



Asymptotic behavior of a system of two difference equations of exponential form

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Abstract

In this paper, we study the boundedness and persistence of the solutions, the global stability of the unique positive equilibrium point and the rate of convergence of a solution that converges to the equilibrium $E = (\bar{x}, \bar{y})$ of the system of two difference equations of exponential form:

$$x_{n+1} = \frac{a + e^{-(bx_n + cy_n)}}{d + bx_n + cy_n}, \quad y_{n+1} = \frac{a + e^{-(by_n + cx_n)}}{d + by_n + cx_n}$$

where a, b, c, d are positive constants and the initial values x_0, y_0 are positive real values.

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1. Introduction and preliminaries

Difference equations have many applications in applied sciences, there are many papers and books that can be found concerning the theory and applications of difference equations, see [1, 4, 6, 7] and the references cited therein. Recently, there has been a great interest in studying the qualitative properties of difference equations and systems of difference equations of exponential form [3, 10, 11, 12, 15, 17, 18]. In [3], the authors studied the boundedness, the asymptotic behavior, the periodic character of the solutions and the stability character of the positive equilibrium of the difference equation:

$$x_{n+1} = a + bx_{n-1}e^{-x_n},$$

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where a, b are positive constants and the initial values x_{-1}, x_0 are positive numbers.

In [10], the authors studied the boundedness, the asymptotic behavior, the periodicity and the stability of the positive solutions of the difference equation:

$$y_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}}$$

where α, β, γ are positive constants and the initial values y_{-1}, y_0 are positive numbers. In [5], the authors studied the boundedness, the asymptotic behavior and the rate of convergence of the positive solutions of the system of two difference equations:

$$x_{n+1} = \frac{a + be^{-x_n}}{c + y_n}, \quad y_{n+1} = \frac{a + be^{-y_n}}{c + x_n}$$

where a, b, c are positive constants and the initial values x_0, y_0 are positive numbers.

In [13], the author investigate the boundedness, the persistence and the asymptotic behavior of the positive solutions of the system of two difference equations of exponential form:

$$\begin{aligned} x_{n+1} &= a + bx_{n-1} + cx_{n-1}e^{-y_n} \\ y_{n+1} &= \alpha + \beta y_{n-1} + \gamma y_{n-1}e^{-x_n} \end{aligned}$$

where $a, b, c, \alpha, \beta, \gamma$ are positive constants and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive numbers.

Motivated by these above papers, we will investigate the boundedness, the persistence and the asymptotic behavior of the positive solutions of the following system of exponential form:

$$x_{n+1} = \frac{a + e^{-(bx_n + cy_n)}}{d + bx_n + cy_n}, \quad y_{n+1} = \frac{a + e^{-(by_n + cx_n)}}{d + by_n + cx_n} \quad (1.1)$$

where a, b, c, d are positive constants and the initial values x_0, y_0 are positive real values. Moreover, we establish the rate of convergence of a solution that converges to the equilibrium $E = (\bar{x}, \bar{y})$ of (1.1).

2. Global behavior of solutions of system (1.1)

In the following lemma we will show that every positive solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of Eq. (1.1) is bounded and persists.

Lemma 2.1. *Every positive solution of Eq. (1.1) is bounded and persists.*

Proof . Let (x_n, y_n) be an arbitrary solution of (1.1). From (1.1) we can see that

$$x_n \leq \frac{a+1}{d}, \quad y_n \leq \frac{a+1}{d}, \quad n = 1, 2, \dots \quad (2.1)$$

In addition, from Eq. (1.1) and Eq. (2.1) we get

$$x_n \geq \frac{a + e^{-\frac{(b+c)(a+1)}{d}}}{d + \frac{(b+c)(a+1)}{d}}, \quad y_n \geq \frac{a + e^{-\frac{(b+c)(a+1)}{d}}}{d + \frac{(b+c)(a+1)}{d}}, \quad n = 2, 3, \dots \quad (2.2)$$

Therefore, from Eq. (2.1) and Eq. (2.2) the proof of lemma is complete. \square

The next lemma establishes an invariant set for the system (1.1).

