L\,r}{M + r}$ for each $\omega \in \Sigma$. As a result, $A(\omega)$ is a continuous and nonlinear random $\mathcal{D}$-contraction on E for $\omega \in \Omega$ a.e.

Next, we show that $B(\omega)$ is a compact and continuous random operator on $\Sigma \times S$ into E. First we show that $B(\omega)$ is continuous random on $\Sigma \times S$. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$ and let $\omega \in \Sigma$ be fixed. Then by dominated convergence theorem for integration, we obtain

$$\begin{aligned}&\lim_{n\to\infty} B(\omega)x_n(t) \\ &= \lim_{n\to\infty}\left[x_0(\omega) - f(t_0, x_0(\omega), \omega) + \int_{t_0}^{t} g(s, x_n(s,\omega), \omega)\, ds\right] \\ &= x_0(\omega) - f(t_0, x_0(\omega), \omega) + \lim_{n\to\infty}\int_{t_0}^{t} g(s, x_n(s,\omega))\, ds \\ &= x_0(\omega) - f(t_0, x_0(\omega), \omega) + \int_{t_0}^{t}\left[\lim_{n\to\infty} g(s, x_n(s,\omega), \omega)\right] ds \\ &= x_0(\omega) - f(t_0, x_0(\omega), \omega) + \int_{t_0}^{t} g(s, x(s,\omega), \omega)\, ds \\ &= B(\omega)x(t)\end{aligned}$$

for all $t \in J$. Moreover, it can be shown as below that $\{B(\omega)x_n\}$ is an equicontinuous sequence of functions in S. Now, following the arguments similar to that given in Granas *et al.* [16], it is proved that $B(\omega)x_n \to B(\omega)x$ for all $\omega \in \Sigma$, that is, $B(\omega)$ is a a continuous random operator on S.

Next, we show that $B(\omega)$ is compact random operator on S. It is enough to show that $B(\omega)(S)$ is a uniformly bounded and equi-continuous set in E for each $\omega \in \Sigma$. Let $x \in S$ be arbitrary and $\omega \in \Sigma$ be fixed. Then by hypothesis (A$_2$),

$$\begin{aligned}|B(\omega)x(t)| &\leq |x_0(\omega) - f(t_0, x_0(\omega), \omega)| + \int_{t_0}^{t} |g(s, x(s,\omega), \omega)|\, ds \\ &\leq |x_0(\omega) - f(t_0, x_0(\omega), \omega)| + \int_{t_0}^{t} h(s)\, ds \\ &\leq F_1 + \|h\|\, a\end{aligned}$$

for all $t \in J$. Taking supremum over t,

$$\|B(\omega)x\| \leq F_1 + \|h\|\, a$$

for all $x \in S$. This shows that $B(\omega)$ is uniformly bounded on S.

Again, let $t_1, t_2 \in J$. Then for any $x \in S$, one has

$$\begin{aligned}&|B(\omega)x(t_1) - B(\omega)x(t_2)| \\ &\qquad = \left|\int_{t_0}^{t_1} g(s, x(s,\omega), \omega)\, ds - \int_{t_0}^{t_2} g(s, x(s,\omega), \omega)\, ds\right| \\ &\qquad \leq \left|\int_{t_2}^{t_1} |g(s, x(s,\omega), \omega)|\, ds\right| \\ &\qquad \leq |p(t_1) - p(t_2)|\end{aligned}$$

where, $p(t) = \int_{t_0}^{t} h(s)\, ds$. Since the function p is continuous on compact J, it is uniformly continuous there. Hence, for $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|t_1 - t_2| < \delta \Longrightarrow |B(\omega)x(t_1) - B(\omega)x(t_2)| < \epsilon$$

uniformly for all $t_1, t_2 \in J$ and for all $x \in S$. This shows that $B(\omega)(S)$ is an equi-continuous set in E. Now the set $B(\omega)(S)$ is uniformly bounded and equicontinuous set in E, so it is compact by Arzelá-Ascoli theorem. As a result, $B(\omega)$ is a continuous and compact random operator on S for each $\omega \in \Sigma$. Therefore, hypothesis (b) of Theorem 3.4 is satisfied.

