Int. J. Nonlinear Anal. Appl. 8 (2017) No. 1, 23-32
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.10557.1508

On genuine Lupaş-Beta operators and modulus of continuity

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(Communicated by M. Eshaghi)

Abstract

In the present article we discuss approximation properties of genuine Lupaş-Beta operators of integral type. We establish quantitative asymptotic formulae and a direct estimate in terms of Ditzian-Totik modulus of continuity. Finally we mention results on the weighted modulus of continuity for the genuine operators.

Keywords: factorial polynomials; Beta basis function; direct estimates; weighted modulus of continuity; K-functionals.
2010 MSC: Primary 11B83; Secondary 35A23.

1. Introduction

Very recently Aral and Gupta [4] considered the Durrmeyer type integral modification of the Lupaş operators, based on generalized Bernstein polynomials. In the year 1995 Lupaş [11] proposed yet another important discrete operator as

$$L_n(f,x) = \sum_{k=0}^{\infty} l_{n,k}(x) f(k/n), x \in [0,\infty) \tag{1.1}$$

where

$$l_{n,k}(x) = 2^{-nx}\frac{(nx)_k}{k!.2^k}.$$

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Received: April 2016 *Revised:* November 2016

Abel [1] considered the general form of the operators (1.1) and established the complete asymptotic expansion of these operators. In the last four decades several operators have been appropriately modified and their approximation behaviour has been discussed in real and complex domain see for instance [5], [7], [8], [12], [13] and [14] etc. In order to modify the operators (1.1), Govil et al. in [6] considered the hybrid operators by taking the weights of Szász basis functions. Also, In this continuation Gupta and Yadav [10] considered other hybrid operators by taking weights of Beta basis functions. But the operators considered in [10] reproduce only the constant functions. Later in [9] Gupta, Rassias and Yadav considered the following form of hybrid operators, which preserve constant as well as linear functions

$$D_n(f,x) = \sum_{k=1}^{\infty} l_{n,k}(x) \int_0^{\infty} b_{n,k-1}(t) f(t)dt + 2^{-nx} f(0),\ x \geq 0$$

where

$$l_{n,k}(x) = 2^{-nx} \frac{(nx)_k}{k!.2^k}\ , \quad b_{n,k-1}(t) = \frac{1}{B(k,n+1)} \frac{t^{k-1}}{(1+t)^{k+n+1}}$$

and $B(m,n)$ being the Beta function. In the present article, we establish a quantitative asymptotic formula in terms of weighted modulus of continuity and a direct result in terms of Ditzian-Totik modulus of continuity. We also show the validity of our operators for the weighted modulus of continuity due to Păltănea [15].

2. Moments

It was observed in [9] that

$$D_n(t^r,x) = \frac{r!(n-r)!.(nx)}{n!} ._2F_1(nx+1, 1-r; 2; -1).$$

Some other methods can be used to find the moments. In this section we find moments by using the factorial polynomials, defined as

$$k^{(m)} = k(k-1)(k-2)...(k-m+1)$$

and the elementary hypergeometric functions ${}_1F_0(a; -; x)$.

Lemma 2.1. *It is observed that*

$$D_n(1,x) = 1, D_n(t,x) = x, D_n(t^2,x) = \frac{nx^2+3x}{n-1}$$

$$D_n(t^3,x) = \frac{n^2x^3+9nx^2+14x}{(n-1)(n-2)}, D_n(t^4,x) = \frac{n^3x^4+18n^2x^3+83nx^2+90x}{(n-1)(n-2)(n-3)},$$

$$D_n(t^5,x) = \frac{n^4x^5+30n^3x^4+275n^2x^3+870nx^2+744x}{(n-1)(n-2)(n-3)(n-4)},$$

and

$$D_n(t^6,x) = \frac{n^5x^6+45n^4x^5+685n^3x^4+4275n^2x^3+10474nx^2+7560x}{(n-1)(n-2)(n-3)(n-4)(n-5)}.$$

Proof . Obviously, we have

$$\int_0^\infty b_{n,k-1}(t)t^m = \int_0^\infty \frac{1}{B(k,n+1)}\frac{t^{k+m-1}}{(1+t)^{k+n+1}}dt = \frac{(k+m-1)!.(n-m)!}{n!.(k-1)!}.$$