Lemma 2.2. Let $\{(x_n, y_n)\}_{n=0}^\infty$ be a positive solution of the system (1.1). Then $\left[\frac{a+e^{-\frac{(b+c)(a+1)}{d}}}{d+\frac{(b+c)(a+1)}{d}}, \frac{a+1}{d}\right] \times \left[\frac{a+e^{-\frac{(b+c)(a+1)}{d}}}{d+\frac{(b+c)(a+1)}{d}}, \frac{a+1}{d}\right]$ is an invariant set for the system (1.1).

Proof . It follows from induction. \square

The following result will be useful in establishing the global attractivity character of the equilibrium of Eq. (1.1).

Theorem 2.3. [2, 7] Let $\mathcal{R} = [a_1, b_1] \times [c_1, d_1]$ and

$$f : \mathcal{R} \longrightarrow [a_1, b_1], g : \mathcal{R} \longrightarrow [c_1, d_1]$$

be a continuous functions such that:

- (a) $f(x, y)$ is decreasing in both variables and $g(x, y)$ is decreasing in both variables for each $(x, y) \in \mathcal{R}$;
- (b) If $(m_1, M_1, m_2, M_2) \in \mathcal{R}^2$ is a solution of

$$\begin{cases} M_1 = f(m_1, m_2), m_1 = f(M_1, M_2) \\ M_2 = g(m_1, m_2), m_2 = g(M_1, M_2) \end{cases} \tag{2.3}$$

then $m_1 = M_1$ and $m_2 = M_2$. Then the following system of difference equations:

$$x_{n+1} = f(x_n, y_n), y_{n+1} = g(x_n, y_n) \tag{2.4}$$

has a unique equilibrium (\bar{x}, \bar{y}) and every solution (x_n, y_n) of the system Eq. (2.4) with $(x_0, y_0) \in \mathcal{R}$ converges to the unique equilibrium (\bar{x}, \bar{y}) . In addition, the equilibrium (\bar{x}, \bar{y}) is globally asymptotically stable.

Now we state the main theorem of this section.

Theorem 2.4. Consider system Eq. (1.1). Suppose that the following relation holds true:

$$d > b + c. \tag{2.5}$$

Then system Eq. (1.1) has a unique positive equilibrium (\bar{x}, \bar{y}) and every positive solution of Eq. (1.1) tends to the unique positive equilibrium (\bar{x}, \bar{y}) as $n \rightarrow \infty$. In addition, the equilibrium (\bar{x}, \bar{y}) is globally asymptotically stable.

Proof . We consider the functions

$$f(u, v) = \frac{a + e^{-(bu+cv)}}{d + bu + cv}, g(u, v) = \frac{a + e^{-(bv+cu)}}{d + bv + cu} \tag{2.6}$$

where

$$u, v \in I = \left[\frac{a + e^{-\frac{(b+c)(a+1)}{d}}}{d + \frac{(b+c)(a+1)}{d}}, \frac{a + 1}{d}\right]. \tag{2.7}$$

It is easy to see that $f(u, v), g(u, v)$ are decreasing in both variables for each $(u, v) \in I \times I$. In addition, from (2.6) and (2.7) we have $f(u, v) \in I, g(u, v) \in I$ as $(u, v) \in I \times I$ and so $f : I \times I \longrightarrow I, g : I \times I \longrightarrow I$.

Now let m_1, M_1, m_2, M_2 be positive real numbers such that

$$\begin{aligned} M_1 &= \frac{a + e^{-(bm_1+cm_2)}}{d + bm_1 + cm_2}, & m_1 &= \frac{a + e^{-(bM_1+cM_2)}}{d + bM_1 + cM_2}, \\ M_2 &= \frac{a + e^{-(bm_2+cm_1)}}{d + bm_2 + cm_1}, & m_2 &= \frac{a + e^{-(bM_2+cM_1)}}{d + bM_2 + cM_1}. \end{aligned} \tag{2.8}$$

Moreover arguing as in the proof of Theorem 2.3, it suffices to assume that

$$m_1 \leq M_1, \quad m_2 \leq M_2. \tag{2.9}$$

From (2.8) we get

$$\begin{aligned} M_1d + bm_1M_1 + cm_2M_1 &= a + e^{-(bm_1+cm_2)}, \\ m_1d + bm_1M_1 + cm_1M_2 &= a + e^{-(bM_1+cM_2)}, \\ M_2d + bm_2M_2 + cm_1M_2 &= a + e^{-(bm_2+cm_1)}, \\ m_2d + bm_2M_2 + cm_2M_1 &= a + e^{-(bM_2+cM_1)}. \end{aligned} \tag{2.10}$$