Next, we show that hypothesis (c) of Theorem 3.4 is satisfied. Let $x \in E$ be fixed and $y \in S$ be arbitrary such that $x = A(\omega)x + B(\omega)y$ for all $\omega \in \Sigma$. Then, by assumption (A$_1$), we have

$$\begin{aligned}|x(t,\omega)| &\leq |A(\omega)x(t)| + |B(\omega)y(t)| \\ &\leq |x_0(\omega) - f(t_0, x_0(\omega), \omega)| + |f(t, x(t,\omega), \omega)| \\ &\qquad + \int_{t_0}^{t} |g(s, y(s,\omega), \omega)|\, ds \\ &\leq |x_0(\omega) - f(t_0, x_0(\omega), \omega)| + \big[|f(t, x(t,\omega), \omega) - f(t, 0, \omega)| \\ &\qquad + |f(t, 0, \omega)|\big] + \int_{t_0}^{t} |g(s, y(s,\omega), \omega)|\, ds \\ &\leq |x_0 - f(t_0, x_0, \omega)| + L + F_0 + \int_{t_0}^{t} h(s)\, ds \\ &\leq F_1 + L + F_0 + \|h\|a.\end{aligned}$$

for all $\omega \in \Sigma$. Taking supremum over t,

$$\|x(\omega)\| \leq F_1 + L + F_0 + \|h\|a,$$

for all $\omega \in \Sigma$ and therefore, $x \in S$.

Thus, all the conditions of Theorem 3.4 are satisfied and hence the operator equation $A(\omega)x + B(\omega)x = x$, $\omega \in \Omega$ a.e., has a random solution in S. As a result, the HRDE (1.1) has a random solution defined on J. This completes the proof. □

4. Maximal and Minimal Random Solutions

In this section, we shall prove the existence of maximal and minimal random solutions for the HRDE (1.1) on $J = [t_0, t_0 + a]$. We need the following definition in what follows.

Definition 4.1. A random solution r of the HRDE (1.1) is said to be maximal if for any other random solution x to the HRDE (1.1) one has for for all $t \in J$, $x(t,\omega) \leq r(t,\omega)$, $\omega \in \Omega$ a.e. Again, a random solution ρ of the HRDE (1.1) is said to be minimal if for for all $t \in J$, $\rho(t,\omega) \leq x(t,\omega)$, $\omega \in \Omega$ a.e., where x is any random solution of the HRDE (1.1) existing on J.

We discuss the case of maximal random solution only, as the case of minimal random solution is similar and can be obtained with the similar arguments with appropriate modifications. Given a arbitrary small real number $\epsilon > 0$, consider the the following initial value problem of HRDE satisfying for $\omega \in \Omega$ a.e.,

$$\left.\begin{aligned} \tfrac{d}{dt}\left[x(t,\omega) - f(t,x(t,\omega),\omega)\right] &= g(t,x(t,\omega),\omega) + \epsilon, \\ x(t_0,\omega) &= x_0(\omega) + \epsilon \end{aligned}\right\} \tag{4.1}$$

where, $f, g : J \times \mathbb{R} \times \Omega \to \mathbb{R}$ and $x_0 : \Omega \to \mathbb{R}$ is measurable.

An existence theorem for the HRDE (4.1) can be stated as follows:

Theorem 4.2. *Assume that the hypotheses (A_0) through (A_3) hold. Then for every small number $\epsilon > 0$, the HRDE* (4.1) *has a random solution defined on J.*

Proof . The proof is similar to Theorem 3.4 and we omit the details. □

Our main existence theorem for maximal random solution for the HRDE (1.1) is as follows.