Thus using the above identity and the fact that

$${}_1F_0\left(a;-;z\right) = \sum_{k=0}^{\infty}(a)_k.\frac{z^k}{k!} = (1-z)^{-a}, |z|<1,$$

we get

$$\begin{aligned}
D_n(1,x) &= \sum_{k=1}^{\infty} l_{n,k}(x) + l_{n,0}(x) = \sum_{k=0}^{\infty} l_{n,k}(x) \\
&= 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{k!.2^k} = 2^{-nx}.{}_1F_0\left(nx;-;\frac{1}{2}\right) \\
&= 2^{-nx}\left(1-\frac{1}{2}\right)^{-nx} = 1.
\end{aligned}$$

Now

$$\begin{aligned}
D_n(t,x) &= \sum_{k=1}^{\infty} l_{n,k}(x)\frac{k}{n} = \frac{2^{-nx}}{n}\sum_{k=1}^{\infty}\frac{(nx)_k}{(k-1)!.2^k} \\
&= \frac{2^{-nx}}{n}\sum_{k=0}^{\infty}\frac{(nx)_{k+1}}{k!.2^{k+1}} = \frac{2^{-nx-1}}{n}\sum_{k=0}^{\infty}\frac{nx.(nx+1)_k}{k!.2^k} \\
&= x.2^{-nx-1}.{}_1F_0\left(nx+1;-;\frac{1}{2}\right) \\
&= x.2^{-nx-1}\left(1-\frac{1}{2}\right)^{-nx-1} = x.
\end{aligned}$$

Writing $k(k+1)$ in terms of factorial polynomials i.e. $k^2+k = k^{(2)}+2k^{(1)}$ and using $(nx)_{k+2} = nx(nx+1)(nx+2)_k$, we have

$$\begin{aligned}
D_n(t^2,x) &= \sum_{k=1}^{\infty} l_{n,k}(x)\frac{k^2+k}{n(n-1)} = \sum_{k=1}^{\infty} l_{n,k}(x)\frac{k^{(2)}+2k^{(1)}}{n(n-1)} \\
&= \frac{2^{-nx}}{n(n-1)}\left[\sum_{k=2}^{\infty}\frac{(nx)_k}{(k-2)!.2^k} + \sum_{k=1}^{\infty}\frac{2(nx)_k}{(k-1)!.2^k}\right] \\
&= \frac{2^{-nx}}{n(n-1)}\left[\sum_{k=0}^{\infty}\frac{(nx)_{k+2}}{k!.2^{k+2}} + \sum_{k=0}^{\infty}\frac{2(nx)_{k+1}}{k!.2^{k+1}}\right] \\
&= \frac{2^{-nx}}{n(n-1)}\left[\sum_{k=0}^{\infty}\frac{nx(nx+1)(nx+2)_k}{k!.2^{k+2}} + \sum_{k=0}^{\infty}\frac{2nx(nx+1)_k}{k!.2^{k+1}}\right]
\end{aligned}$$