From (2.10) we obtain

$$\begin{aligned} d(M_1 - m_1) + cM_1(m_2 - M_2) + cM_2(M_1 - m_1) &= e^{-(bm_1+cm_2)} - e^{-(bM_1+cM_2)}, \\ d(M_2 - m_2) + cM_2(m_1 - M_1) + cM_1(M_2 - m_2) &= e^{-(bm_2+cm_1)} - e^{-(bM_2+cM_1)}. \end{aligned} \tag{2.11}$$

Then by adding the two relations Eq. (2.11) we obtain

$$\begin{aligned} &d(M_1 - m_1) + d(M_2 - m_2) \\ &= e^{-(bm_1+cm_2+bM_1+cM_2)+\theta_1} [b(M_1 - m_1) + c(M_2 - m_2)] \\ &+ e^{-(bm_2+cm_1+bM_2+cM_1)+\theta_2} [b(M_2 - m_2) + c(M_1 - m_1)], \end{aligned} \tag{2.12}$$

where $bm_1 + cm_2 \leq \theta_1 \leq bM_1 + cM_2, \quad bm_2 + cm_1 \leq \theta_2 \leq bM_2 + cM_1$.

Therefore from Eq. (2.12) we have

$$\begin{aligned} &(M_1 - m_1)(d - be^{-(bm_1+cm_2+bM_1+cM_2)+\theta_1} - ce^{-(bm_2+cm_1+bM_2+cM_1)+\theta_2}) \\ &+ (M_2 - m_2)(d - ce^{-(bm_1+cm_2+bM_1+cM_2)+\theta_1} - be^{-(bm_2+cm_1+bM_2+cM_1)+\theta_2}) = 0. \end{aligned} \tag{2.13}$$

Then using (2.5), (2.9) and (2.13), gives us $m_1 = M_1$ and $m_2 = M_2$. Hence from Theorem.2.3 system Eq. (1.1) has a unique positive equilibrium (\bar{x}, \bar{y}) and every positive solution of Eq. (1.1) tends to the unique positive equilibrium (\bar{x}, \bar{y}) as $n \rightarrow \infty$. In addition, the equilibrium (\bar{x}, \bar{y}) is globally asymptotically stable. This completes the proof of the theorem. \square

3. Rate of convergence

In this section we give the rate of convergence of a solution that converges to the equilibrium $E = (\bar{x}, \bar{y})$ of the systems (1.1) for all values of parameters. The rate of convergence of solutions that converge to an equilibrium has been obtained for some two-dimensional systems in [8] and [9].

The following results give the rate of convergence of solutions of a system of difference equations

$$\mathbf{x}_{n+1} = [A + B(n)]\mathbf{x}_n \tag{3.1}$$

where \mathbf{x}_n is a k -dimensional vector, $A \in \mathbf{C}^{k \times k}$ is a constant matrix, and $B : \mathbb{Z}^+ \rightarrow \mathbf{C}^{k \times k}$ is a matrix function satisfying

$$\|B(n)\| \rightarrow 0 \text{ when } n \rightarrow \infty, \tag{3.2}$$

where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm; $\|\cdot\|$ also denotes the Euclidean norm in \mathbb{R}^2 given by

$$\|\mathbf{x}\| = \|(x, y)\| = \sqrt{x^2 + y^2}. \tag{3.3}$$

Theorem 3.1. ([14]) Assume that condition (3.2) holds. If \mathbf{x}_n is a solution of system (3.1), then either $\mathbf{x}_n = 0$ for all large n or

$$\rho = \lim_{n \rightarrow \infty} \sqrt[n]{\|\mathbf{x}_n\|} \tag{3.4}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A .

Theorem 3.2. ([14]) Assume that condition (3.2) holds. If \mathbf{x}_n is a solution of system (3.1), then either $\mathbf{x}_n = 0$ for all large n or

$$\rho = \lim_{n \rightarrow \infty} \frac{\|\mathbf{x}_{n+1}\|}{\|\mathbf{x}_n\|} \tag{3.5}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A .