Theorem 4.3. *Assume that the hypotheses (A_0) through (A_3) hold. Then the HRDE* (1.1) *has a maximal random solution defined on J.*

Proof . Let $\{\epsilon_n\}_0^\infty$ be a decreasing sequence of positive real numbers such that $\lim_{n\to\infty} \epsilon_n = 0$. Then for any random solution u of the HRDE (1.1), by Theorem 2.1, one has

$$u(t,\omega) < r(t,\epsilon_n,\omega),\ \omega \in \Omega \text{ a.e.,} \tag{4.2}$$

for all $t \in J$ and $n \in \mathbb{N} \cup \{0\}$, where $r(t,\epsilon_n,\omega)$ is a random solution of the following HRDE satisfying for $\omega \in \Omega$ a.e.,

$$\left.\begin{aligned} \tfrac{d}{dt}\left[x(t,\omega) - f(t,x(t,\omega),\omega)\right] &= g(t,x(t,\omega),\omega) + \epsilon_n, \\ x(t_0,\omega) &= x_0(\omega) + \epsilon_n \end{aligned}\right\} \tag{4.3}$$

defined on J.

Since, by Theorems 3.4 and 3.8, $\{r(t,\epsilon_n,\omega)\}$ is a decreasing sequence of positive real numbers, the limit

$$r(t,\omega) = \lim_{n\to\infty} r(t,\epsilon_n,\omega) \tag{4.4}$$

exists for each $\omega \in \Sigma$, where $\Sigma \subset \Omega$ is defined as in the proof of Theorem 3.4. We show that the convergence in (4.6) is uniform on J. To finish, it is enough to prove that the sequence $\{r(t,\epsilon_n,\omega)\}$ is equi-continuous in $C(J,\mathbb{R})$. Let $\omega \in \Sigma$ be fixed and let $t_1, t_2 \in J$ be arbitrary. Then,

$$\begin{aligned}
&|r(t_1,\epsilon_n,\omega) - r(t_2,\epsilon_n,\omega)| \\
&\quad \le \big|f(t_1, r(t_1,\epsilon_n,\omega),\omega) - f(t_2, r(t_2,\epsilon_n),\omega)\big| \\
&\qquad + \left|\int_{t_0}^{t_1} g(s, r_{\epsilon_n}(s,\omega),\omega)\,ds - \int_{t_0}^{t_2} g(s, r_{\epsilon_n}(s,\omega),\omega)\,ds\right| \\
&\qquad + \left|\int_{t_0}^{t_1} \epsilon_n\,ds - \int_{t_0}^{t_2} \epsilon_n\,ds\right| \\
&\quad = \big|f(t_1, r(t_1,\epsilon_n,\omega),\omega) - f(t_2, r(t_2,\epsilon_n,\omega),\omega)\big| \\
&\qquad + \left|\int_{t_1}^{t_2} g(s, r_{\epsilon_n}(s,\omega),\omega)\,ds\right| + \left|\int_{t_1}^{t_2} \epsilon_n\,ds\right| \\
&\quad \le \big|f(t_1, r(t_1,\epsilon_n,\omega),\omega) - f(t_2, r(t_2,\epsilon_n,\omega),\omega)\big| \\
&\qquad + \left|\int_{t_1}^{t_2} h(s)\,ds\right| + \left|\int_{t_1}^{t_2} \epsilon_n\,ds\right| \\
&\quad = \big|f(t_1, r(t_1,\epsilon_n,\omega),\omega) - f(t_2, r(t_2,\epsilon_n,\omega),\omega)\big| \\
&\qquad + \left|\int_{t_1}^{t_2} h(s)\,ds\right| + |t_1 - t_2|\,\epsilon_n \\
&\quad = \big|f(t_1, r(t_1,\epsilon_n,\omega),\omega) - f(t_2, r(t_2,\epsilon_n,\omega),\omega)\big| \\
&\qquad + |p(t_1) - p(t_2)| + |t_1 - t_2|\,\epsilon_n
\end{aligned} \tag{4.5}$$

where, $p(t) = \int_{t_0}^{t} h(s)\,ds$.