and so

$$\begin{aligned}
D_n(t^2,x) &= \frac{2^{-nx-2}nx(nx+1)}{n(n-1)}\sum_{k=0}^{\infty}\frac{(nx+2)_k}{k!.2^k}+\frac{2^{-nx}nx}{n(n-1)}\sum_{k=0}^{\infty}\frac{(nx+1)_k}{k!.2^k}\\
&= \frac{2^{-nx-2}x(nx+1)}{(n-1)}.{}_1F_0\left(nx+2;-;\frac{1}{2}\right)+\frac{2^{-nx}x}{(n-1)}.{}_1F_0\left(nx+1;-;\frac{1}{2}\right)\\
&= \frac{2^{-nx-2}x(nx+1)}{(n-1)}.\left(1-\frac{1}{2}\right)^{-nx-2}+\frac{2^{-nx}x}{(n-1)}.\left(1-\frac{1}{2}\right)^{-nx-1}\\
&= \frac{x(nx+1)}{(n-1)}+\frac{2x}{(n-1)}=\frac{nx^2+3x}{(n-1)}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
D_n(t^3,x) &= \sum_{k=1}^{\infty} l_{n,k}(x)\frac{k^3+3k^2+2k}{n(n-1)(n-2)}\\
&= \sum_{k=1}^{\infty} l_{n,k}(x)\frac{k^{(3)}+6k^{(2)}+6k^{(1)}}{n(n-1)(n-2)}\\
&= \frac{2^{-nx}}{n(n-1)(n-2)}\left[\sum_{k=3}^{\infty}\frac{(nx)_k}{(k-3)!.2^k}+6\sum_{k=2}^{\infty}\frac{(nx)_k}{(k-2)!.2^k}+6\sum_{k=1}^{\infty}\frac{(nx)_k}{(k-1)!.2^k}\right]\\
&= \frac{2^{-nx}}{n(n-1)(n-2)}\left[\sum_{k=0}^{\infty}\frac{(nx)_{k+3}}{k!.2^{k+3}}+6\sum_{k=0}^{\infty}\frac{(nx)_{k+2}}{k!.2^{k+2}}+6\sum_{k=0}^{\infty}\frac{(nx)_{k+1}}{k!.2^{k+1}}\right]\\
&= \frac{2^{-nx-3}nx(nx+1)(nx+2)}{n(n-1)(n-2)}\sum_{k=0}^{\infty}\frac{(nx+3)_k}{k!.2^k}+6\frac{2^{-nx-2}nx(nx+1)}{n(n-1)(n-2)}\sum_{k=0}^{\infty}\frac{(nx+2)_k}{k!.2^k}\\
&\quad +6\frac{2^{-nx-1}nx}{n(n-1)(n-2)}\sum_{k=0}^{\infty}\frac{(nx+1)_k}{k!.2^k}\\
&= \frac{x(nx+1)(nx+2)}{(n-1)(n-2)}+\frac{6x(nx+1)}{(n-1)(n-2)}+\frac{6x}{(n-1)(n-2)}.
\end{aligned}$$

Next, using

$$k^4+6k^3+11k^2+6k=k^{(4)}+12k^{(3)}+36k^{(2)}+18k^{(1)},$$

we get

$$\begin{aligned}
D_n(t^4,x) &= \sum_{k=1}^{\infty} l_{n,k}(x)\frac{k^4+6k^3+11k^2+6k}{n(n-1)(n-2)(n-3)}\\
&= \frac{x(nx+1)(nx+2)(nx+3)}{(n-1)(n-2)(n-3)}+\frac{12x(nx+1)(nx+2)}{(n-1)(n-2)(n-3)}\\
&\quad +\frac{36x(nx+1)}{(n-1)(n-2)(n-3)}+\frac{18x}{(n-1)(n-2)(n-3)}.
\end{aligned}$$

Further using

$$k^5+10k^4+35k^3+50k^2+24k=k^{(5)}+20k^{(4)}+120k^{(3)}+240k^{(2)}+120k^{(1)},$$

we have

$$\begin{aligned}
D_n(t^5,x) &= \sum_{k=1}^{\infty} l_{n,k}(x)\frac{k^5+10k^4+35k^3+50k^2+24k}{n(n-1)(n-2)(n-3)(n-4)}\\
&= \frac{x(nx+1)(nx+2)(nx+3)(nx+4)}{(n-1)(n-2)(n-3)(n-4)} + \frac{20x(nx+1)(nx+2)(nx+3)}{(n-1)(n-2)(n-3)(n-4)}\\
&\quad + \frac{120x(nx+1)(nx+2)}{(n-1)(n-2)(n-3)(n-4)} + \frac{240x(nx+1)}{(n-1)(n-2)(n-3)(n-4)}\\
&\quad + \frac{120x}{(n-1)(n-2)(n-3)(n-4)}.
\end{aligned}$$