The equilibrium point of the system (1.1) satisfies the following system of equations

$$\begin{cases} \bar{x} = \frac{a + e^{-(b\bar{x}+c\bar{y})}}{d + b\bar{x} + c\bar{y}} \\ \bar{y} = \frac{a + e^{-(b\bar{y}+c\bar{x})}}{d + b\bar{y} + c\bar{x}} \end{cases} \tag{3.6}$$

The map T associated to the system (1.1) is

$$T(x, y) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} = \begin{pmatrix} \frac{a + e^{-(bx+cy)}}{d + bx + cy} \\ \frac{a + e^{-(by+cx)}}{d + by + cx} \end{pmatrix} \tag{3.7}$$

The Jacobian matrix of T is

$$J_T(x, y) = \begin{pmatrix} \frac{-b[a + (d + bx + cy + 1)e^{-(bx+cy)}]}{(d + bx + cy)^2} & \frac{-c[a + (d + bx + cy + 1)e^{-(bx+cy)}]}{(d + bx + cy)^2} \\ \frac{-c[a + (d + by + cx + 1)e^{-(by+cx)}]}{(d + by + cx)^2} & \frac{-b[a + (d + by + cx + 1)e^{-(by+cx)}]}{(d + by + cx)^2} \end{pmatrix} \tag{3.8}$$

By using the system (3.6), value of the Jacobian matrix of T at the equilibrium point $E = (\bar{x}, \bar{y})$ is

$$J_T(\bar{x}, \bar{y}) = \begin{pmatrix} \frac{-b[a + (d + b\bar{x} + c\bar{y} + 1)e^{-(b\bar{x}+c\bar{y})}]}{(d + b\bar{x} + c\bar{y})^2} & \frac{-c[a + (d + b\bar{x} + c\bar{y} + 1)e^{-(b\bar{x}+c\bar{y})}]}{(d + b\bar{x} + c\bar{y})^2} \\ \frac{-c[a + (d + b\bar{y} + c\bar{x} + 1)e^{-(b\bar{y}+c\bar{x})}]}{(d + b\bar{y} + c\bar{x})^2} & \frac{-b[a + (d + b\bar{y} + c\bar{x} + 1)e^{-(b\bar{y}+c\bar{x})}]}{(d + b\bar{y} + c\bar{x})^2} \end{pmatrix} \tag{3.9}$$

Our goal in this section is to determine the rate of convergence of every solution of the system (1.1) in the regions where the parameters $a, b, c, d \in (0, \infty)$, $(d > b + c)$ and initial conditions x_0 and y_0 are arbitrary, nonnegative numbers.

Theorem 3.3. *The error vector $\mathbf{e}_n = \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix} = \begin{pmatrix} x_n - \bar{x} \\ y_n - \bar{y} \end{pmatrix}$ of every solution $(x_n, y_n) \neq (\bar{x}, \bar{y})$ of (1.1) satisfies both of the following asymptotic relations:*

$$\lim_{n \rightarrow \infty} \sqrt[n]{\|\mathbf{e}_n\|} = |\lambda_i(J_T(E))| \text{ for some } i = 1, 2, \tag{3.10}$$

and

$$\lim_{n \rightarrow \infty} \frac{\|\mathbf{e}_{n+1}\|}{\|\mathbf{e}_n\|} = |\lambda_i(J_T(E))| \text{ for some } i = 1, 2, \tag{3.11}$$

where $|\lambda_i(J_T(E))|$ is equal to the modulus of one of the eigenvalues of the Jacobian matrix evaluated at the equilibrium $J_T(E)$.