Since $f(\cdot,\cdot,\omega)$ is continuous on compact set $J \times [-N,N]$, it is uniformly continuous there. Hence,

$$\big|f(t_1, r(t_1,\epsilon_n,\omega),\omega) - f(t_2, r(t_2,\epsilon_n,\omega),\omega)\big| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $n \in \mathbb{N}$. Similarly, since the function p is continuous on compact set J, it is uniformly continuous and hence

$$\big|p(t_1) - p(t_2)\big| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $t_1, t_2 \in J$. Therefore, from the above inequality (4.5), it follows that

$$|r(t_1,\epsilon_n,\omega) - r(t_2,\epsilon_n,\omega)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $n \in \mathbb{N}$. Therefore,

$$r(t,\epsilon_n,\omega) \to r(t,\omega) \quad \text{as} \quad n \to \infty$$

uniformly for all $t \in J$. Next, we show that the function $r(t,\omega)$ is a random solution of the HRDE (3.4) defined on J. Now, since $r(t,\epsilon_n,\omega)$ is a random solution of the HRDE (4.3), we have

$$\left.\begin{aligned}
r(t,\epsilon_n,\omega) &= [x_0(\omega) + \epsilon_n - f(t_0, x_0(\omega) + \epsilon_n,\omega)] \\
&\quad + f(t, r(t,\epsilon_n,\omega),\omega) + \int_{t_0}^{t} g(s, r_{\epsilon_n}(s,\omega),\omega)\,ds
\end{aligned}\right\} \tag{4.6}$$

for all $t \in J$. Taking the limit as $n \to \infty$ in the above equation (4.6) yields

$$\begin{aligned} r(t,\omega) &= x_0(\omega) - f(t_0, x_0(\omega), \omega) \\ &\quad + f(t, r(t,\omega), \omega) + \int_{t_0}^{t} g(s, r(s,\omega), \omega)\, ds \end{aligned}$$

for $t \in J$ and $\omega \in \sigma$. Thus, the function r is a random solution of the HRDE (1.1) on J. Finally, form the inequality (4.4) it follows that

$$u(t,\omega) \le r(t,\omega), \;\; \omega \in \Omega \;\text{ a.e. },$$

for all $t \in J$. Hence the HRDE (1.1) has a maximal random solution on J. This completes the proof. □

5. Comparison Principle

The main problem of the random differential inequalities is to estimate the qualitative information about the solution set of the random differential inequalities related to the HRDE (1.1)and obtaining an upper bound for such random solution has widely been discussed in the literature. In this section we prove that the maximal and minimal random solutions serve as the bounds for the solutions of the related differential inequality to HRDE (1.1) on $J = [t_0, t_0 + a]$.

Theorem 5.1. *Assume that the hypotheses (A_0) through (A_3) hold. Further, if there exists a measurable function $u : \Omega \to C(J, \mathbb{N})$ satisfying for $\omega \in \Omega$ a.e.,*

$$\left.\begin{aligned} \tfrac{d}{dt}\left[u(t,\omega) - f(t, u(t,\omega), \omega)\right] &\le g(t, u(t,\omega), \omega), \;\; t \in J \\ u(t_0,\omega) &\le x_0(\omega). \end{aligned}\right\} \tag{5.1}$$

Then,

$$u(t,\omega) \le r(t,\omega), \;\; \omega \in \Omega \text{ a.e.} \tag{5.2}$$

where, r is a maximal random solution of the HRDE (1.1) defined on J.

Proof . Let $\epsilon > 0$ be arbitrary small. Then, by Theorem 4.3, $r(t,\epsilon,\omega)$ is a maximal random solution of the HRDE (4.1) and that the limit

$$r(t,\omega) = \lim_{\epsilon \to 0} r(t,\epsilon,\omega) \tag{5.3}$$

is uniform on J and the function r is a maximal random solution of the HRDE (1.1) on J. Hence, for $\omega \in \Omega$ a.e., we obtain

$$\left.\begin{aligned} \tfrac{d}{dt}\left[r(t,\epsilon,\omega) - f(t, r(t,\epsilon,\omega), \omega)\right] &= g(t, r(t,\epsilon,\omega), \omega) + \epsilon, \;\; t \in J \\ r(t_0,\epsilon,\omega) &= x_0(\omega) + \epsilon. \end{aligned}\right\} \tag{5.4}$$

From above inequality (5.4) it follows that

$$\left.\begin{aligned} \tfrac{d}{dt}\left[r(t,\epsilon,\omega) - f(t, r(t,\epsilon,\omega), \omega)\right] &> g(t, r(t,\epsilon,\omega), \omega), \;\; t \in J \\ r(t_0,\epsilon) &> x_0(\omega) \end{aligned}\right\} \tag{5.5}$$

for $\omega \in \Omega$ a.e. Now we apply Theorem 2.1 to the inequalities (5.1) and (5.5) and conclude that

$$u(t,\omega) < r(t,\epsilon,\omega), \;\; \omega \in \Omega \;\text{ a.e. }, \tag{5.6}$$

for all $t \in J$. This further in view of limit (5.3) implies that inequality (5.2) holds on J. This completes the proof. □

