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Quadratic ρ-functional inequalities in β-homogeneous normed spaces

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Abstract

In [12], Park introduced the quadratic ρ-functional inequalities

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\quad \le \left\|\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\|, \end{aligned} \tag{0.1}$$

where ρ is a fixed complex number with $|\rho|<1$, and

$$\begin{aligned} &\left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\| \\ &\quad \le \|\rho(f(x+y)+f(x-y)-2f(x)-2f(y))\|, \end{aligned} \tag{0.2}$$

where ρ is a fixed complex number with $|\rho|<\frac{1}{2}$.

In this paper, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequalities (0.1) and (0.2) in β-homogeneous complex Banach spaces and prove the Hyers-Ulam stability of quadratic ρ-functional equations associated with the quadratic ρ-functional inequalities (0.1) and (0.2) in β-homogeneous complex Banach spaces.

Keywords: Hyers-Ulam stability; β-homogeneous space; quadratic ρ-functional equation; quadratic ρ-functional inequality.
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1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [18] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [14] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [17] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group.

The functional equation

$$2f\left(\frac{x+y}{2}\right) + 2\left(\frac{x-y}{2}\right) = f(x) + f(y)$$

is called a *Jensen type quadratic equation*. See [9, 10, 11] for the stability problems.

In [6], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [15]. Gilányi [7] and Fechner [4] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [13] proved the Hyers-Ulam stability of additive functional inequalities.

Definition 1.1. Let X be a linear space. A nonnegative valued function $\|\cdot\|$ is an F-norm if it satisfies the following conditions:

(FN_1) $\|x\| = 0$ if and only if $x = 0$;
(FN_2) $\|\lambda x\| = \|x\|$ for all $x \in X$ and all λ with $|\lambda| = 1$;
(FN_3) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$;
(FN_4) $\|\lambda_n x\| \to 0$ provided $\lambda_n \to 0$;
(FN_5) $\|\lambda x_n\| \to 0$ provided $x_n \to 0$.

Then $(X, \|\cdot\|)$ is called an F^*-space. An F-space is a complete F^*-space.

An F-norm is called β-homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathbb{C}$ (see [16]).

In Section 2, we investigate the quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in β-homogeneous complex Banach spaces.

We moreover prove the Hyers-Ulam stability of a quadratic ρ-functional equation associated with the quadratic ρ-functional inequality (0.1) in β-homogeneous complex Banach spaces.

In Section 3, we investigate the quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in β-homogeneous complex Banach spaces. We moreover prove the Hyers-Ulam stability of a quadratic ρ-functional equation associated with the quadratic ρ-functional inequality (0.2) in β-homogeneous complex Banach spaces.

Throughout this paper, let β_1, β_2 be positive real numbers with $\beta_1 \leq 1$ and $\beta_2 \leq 1$. Assume that X is a β_1-homogeneous real or complex normed space with norm $\|\cdot\|$ and that Y is a β_2-homogeneous complex Banach space with norm $\|\cdot\|$.

2. Quadratic ρ-functional inequality (0.1)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < 1$.

In this section, we investigate the quadratic ρ-functional inequality (0.1) in β-homogeneous complex Banach spaces.

Lemma 2.1. *A mapping $f : X \to Y$ satisfies*

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\quad \leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \end{aligned} \tag{2.1}$$

for all $x, y \in X$ if anf only if $f : X \to Y$ is quadratic.

Proof . Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $\|2f(0)\| \leq |\rho|^{\beta_2}\|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \leq 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\quad \leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &\quad = \frac{|\rho|^{\beta_2}}{2^{\beta_2}}\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

The converse is obviously true. □

Corollary 2.2. A mapping $f : X \to Y$ satisfies

$$\begin{aligned} &f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ &\quad = \rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right) \end{aligned} \tag{2.3}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is quadratic.

The functional equation (2.3) is called a *quadratic ρ-functional equation.*

We prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (2.1) in β-homogeneous complex Banach spaces.

Theorem 2.3. *Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\le \left\|\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| + \theta(\|x\|^r+\|y\|^r) \end{aligned} \tag{2.4}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x)-h(x)\| \le \frac{2\theta}{2^{\beta_1 r}-4^{\beta_2}}\|x\|^r \tag{2.5}$$

for all $x \in X$.