Proof . First, we will find a system satisfied by the error terms. The error terms are given as

$$\begin{aligned} x_{n+1} - \bar{x} &= \frac{a + e^{-(bx_n+cy_n)}}{d + bx_n + cy_n} - \frac{a + e^{-(b\bar{x}+c\bar{y})}}{d + b\bar{x} + c\bar{y}} \\ &= \frac{a(d + b\bar{x} + c\bar{y}) - a(d + bx_n + cy_n)}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})} \\ &\quad + \frac{(d + b\bar{x} + c\bar{y})e^{-(bx_n+cy_n)} - (d + bx_n + cy_n)e^{-(b\bar{x}+c\bar{y})}}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})} \\ &= \frac{b(\bar{x} - x_n) + c(\bar{y} - y_n) + [b(\bar{x} - x_n) + c(\bar{y} - y_n)]e^{-(bx_n+cy_n)}}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})} \\ &\quad + \frac{(d + bx_n + cy_n)[e^{-(bx_n+cy_n)} - e^{-(b\bar{x}+c\bar{y})}]}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})} \\ &= \frac{-b(x_n - \bar{x}) - c(y_n - \bar{y}) + [b(\bar{x} - x_n) + c(\bar{y} - y_n)]e^{-(bx_n+cy_n)}}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})} \\ &\quad + \frac{(d + bx_n + cy_n)e^{-(b\bar{x}+c\bar{y})}[e^{-(bx_n - b\bar{x} + cy_n - c\bar{y})} - 1]}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})} \\ &= \frac{-b(x_n - \bar{x}) - c(y_n - \bar{y}) + [b(\bar{x} - x_n) + c(\bar{y} - y_n)]e^{-(bx_n+cy_n)}}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})} \\ &\quad + \frac{(d + bx_n + cy_n)e^{-(b\bar{x}+c\bar{y})}[-b(x_n - \bar{x}) - c(y_n - \bar{y})]}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})} \\ &\quad + \frac{\mathcal{O}_1((x_n - \bar{x})) + \mathcal{O}_2((y_n - \bar{y}))}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})} \\ &= \frac{-b[a + e^{-(bx_n+cy_n)} + (d + bx_n + cy_n)e^{-(b\bar{x}+c\bar{y})}]}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})}(x_n - \bar{x}) \\ &\quad + \frac{-c[a + e^{-(bx_n+cy_n)} + (d + bx_n + cy_n)e^{-(b\bar{x}+c\bar{y})}]}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})}(y_n - \bar{y}) \\ &\quad + \frac{1}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})}\mathcal{O}_1((x_n - \bar{x})) \\ &\quad + \frac{1}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})}\mathcal{O}_2((y_n - \bar{y})) \end{aligned} \tag{3.12}$$

By calculating similarly, we get

$$\begin{aligned}
 y_{n+1} - \bar{y} &= \frac{-b [a + e^{-(by_n+cx_n)} + (d + by_n + cx_n)e^{-(b\bar{y}+c\bar{x})}]}{(d + by_n + cx_n)(d + b\bar{y} + c\bar{x})} (x_n - \bar{x}) \\
 &+ \frac{-c [a + e^{-(by_n+cx_n)} + (d + by_n + cx_n)e^{-(b\bar{y}+c\bar{x})}]}{(d + by_n + cx_n)(d + b\bar{y} + c\bar{x})} (y_n - \bar{y}) \\
 &+ \frac{1}{(d + by_n + cx_n)(d + b\bar{y} + c\bar{x})} \mathcal{O}_3((x_n - \bar{x})) \\
 &+ \frac{1}{(d + by_n + cx_n)(d + b\bar{y} + c\bar{x})} \mathcal{O}_4((y_n - \bar{y}))
 \end{aligned} \tag{3.13}$$

From (3.12) and (3.13) we have

$$\begin{aligned}
 x_{n+1} - \bar{x} &\approx \frac{-b [a + e^{-(bx_n+cy_n)} + (d + bx_n + cy_n)e^{-(b\bar{x}+c\bar{y})}]}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})} (x_n - \bar{x}) \\
 &+ \frac{-c [a + e^{-(bx_n+cy_n)} + (d + bx_n + cy_n)e^{-(b\bar{x}+c\bar{y})}]}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})} (y_n - \bar{y}) \\
 y_{n+1} - \bar{y} &\approx \frac{-b [a + e^{-(by_n+cx_n)} + (d + by_n + cx_n)e^{-(b\bar{y}+c\bar{x})}]}{(d + by_n + cx_n)(d + b\bar{y} + c\bar{x})} (x_n - \bar{x}) \\
 &+ \frac{-c [a + e^{-(by_n+cx_n)} + (d + by_n + cx_n)e^{-(b\bar{y}+c\bar{x})}]}{(d + by_n + cx_n)(d + b\bar{y} + c\bar{x})} (y_n - \bar{y}).
 \end{aligned} \tag{3.14}$$