Theorem 5.2. Assume that the hypotheses (A_0) through (A_3) hold. Further, if there exists a measurable function $v : \Omega \to C(J, \mathbb{R})$ satisfying for $\omega \in \Omega$ a.e.,

$$\left.\begin{array}{rcl} \frac{d}{dt}\left[v(t,\omega) - f(t, v(t,\omega),\omega)\right] & \geq & g(t, v(t,\omega),\omega), \ \ t \in J \\ v(t_0,\omega) & \geq & x_0(\omega). \end{array}\right\} \tag{5.7}$$

Then,

$$\rho(t,\omega) \leq v(t,\omega), \ \omega \in \Omega \quad \text{a.e. }, \tag{5.8}$$

for all $t \in J$, where ρ is a minimal random solution of the HRDE (1.1) defined on J.

Note that Theorem 5.1 is useful to prove the boundedness and uniqueness of the random solutions for the HRDE (1.1) on J. A result in this direction is

Theorem 5.3. *Assume that the hypotheses (A_0) through (A_3) hold. Suppose that there exists a function $G : J \times \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ satisfying for $\omega \in \Omega$ a.e.,*

$$\begin{aligned} &\big|g(t, x_1(t,\omega),\omega) - g(t, x_2(t,\omega),\omega)\big| \\ &\qquad \leq G\left(t, \big|\big(x_1(t,\omega) - f(t, x_1(t,\omega),\omega)\big) - \big(x_2(t,\omega) - f(t, x_2(t,\omega),\omega)\big)\big|\right) \end{aligned} \tag{5.9}$$

for all $t \in J$ and $x_1, x_2 \in E$. If identically zero random function is the only random solution of the differential equation satisfying for $\omega \in \Omega$ a.e.,

$$m'(t,\omega) = G(t, m(t,\omega),\omega), \ t \in J, \ \ m(t_0,\omega) = 0, \tag{5.10}$$

then the HRDE (1.1) has a unique random solution defined on J.

Proof . By Theorem 3.8, the HRDE (1.1) has a random solution defined on J. Suppose that there are two random solutions u_1 and u_2 of the HRDE (1.1) existing on J. Define a function $m : \Omega \to C(J, \mathbb{R}_+)$ by

$$m(t,\omega) = \big|\big(u_1(t,\omega) - f(t, u_1(t,\omega),\omega)\big) - \big(u_2(t,\omega) - f(t, u_2(t,\omega),\omega)\big)\big|. \tag{5.11}$$

Let $\omega \in \Sigma$ be fixed, where Σ is defined as in the proof of Theorem 3.8. As

$$\big(|x(t,\omega)|\big)' \leq |x'(t,\omega)|$$

for $t \in J$, we have that

$$\begin{aligned} m'(t,\omega) &\leq \left|\frac{d}{dt}\left[u_1(t,\omega) - f(t, u_1(t,\omega),\omega)\right] - \frac{d}{dt}\left[u_2(t,\omega) - f(t, u_2(t,\omega),\omega)\right]\right| \\ &\leq |g(t, u_1(,\omega),\omega) - g(t, u_2(t,\omega),\omega)| \\ &\leq G\left(t, \big|\big(u_1(t,\omega) - f(t, u_1(t,\omega),\omega)\big) - \big(u_2(t,\omega) - f(t, u_2(t,\omega),\omega)\big)\big|, \omega\right) \\ &= G(t, m(t,\omega),\omega) \end{aligned}$$

for all $t \in J$; and that $m(t_0,\omega) = 0$.