Finally using

$$k^6+15k^5+85k^4+225k^3+274k^2+120k = k^{(6)}+30k^{(5)}+300k^{(4)}+1200k^{(3)}+1800k^{(2)}+720k^{(1)},$$

we get

$$\begin{aligned}
D_n(t^6,x) &= \sum_{k=1}^{\infty} l_{n,k}(x)\frac{k^6+15k^5+85k^4+225k^3+274k^2+120k}{n(n-1)(n-2)(n-3)(n-4)(n-5)}\\
&= \frac{x(nx+1)(nx+2)(nx+3)(nx+4)(nx+5)}{(n-1)(n-2)(n-3)(n-4)(n-4)(n-5)} + \frac{30x(nx+1)(nx+2)(nx+3)(nx+4)}{(n-1)(n-2)(n-3)(n-4)(n-5)}\\
&\quad + \frac{300x(nx+1)(nx+2)(nx+3)}{(n-1)(n-2)(n-3)(n-4)(n-5)} + \frac{1200x(nx+1)(nx+2)}{(n-1)(n-2)(n-3)(n-4)(n-5)}\\
&\quad + \frac{1800x(nx+1)}{(n-1)(n-2)(n-3)(n-4)(n-5)} + \frac{720x}{(n-1)(n-2)(n-3)(n-4)(n-5)}.
\end{aligned}$$

□

Remark 2.2. If $\mu_{n,m}(x) = D_n((t-x)^m,x)$, then by simple computation, we have

$$\mu_{n,1}(x) = 0, \mu_{n,2}(x) = \frac{x(x+3)}{n-1},$$

$$\begin{aligned}
\mu_{n,4}(x) &= D_n((t-x)^4,x) = D_n(t^4,x) - 4xD_n(t^3,x) + 6x^2D_n(t^2,x) - 4x^3D_n(t,x) + x^4\\
&= \frac{3(n+6)x^4+18(n+6)x^3+(27n+168)x^2+90x}{(n-1)(n-2)(n-3)},
\end{aligned}$$

$$\begin{aligned}
\mu_{n,6}(x) &= D_n(t^6,x) - 6xD_n(t^5,x) + 15x^2D_n(t^4,x) - 20x^3D_n(t^3,x)\\
&\quad + 15x^4D_n(t^2,x) - 6x^5D_n(t,x) + x^6\\
&= D_n(t^6,x) - 6xD_n(t^5,x) + 15x^2D_n(t^4,x) - 20x^3D_n(t^3,x) + 15x^4D_n(t^2,x) - 5x^6\\
&= \frac{n^5x^6+45n^4x^5+685n^3x^4+4275n^2x^3+10474nx^2+7560x}{(n-1)(n-2)(n-3)(n-4)(n-5)}\\
&\quad - 6x\frac{n^4x^5+30n^3x^4+275n^2x^3+870nx^2+744x}{(n-1)(n-2)(n-3)(n-4)} + 15x^2\frac{n^3x^4+18n^2x^3+83nx^2+90x}{(n-1)(n-2)(n-3)}\\
&\quad - 20x^3\frac{n^2x^3+9nx^2+14x}{(n-1)(n-2)} + 15x^4\frac{nx^2+3x}{(n-1)} - 5x^6
\end{aligned}$$

and so

$$\mu_{n,6}(x) = \frac{1}{(n-1)(n-2)(n-3)(n-4)(n-5)}\Big[(15n^2+430n+600)x^6 + (135n^2+3870n+5400)x^5 + (405n^2+11740n+16800)x^4 + (405n^2+13950n+27000)x^3 + (6010n+22320)x^2 + 7560x\Big].$$

We may note here that

$$\mu_{n,m}(x) = O(n^{-[(m+1)/2]}).$$

3. Direct Estimates

Let $C_{x^2}[0,\infty) = C[0,\infty) \cap B_{x^2}[0,\infty)$, where $B_{x^2}[0,\infty)$ is the set of all functions f defined on $\mathbb{R}^+$ satisfying the condition $|f(x)| \le M_f(1+x^2)$ with some constant M_f, depending only on f, but independent of x and by $C^k_{x^2}[0,\infty)$, we denote subspace of all continuous functions $f \in B_{x^2}[0,\infty)$ for which $\lim\limits_{x\to\infty} \frac{f(x)}{1+x^2}$ is finite.

The weighted modulus of continuity $\Omega(f,\delta)$ defined on an infinite interval $\mathbb{R}^+ = [0,\infty)$ (see [2]) is given by

$$\mathbf{\Omega}(f,\delta) = \sup_{|h|<\delta,\ x\in\mathbb{R}^+} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)} \text{ for each } f\in C_{x^2}[0,\infty).$$

Some elementary properties of $\Omega(f,\delta)$ are collected in the following lemma.