Proof . Letting $x = y = 0$ in (2.4), we get $\|2f(0)\| \le |\rho|^{\beta_2}\|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.4), we get

$$\|f(2x)-4f(x)\| \le 2\theta\|x\|^r \tag{2.6}$$

for all $x \in X$. So $\left\|f(x)-4f\left(\frac{x}{2}\right)\right\| \le \frac{2}{2^{\beta_1 r}}\theta\|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned} \left\|4^l f\left(\frac{x}{2^l}\right)-4^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|4^j f\left(\frac{x}{2^j}\right)-4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \frac{2}{2^{\beta_1 r}}\sum_{j=l}^{m-1}\frac{4^{\beta_2 j}}{2^{\beta_1 r j}}\theta\|x\|^r \end{aligned} \tag{2.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.7), we get (2.5).

It follows from (2.4) that

$$\begin{aligned} &\|h(x+y)+h(x-y)-2h(x)-2h(y)\| \\ &= \lim_{n\to\infty} 4^{\beta_2 n}\left\|f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty} 4^{\beta_2 n}|\rho|^{\beta_2}\left\|2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right\| \\ &+ \lim_{n\to\infty}\frac{4^{\beta_2 n}\theta}{2^{\beta_1 n r}}(\|x\|^r+\|y\|^r) \\ &= |\rho|^{\beta_2}\left\|2h\left(\frac{x+y}{2}\right)+2h\left(\frac{x-y}{2}\right)-h(x)-h(y)\right\| \end{aligned}$$

for all $x, y \in X$. So

$$\|h(x+y)+h(x-y)-2h(x)-2h(y)\| \le \left\|\rho\left(2h\left(\frac{x+y}{2}\right)+2h\left(\frac{x-y}{2}\right)-h(x)-h(y)\right)\right\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.5). Then we have

$$\begin{aligned}\|h(x)-T(x)\| &= 4^{\beta_2 n}\left\|h\left(\frac{x}{2^n}\right)-T\left(\frac{x}{2^n}\right)\right\| \\ &\le 4^{\beta_2 n}\left(\left\|h\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right\|+\left\|T\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right\|\right) \\ &\le \frac{4\cdot 4^{\beta_2 n}}{(2^{\beta_1 r}-4^{\beta_2})2^{\beta_1 nr}}\theta\|x\|^r,\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique quadratic mapping satisfying (2.5). □

Theorem 2.4. *Let $r < \frac{2\beta_2}{\beta_1}$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.4). *Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x)-h(x)\| \le \frac{2\theta}{4^{\beta_2}-2^{\beta_1 r}}\|x\|^r \tag{2.8}$$

for all $x \in X$.

Proof . It follows from (2.6) that $\left\|f(x)-\frac{1}{4}f(2x)\right\| \le \frac{2\theta}{4^{\beta_2}}\|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{4^l}f(2^l x)-\frac{1}{4^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f(2^j x)-\frac{1}{4^{j+1}}f(2^{j+1}x)\right\| \\ &\le \sum_{j=l}^{m-1}\frac{2^{\beta_1 rj}}{4^{\beta_2 j}}\frac{2\theta}{4^{\beta_2}}\|x\|^r\end{aligned} \tag{2.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.9) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.8).

The rest of the proof is similar to the proof of Theorem 2.3. □

By the triangle inequality, we have

$$\begin{aligned}&\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\quad -\left\|\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\ &\le \|f(x+y)+f(x-y)-2f(x)-2f(y) \\ &\quad -\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\bigg\|.\end{aligned}$$

As corollaries of Theorems 2.3 and 2.4, we obtain the Hyers-Ulam stability results for the quadratic ρ-functional equation (2.3) in β-homogeneous complex Banach spaces.

Corollary 2.5. Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that

$$\begin{aligned}&\|f(x+y)+f(x-y)-2f(x)-2f(y)\\&-\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\bigg\| \le \theta(\|x\|^r+\|y\|^r)\end{aligned} \tag{2.10}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying (2.5).

Corollary 2.6. Let $r < \frac{2\beta_2}{\beta_1}$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (2.10). Then there exists a unique quadratic mapping $h : X \to Y$ satisfying (2.8).

Remark 2.7. If ρ is a real number such that $-1 < \rho < 1$ and Y is a β_2-homogeneous real Banach space, then all the assertions in this section remain valid.

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