Now, we apply Theorem 5.1 with $f \equiv 0$ to get that $m(t,\omega) = 0$ for all $t \in J$. This gives

$$u_1(t,\omega) - f(t, u_1(t,\omega)) = u_2(t,\omega) - f(t, u_2(t,\omega),\omega)$$

for all $t \in J$. Finally, in view of hypothesis (A_0) we conclude that $u_1(t,\omega) = u_2(t,\omega)$ on J. This completes the proof. □

6. Extremal Random Solutions in Vector Segments

Sometimes it is desirable to have knowledge of existence of extremal random solutions for the HRDE (1.1) in a vector segment defined on J. Therefore, in this section we shall prove the existence of maximal and minimal random solutions for HRDE (1.1) between the given upper and lower solutions on $J = [t_0, t_0 + a]$. We use a hybrid random fixed point theorem of Dhage [29,30] in an ordered separable Banach space for establishing our results. We need the following preliminaries in the sequel.

A non-empty closed set $\mathcal{K}$ in a Banach space E is called a **cone** with vertex 0, if (i) $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$, (ii) $\lambda\mathcal{K} \subseteq \mathcal{K}$ for $\lambda \in \mathbb{N}, \lambda \geq 0$ and (iii) $\{-\mathcal{K}\} \cap \mathcal{K} = 0$, where 0 is the zero element of E. We introduce an order relation $\leq$ in E as follows. Let $x, y \in E$. Then $x \leq y$ if and only if $y - x \in \mathcal{K}$. A cone K is called to be **normal** if the norm $\|\cdot\|$ is semi-monotone increasing on K, that is, there is a constant $N > 0$ such that $\|x\| \leq N\|y\|$ for all $x, y \in \mathcal{K}$ with $x \leq y$. It is known that if the cone $\mathcal{K}$ is normal in E, then every order-bounded set in E is norm-bounded. The details of cones and their properties appear in Heikkilä and Lakshmikantham [18].

Given two measurable functions $a, b : \Omega \to E$, $a \leq b$, we define a random interval $[a, b]$ in E by

$$\begin{aligned} [a, b] &= \{x \in E \mid a(\omega) \leq x \leq b(\omega)\ \forall\, \omega \in \Omega\} \\ &= \bigcap_{\omega \in \Omega} [a(\omega), b(\omega)]. \end{aligned}$$

Definition 6.1. An operator $Q : \Omega \times E \to E$ is called nondecreasing if $Q(\omega)x \leq Q(\omega)y$ for all $\omega \in \Omega$ and for all $x, y \in E$ for which $x \leq y$.

We use the following random fixed point theorems of Dhage [4] for proving the existence of extremal random solutions for the BVP (1.1) under certain monotonicity conditions.

Theorem 6.2. [*Dhage [4]*] *Let $\mathcal{K}$ be a cone in a Banach space E and let $a, b \in E$. be such that $a \leq b$. Suppose that $A, B : \Omega \times [a, b] \to E$ are two continuous and nondecreasing random operators such that for $\omega \in \Omega$ a.e.,*

(a) $A(\omega)$ is nonlinear contraction,

(b) $B(\omega)$ is complact, and

(c) $A(\omega)x + B(\omega)x \in [a, b]$ for each $x \in [a, b]$.

Further, if the cone $\mathcal{K}$ is normal, then the random operator equation $A(\omega)x + B(\omega)x = x$ $\omega \in \Omega$ a.e., has a least and a greatest random solution in $[a, b]$.

Proof . The proof is similar to Theorem 3.4 and now the conclusion follows by an application of a hybrid random fixed point theorem proved in Dhage [4]. □

We equip the space $C(J, \mathbb{N})$ with the order relation $\leq$ with the help of the cone K in it defined by

$$\mathcal{K} = \big\{x \in C(J, \mathbb{R}) \,:\, x(t) \geq 0 \text{ for all } t \in J\big\}. \tag{6.1}$$

It is well known that the cone K is a normal in $C(J, \mathbb{R})$. We need the following definitions in the sequel.