Lemma 3.1. [2] Let $f \in C^k_{x^2}[0,\infty)$. Then,

i) $\Omega(f,\delta)$ is a monotonically increasing function of δ, $\delta \ge 0$.

ii) For every $f \in C^k_{x^2}[0,\infty)$, $\lim\limits_{\delta\to 0}\Omega(f,\delta) = 0$.

iii) For each $\lambda > 0$, we have $\Omega(f,\lambda\delta) \le 2(1+\lambda)(1+\delta^2)\Omega(f,\delta)$.

We now estimate the following quantitative Voronovskaja type asymptotic Formula:

Theorem 3.2. *Let $f'' \in C^k_{x^2}[0,\infty)$, and $x > 0$. Then, we have*

$$\left|D_n(f,x) - f(x) - \frac{x(x+3)}{2(n-1)}f''(x)\right| \le 8\left(1+x^2\right)\ \mathcal{O}\left(n^{-1}\right)\Omega\left(f'', \frac{1}{\sqrt{n}}\right).$$

Proof . By the Taylor's formula, there exist η lying between x and y such that

$$f(y) = f(x) + f'(x)(y-x) + \frac{f''(x)}{2}(y-x)^2 + h(y,x)(y-x)^2,$$

where

$$h(y,x) := \frac{f''(\eta) - f''(x)}{2}$$

and h is a continuous function which vanishes at 0. Applying the operator D_n to above equality, we get

$$D_n(f,x) - f(x) = \frac{f''(x)}{2}\left[\frac{x(x+3)}{n-1}\right] + D_n\left(h(y,x)(y-x)^2, x\right).$$

Also, we can write that

$$\left|D_n\left(f,x\right)-f\left(x\right)-\frac{f''\left(x\right)}{2}\left(\frac{x(x+3)}{n-1}\right)\right|\leq D_n\left(\left|h\left(y,x\right)\right|\left(y-x\right)^2,x\right)$$

To estimate last inequality using Lemma 3.1 and the inequality $|\eta-x|\leq|y-x|$, we can write that

$$|h\left(y,x\right)|\leq\left(1+(y-x)^2\right)\left(1+x^2\right)\left(1+\frac{|y-x|}{\delta}\right)\left(1+\delta^2\right)\Omega\left(f'',\delta\right).$$

Also,

$$|h\left(y,x\right)|\leq\left\{\begin{array}{c}2\left(1+x^2\right)\left(1+\delta^2\right)^2\Omega\left(f'',\delta\right),|y-x|<\delta\\ \left(1+(y-x)^2\right)\left(1+x^2\right)\left(1+\frac{|y-x|}{\delta}\right)\left(1+\delta^2\right)\Omega\left(f'',\delta\right),|y-x|\geq\delta\end{array}\right.$$

Now choosing $\delta<1$, we have

$$\begin{aligned}|h\left(y,x\right)|&\leq 2\left(1+x^2\right)\left(1+\frac{\left(y-x\right)^4}{\delta^4}\right)\left(1+\delta^2\right)^2\Omega\left(f'',\delta\right)\\&\leq 8\left(1+x^2\right)\left(1+\frac{\left(y-x\right)^4}{\delta^4}\right)\Omega\left(f'',\delta\right).\end{aligned}$$

Using Remark 2.2, we deduce that

$$\begin{aligned}D_n\left(\left|h\left(y,x\right)\right|\left(y-x\right)^2,x\right)&=8\left(1+x^2\right)\Omega\left(f'',\delta\right)\left\{D_n((t-x)^2,x)+\frac{1}{\delta^4}D_n((t-x)^6,x)\right\}\\&=8\left(1+x^2\right)\Omega\left(f'',\delta\right)\left\{\mathcal{O}\left(n^{-1}\right)+\frac{1}{\delta^4}\mathcal{O}\left(n^{-3}\right)\right\}.\end{aligned}$$