Set

$$e_n^1 = x_n - \bar{x} \text{ and } e_n^2 = y_n - \bar{y}.$$

Then system (3.14) can be represented as:

$$\begin{aligned}
 e_{n+1}^1 &\approx a_n e_n^1 + b_n e_n^2 \\
 e_{n+1}^2 &\approx c_n e_n^1 + d_n e_n^2
 \end{aligned}$$

where

$$\begin{aligned}
 a_n &= \frac{-b [a + e^{-(bx_n+cy_n)} + (d + bx_n + cy_n)e^{-(b\bar{x}+c\bar{y})}]}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})}, \\
 b_n &= \frac{-c [a + e^{-(bx_n+cy_n)} + (d + bx_n + cy_n)e^{-(b\bar{x}+c\bar{y})}]}{(d + bx_n + cy_n)(d + b\bar{x} + c\bar{y})}, \\
 c_n &= \frac{-b [a + e^{-(by_n+cx_n)} + (d + by_n + cx_n)e^{-(b\bar{y}+c\bar{x})}]}{(d + by_n + cx_n)(d + b\bar{y} + c\bar{x})}, \\
 d_n &= \frac{-c [a + e^{-(by_n+cx_n)} + (d + by_n + cx_n)e^{-(b\bar{y}+c\bar{x})}]}{(d + by_n + cx_n)(d + b\bar{y} + c\bar{x})}.
 \end{aligned}$$

Taking the limits of a_n, b_n, c_n and d_n as $n \rightarrow \infty$, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \frac{-b [a + (d + b\bar{x} + c\bar{y} + 1)e^{-(b\bar{x}+c\bar{y})}]}{(d + b\bar{x} + c\bar{y})^2} := A_1, \\ \lim_{n \rightarrow \infty} b_n &= \frac{-c [a + (d + b\bar{x} + c\bar{y} + 1)e^{-(b\bar{x}+c\bar{y})}]}{(d + b\bar{x} + c\bar{y})^2} := B_1, \\ \lim_{n \rightarrow \infty} c_n &= \frac{-b [a + (d + b\bar{y} + c\bar{x} + 1)e^{-(b\bar{y}+c\bar{x})}]}{(d + b\bar{y} + c\bar{x})^2} := C_1, \\ \lim_{n \rightarrow \infty} d_n &= \frac{-b [a + (d + b\bar{y} + c\bar{x} + 1)e^{-(b\bar{y}+c\bar{x})}]}{(d + b\bar{y} + c\bar{x})^2} := D_1, \end{aligned}$$

that is

$$\begin{aligned} a_n &= A_1 + \alpha_n, \quad b_n = B_1 + \beta_n, \\ c_n &= C_1 + \gamma_n, \quad d_n = D_1 + \delta_n, \end{aligned}$$

where $\alpha_n \rightarrow 0, \beta_n \rightarrow 0, \gamma_n \rightarrow 0$ and $\delta_n \rightarrow 0$ as $n \rightarrow \infty$.

Now, we have system of the form (3.1):

$$\mathbf{e}_{n+1} = (A + B(n))\mathbf{e}_n,$$

where $A = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, B(n) = \begin{pmatrix} \alpha_n & \beta_n \\ \delta_n & \gamma_n \end{pmatrix}$ and

$$\|B(n)\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus, the limiting system of error terms can be written as:

$$\begin{pmatrix} e_{n+1}^1 \\ e_{n+1}^2 \end{pmatrix} = A \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix}.$$

The system is exactly linearized system of (1.1) evaluated at the equilibrium $E = (\bar{x}, \bar{y})$. Then Theorem 3.1 and Theorem 3.2 imply the result. \square

4. Examples

In order to verify our theoretical results and to support our theoretical discussion, we consider several interesting numerical examples. These examples represent different types of qualitative behavior of solutions of the systems (1.1). All plots in this section are drawn with Matlab.