Definition 6.3. A measurable function $a : \Omega \to C(J, \mathbb{R})$ is called a lower random solution of the HRDE (1.1) defined on J if the map $t \mapsto x - f(t, x, \omega)$ is continuous for each $x \in \mathbb{R}$ and satisfies for each $\omega \in \Omega$,

$$\left.\begin{array}{rcl} \frac{d}{dt}\left[a(t,\omega) - f(t, a(t,\omega), \omega)\right] & \leq & g(t, a(t,\omega), \omega), \;\; t \in J \\ a(t_0, \omega) & \leq & x_0(\omega). \end{array}\right\} \tag{6.2}$$

Similarly, a measurable function $b : \Omega \to C(J, \mathbb{R})$ is called an upper random solution of the HRDE (1.1) defined on J if if the map $t \mapsto x - f(t, x, \omega)$ is continuous for each $x \in \mathbb{R}$ and satisfies for each $\omega \in \Omega$,

$$\left.\begin{array}{rcl} \frac{d}{dt}\left[b(t,\omega) - f(t, b(t,\omega), \omega)\right] & \geq & g(t, b(t,\omega), \omega), \;\; t \in J \\ b(t_0, \omega) & \geq & x_0(\omega). \end{array}\right\} \tag{6.3}$$

A random solution to the HRDE (1.1) is a lower as well as an upper random solution for the HRDE (1.1) defined on $J \times \Omega$ and vice versa.

We consider the following set of assumptions:

(B_1) The HRDE (1.1) has a lower random solution $a(\omega)$ and an upper random solution $b(\omega)$ for $\omega \in \Omega$ a.e. with $a \leq b$.

(B_2) The function $x \mapsto x - f(t, x, \omega)$ is increasing for $\omega \in \Omega$ a.e. in the interval $\Big[\min_{t\in J} a(t,\omega), \max_{t \in J} b(t,\omega)\Big]$ for all $t \in J$.

(B_3) The functions $f(t, x, \omega)$ and $g(t, x, \omega)$ are nondecreasing in x for $\omega \in \Omega$ a.e., and for all $t \in J$.

(B_4) There exists a continuous function $h \in C(J, \mathbb{R})$ such that

$$g(t, b(t,\omega), \omega) \leq h(t), \;\; \omega \in \Omega \;\; \text{a.e. .}$$

for all $t \in J$.

Remark 6.4. Note that the hypotheses (B_1) to (B_4) are natural and widely used in the literature by several authors (see Dhage [4, 5], Heikillá and Lakshmikatham [18] and the references given therein).

Theorem 6.5. *Suppose that the assumptions (A_1) and (B_1) through (B_4) hold. Then the HRDE* (1.1) *has a minimal and a maximal random solution in* $[a, b]$ *defined on* J.

Proof . Now, the HRDE (1.1) is equivalent to hybrid random integral equation (3.7) defined on J. Let $E = C(J, \mathbb{R})$. Define two operators A and B on $\Omega \times [a, b]$ by (3.8) and (3.9) respectively. Then the random integral equation (3.7) is transformed into a random operator equation as

$$A(\omega)x(t) + B(\omega)x(t) = x(t, \omega) \;\; \omega \in \Omega \;\; \text{a.e.}$$

in the ordered Banach space E. Denote

$$\Delta = \{\omega \in \Omega \mid \text{The HRDE (1.1) has a lower random solution } a \text{ and an upper random solution } b\},$$

$$\Theta = \left\{\omega \in \Omega \,\middle|\, \text{The function } x \mapsto x - f(t, x, \omega) \text{ is increasing in } \left[\min_{t \in J} a(t,\omega), \max_{t\in J} b(t, \omega)\right] \text{ for all } t \in J \text{ with } a \leq b\right\},$$

$$\Lambda = \{\omega \in \Omega \mid \text{The functions } f(t,x,\omega) \text{ and } g(t,x,\omega) \text{ are nondecreasing in } x \text{ for all } t \in \mathbb{R}\},$$

$$\Pi = \{\omega \in \Omega \mid \text{ There exists a continuous function } h \in C(J,\mathbb{R}) \text{ such that } g(t,b(t,\omega),\omega) \leq h(t) \text{ for all } t \in J\}.$$

Let $\Sigma = \mathcal{N}_A \cap \mathcal{N}_B \cap \Delta \cap \Theta \cap \Lambda \cap \Pi$. Then $\Sigma^c = \mathcal{N}_A^c \cup \mathcal{N}_B^c \cup \Delta^c \cup \Theta^c \cup \Lambda^c \cup \Pi^c$ and consequently

$$\mu(\Sigma^c) = \mu\left(\mathcal{N}_A^c \cup \mathcal{N}_B^c \cup \Delta^c \cup \Theta^c \cup \Lambda^c \cup \Pi^c\right) = 0.$$