Finally, choosing $\delta=1/\sqrt{n}$, we have

$$D_n\left(\left|h\left(y,x\right)\right|\left(y-x\right)^2,x\right)=8\left(1+x^2\right)\Omega\left(f'',\delta\right).\mathcal{O}\left(n^{-1}\right).$$

This completes the proof of the theorem. □

4. Direct Result in Terms of DT-Modulus of Continuity.

By $C_B[0,\infty)$, we denote the class of all real valued continuous and bounded functions f on $[0,\infty)$. The second order Ditzian-Totik modulus of smoothness is defined by:

$$\omega_\varphi^2(f,\delta)=\sup_{0\leq h\leq\delta}\ \sup_{x\pm h\varphi(x)\in[0,\infty)}|f(x+h\varphi(x))-2f(x)+f(x-h\varphi(x))|,$$

$\varphi(x)=\sqrt{x(x+3)}$, $x\geq 0$. The corresponding K-functional is:

$$K_{2,\varphi}(f,\delta^2)=\inf_{h\in W_\infty^2(\varphi)}\{||f-h||+\delta^2||\varphi^2h''||\},$$

where $W_\infty^2(\varphi)=\{h\in C_B\left[0,\infty\right):h'\in AC_{loc}\left[0,\infty\right):\varphi^2h''\in C_B\left[0,\infty\right)\}$. By Th. 2.1.1 of [3], it follows that

$$C^{-1}\omega_\varphi^2(f,\delta)\leq K_{2,\varphi}(f,\delta^2)\leq C\omega_\varphi^2(f,\delta),$$

for some absolute constant $C>0$.

Our following direct result is in terms of Ditzian-Totik modulus of smoothness:

Theorem 4.1. *If $f \in C_B[0,\infty)$ and $n \in \mathbb{N}$, then we have the following inequality:*

$$\|D_n\,(f,x) - f\,(x)\,\| \leq 4\,\omega_\varphi^2\bigg(f, \frac{1}{\sqrt{n}}\bigg).$$

Proof . We set $\varphi(x) = \sqrt{x(x+3)}$, $W_\varphi^2\,[0,\infty) = \{g \in AC_{loc}\,[0,\infty) : \varphi^2 g'' \in C_B\,[0,\infty)\}$ then,

$$\frac{|t-u|}{\varphi^2(u)} \leq \frac{|t-x|}{\varphi^2(x)} \quad \text{for } u \text{ between } x \text{ and } t,$$

and for $g \in W_\varphi^2\,[0,\infty)$, by Taylor's formula, we have:

$$g(t) = g(x) + g'(x)\,(t-x) + \int_x^t g''(u)\,(t-u)\,du.$$

Applying the operator D_n to above equality and then taking modulus, we get:

$$\begin{aligned}|D_n(g,x) - g(x)| &\leq D_n\bigg(\bigg|\int_x^t (t-u)g''(u)du\bigg|, x\bigg)\\ &\leq \|\varphi^2 g''\|\frac{D_n\,((t-x)^2, x)}{x(x+3)} = \frac{1}{n-1}\|\varphi^2 g''\|.\end{aligned}$$

Now for $f \in C_B\,[0,\infty)$, we have

$$\begin{aligned}|D_n(f,x) - f(x)| &= |D_n(f-g,x) - (f-g)(x)| + |D_n(g,x) - g(x)|\\ &\leq 4\|f-g\| + \frac{1}{n-1}\|\varphi^2 g''\|\\ &\leq 4\bigg\{\|f-g\| + \frac{1}{n-1}\|\varphi^2 g''\|\bigg\}.\end{aligned}$$

Hence, by definition of $K_{2,\varphi}(f,\delta^2)$, we have the inequality:

$$\|D_n(f,x) - f(x)\| \leq 4\,K_{2,\varphi}\left(f, \frac{1}{n}\right) \leq 4\,\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right).$$