Example 4.1. Let $a = 20, b = 0.001, c = 0.5, d = 0.8, (d > b + c)$. Then system (1.1) can be written as

$$x_{n+1} = \frac{20 + e^{-(0.001x_n+0.5y_n)}}{0.8 + 0.001x_n + 0.5y_n}, \quad y_{n+1} = \frac{20 + e^{-(0.001y_n+0.5x_n)}}{0.8 + 0.001y_n + 0.5x_n} \tag{4.1}$$

with initial conditions $x_0 = 3$ and $y_0 = 6$.

In this case, the unique positive equilibrium point of the system (4.1) is given by

$$(\bar{x}, \bar{y}) = (5.579648535, 5.579648535).$$

Moreover, in Figure 1, the plot of x_n is shown in Figure 1 (a), the plot of y_n is shown in Figure 1 (b), and an attractor of the system (4.1) is shown in Figure 1 (c).

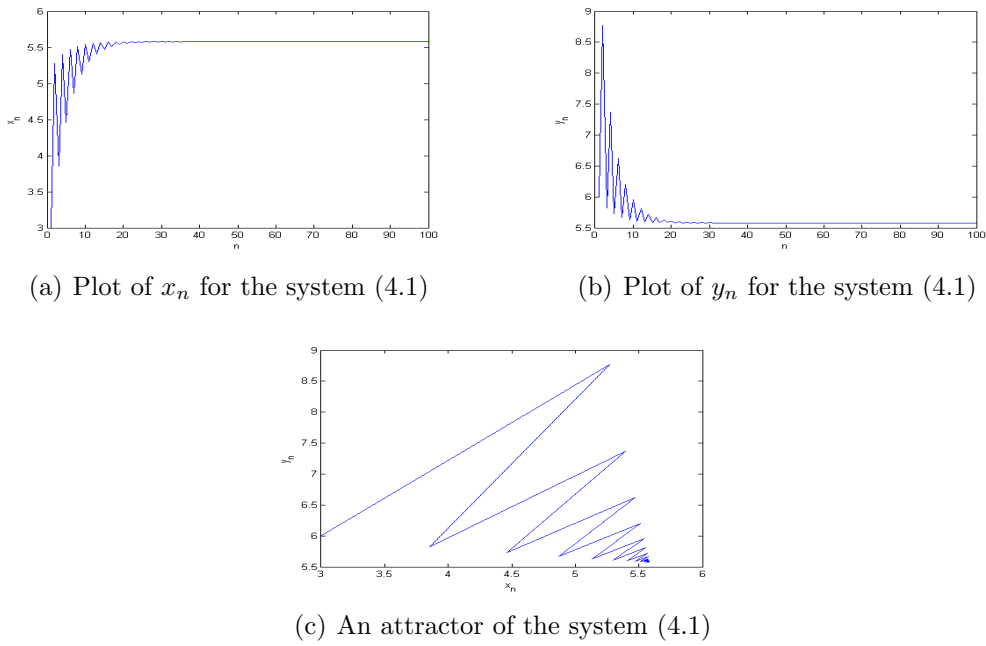


Figure 1: Plots for the system (4.1)

Example 4.2. Let $a = 30, b = 0.0007, c = 0.8, d = 0.95, (d > b + c)$. Then system (1.1) can be written as

$$x_{n+1} = \frac{30 + e^{-(0.0007x_n + 0.8y_n)}}{0.95 + 0.0007x_n + 0.8y_n}, \quad y_{n+1} = \frac{30 + e^{-(0.0007y_n + 0.8x_n)}}{0.95 + 0.0007y_n + 0.8x_n} \quad (4.2)$$

with initial conditions $x_0 = 8$ and $y_0 = 5$.