Let $\omega \in \Sigma$ be fixed element. Notice that hypothesis (B_1) implies $A, B : \Sigma \times [a,b] \to E$. Since the cone K in E is normal, $[a,b]$ is a norm-bounded set in E. Now it is shown, as in the proof of Theorem 3.8, that the operators $A(\omega)$ is a nonlinear $\mathcal{D}$-contraction and $B(\omega)$ is a completely continuous random operator on $\Sigma \times [a,b]$ into E. Again, the hypothesis (B_3) implies that $A(\omega)$ and $B(\omega)$ are nondecreasing on $[a,b]$. To see this, let $x, y \in [a,b]$ be such that $x \leq y$. Then, by hypothesis (B_3),

$$A(\omega)x(t) = f(t,x(t,\omega),\omega) \leq f(t,y(t,\omega),\omega) = A(\omega)y(t)$$

for all $t \in J$. Similarly, we have

$$\begin{aligned} B(\omega)x(t) &= x_0 - f(t_0,x_0,\omega) + \int_{t_0}^{t} g(s,x(s,\omega),\omega)\,ds \\ &\leq x_0 - f(t_0,x_0,\omega) + \int_{t_0}^{t} g(s,y(s,\omega),\omega)\,ds \\ &= B(\omega)y(t) \end{aligned}$$

for all $t \in J$. So $A(\omega)$ and $B(\omega)$ are nondecreasing random operators on $[a,b]$ for each $\omega \in \Sigma$. Further, we obtain

$$\begin{aligned} a(t,\omega) &\leq x_0(\omega) - f(t_0,x_0(\omega),\omega) + f(t,a(t,\omega),\omega) + \int_{t_0}^{t} g(s,a(s,\omega),\omega)\,ds \\ &\leq x_0(\omega) - f(t_0,x_0(\omega),\omega) + f(t,x(t,\omega),\omega) + \int_{t_0}^{t} g(s,x(s,\omega),\omega)\,ds \\ &\leq x_0(\omega) - f(t_0,x_0(\omega),\omega) + f(t,b(t,\omega),\omega) + \int_{t_0}^{t} g(s,b(s,\omega),\omega)\,ds \\ &\leq b(t,\omega), \end{aligned}$$

for all $t \in J$ and $x \in [a,b]$. As a result $a(t,\omega) \leq A(\omega)x(t) + B(\omega)x(t) \leq b(t,\omega)$ for all $t \in J$ and $x \in [a,b]$. Hence, $A(\omega)x + B(\omega)x \in [a,b]$ for all $x \in [a,b]$ and $\omega \in \Sigma$.

Now, we apply Theorem 6.2 to the operator equation $A(\omega)x + B(\omega)x = x$, $\omega \in \Omega$ a.e. to yield that the HRDE (1.1) has a minimal and a maximal random solution in $[a,b]$ defined on J. This completes the proof. □

When $f \equiv 0$ in our results of this chapter, we obtain the differential inequalities and other related results given in Lakshmikantham and Ladde [22] for the IVP of ordinary nonlinear random differential equation satisfying for $\omega \in \Omega$ a.e.,

$$x'(t,\omega) = g(t,x(t,\omega),\omega), \quad x(t_0,\omega) = x_0(\omega). \tag{6.4}$$

for all $t \in J$.

Remark 6.6. The hybrid random differential equations is a rich area for variety of nonlinear ordinary as well as partial random differential as well as integral equations which occur in several engineering and biological problems. See for the details Tsokos and Padgett [26, 27] and the references cited therein. Here, in this paper, we have considered a very simple hybrid random differential equation involving two nonlinearities, however, a more complex hybrid random differential equation can also be studied on the similar lines with appropriate modifications. Again, the results proved in this paper are very fundamental in nature and therefore, all the other problems for the hybrid random differential equation in question are still open. In a forthcoming paper we plan to prove some approximation results for the hybrid random differential equation considered in this paper.

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