□

5. Applications to Weighted Modulus of Continuity

Păltănea in [15] considered the weighted modulus of continuity $\omega_\varphi(f;h)$:

$$\omega_\varphi(f;h) = \sup\left\{|f(x) - f(y)| : x \geq 0, y \geq 0, |x-y| \leq h\varphi\left(\frac{x+y}{2}\right)\right\}, h \geq 0$$

where $\varphi(x) = \frac{\sqrt{x}}{1+x^m}, x \in [0,\infty), m \in \mathbb{N}, m \geq 2$. We consider here those functions, for which we have the property

$$\lim_{h\to 0} \omega_\varphi(f;h) = 0.$$

It is easy to verify that this property is fulfilled for f an algebraic polynomial of degree $\leq m$. This follows from Theorem 2 in [15], which states that $\lim_{h\to 0} \omega_\varphi(f;h) = 0$ whenever the function $f \circ e_2$ is uniformly continuous on $[0,1]$ and the function $f \circ e_v, v = \frac{2}{2m+1}$ is uniformly continuous on $[1,\infty)$, where $e_v(x) = x^v, x \geq 0$. By E, we denote the subspace of $C[0,\infty)$ which contains the polynomials.

Let us denote by $W_\varphi[0,\infty)$ the subspace of all real functions defined on $[0,\infty)$, for which the two conditions mentioned above hold true. Recently Tachev and Gupta [16] established quantitative estimates in terms of the above Păltănea's modulus of continuity. The operators discussed here also preserve linear functions, so can be applied with this modulus of continuity. By the same arguments as in [16] the terms $C_{n,2,m}(x)$, $A_{n,m,x}$ are bounded for fixed x and m, when $n \to \infty$.

Also, we can write

$$\begin{aligned} &D_n\left(\left[1+\left(x+\frac{|t-x|}{2}\right)^m\right]^2; x\right) \\ &= 1 + 2.\sum_{k=0}^{m}\binom{m}{k}x^k D_n(|t-x|^{m-k},x).\frac{1}{2^{m-k}} + \sum_{k=0}^{2m}\binom{2m}{k}x^k D_n(|t-x|^{2m-k},x)\frac{1}{2^{2m-k}} \\ &= A_{n,m,x}. \end{aligned} \tag{5.1}$$

It is easy to verify that for fixed x and m, the term $A_{n,m,x}$ defined in (5.1) is bounded when $n \to \infty$.

We apply Theorem 2.2 and Theorem 2.3 of [16] and for our operator, to obtain the following results:

Theorem 5.1. If $f \in C^2[0,\infty) \cap E$ and $f'' \in W_\varphi[0,\infty)$, then we have for $x \in (0,\infty)$ that

$$\begin{aligned} \left|D_n(f;x) - f(x) - \frac{x(x+3)}{2(n-1)}f''(x)\right| &\leq \frac{1}{2}\left[\frac{x(x+3)}{n-1} + \sqrt{2A_{n,m,x}}\right] \\ &\quad .\omega_\varphi\left(f''; \sqrt{\frac{\mu_{n,6}(x)}{x}}\right), \end{aligned}$$

where $A_{n,m,x}$ is given by (5.1) and $\mu_{n,6}(x)$ is given as in Remark 2.2.

To obtain the quantitative variant of Voronovskaja's theorem for our operator is as follows:

Theorem 5.2. If $f \in C^2[0,\infty) \cap E$ and $f'' \in W_\varphi[0,\infty)$, then we have for $x \in (0,\infty)$ that

$$\begin{aligned} &\left|(n-1)\left[D_n(f;x) - f(x) - \frac{x(x+3)}{n-1}f''(x)\right]\right| \\ &\leq \frac{1}{2}\left[x(x+3) + \sqrt{2}\sqrt{x}(x+3)C_{n,2,m}(x)\right]\omega_\varphi\left(f; \sqrt{\frac{\mu_{n,4}(x)}{\mu_{n,2}(x)}}\right), \end{aligned}$$

Where $\mu_{n,4}(x), \mu_{n,2}(x)$ are given as in Remark 2.2 and

$$C_{n,2,m}(x) = 1 + \frac{1}{D_n(|t-x|^3,x)}.\sum_{k=0}^{m}\binom{m}{k}x^{m-k}\frac{D_n(|t-x|^{k+3},x)}{2^k}.$$

We suppose for the operators D_n that

$$\frac{D_n(|t-x|^k,x)}{D_n(|t-x|^3,x)},\ 4\leq k\leq m$$

is a bounded ratio for fixed x and m, when $n\to\infty$.

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