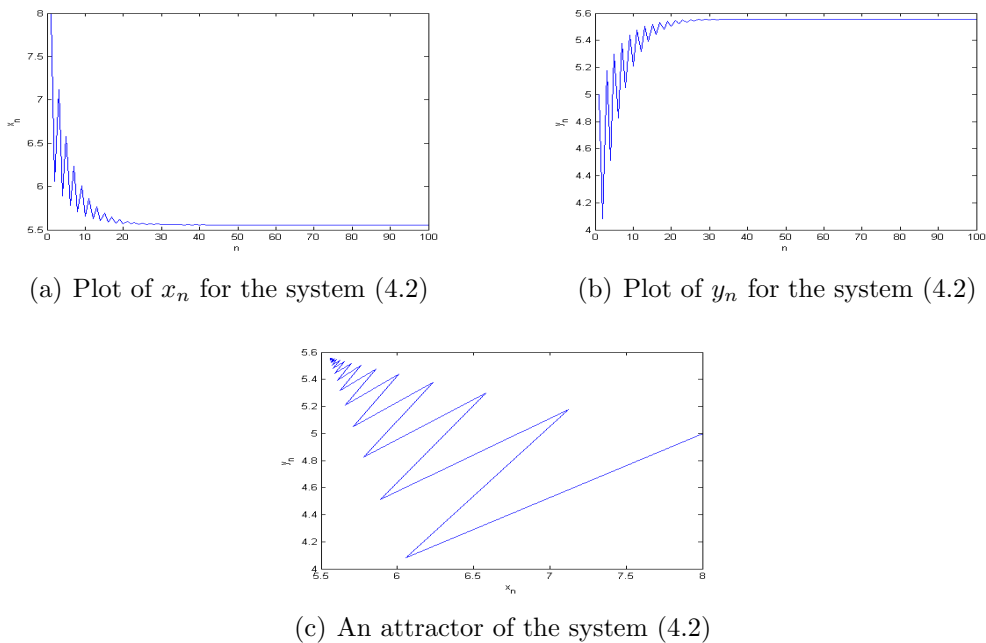


Figure 2: Plots for the system (4.2)

In this case, the unique positive equilibrium point of the system (4.2) is given by

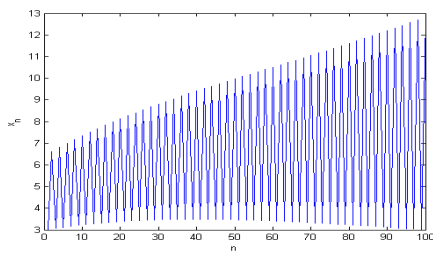
$$(\bar{x}, \bar{y}) = (5.557681533, 5.557681533).$$

Moreover, in Figure 2, the plot of x_n is shown in Figure 2 (a), the plot of y_n is shown in Figure 2 (b), and an attractor of the system (4.2) is shown in Figure 2 (c).

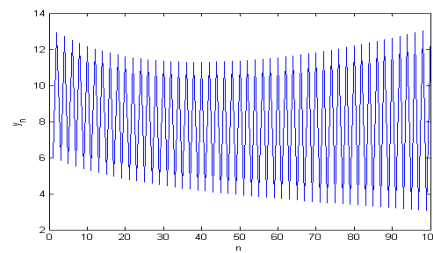
Example 4.3. Let $a = 20, b = 0.01, c = 0.5, d = 0.002, (d < b + c)$. Then system (1.1) can be written as

$$x_{n+1} = \frac{20 + e^{-(0.01x_n+0.5y_n)}}{0.002 + 0.01x_n + 0.5y_n}, \quad y_{n+1} = \frac{20 + e^{-(0.01y_n+0.5x_n)}}{0.002 + 0.01y_n + 0.5x_n} \tag{4.3}$$

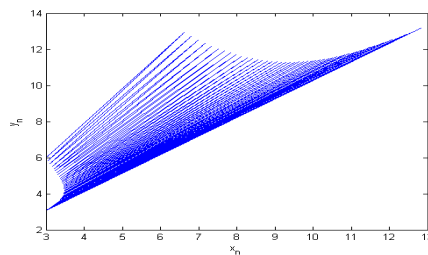
with initial conditions $x_0 = 3$ and $y_0 = 6$.



(a) Plot of x_n for the system (4.3)



(b) Plot of y_n for the system (4.3)



(c) Phase portrait of the system (4.3)

Figure 3: Plots for the system (4.3)

In this case, the unique positive equilibrium point of the system (4.3) is unstable. Moreover, in Figure 3, the plot of x_n is shown in Figure 3 (a), the plot of y_n is shown in Figure 3 (b), and a phase portrait of the system (4.3) is shown in Figure 3 (c).

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