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On some generalisations of Brown's conjecture

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Abstract

Let P be a complex polynomial of the form $P(z) = z\prod_{k=1}^{n-1}(z - z_k)$,where $|z_k| \geq 1, 1 \leq k \leq n-1$ then $P'(z) \neq 0$. If $|z| < \frac{1}{n}$. In this paper, we present some interesting generalisations of this result.

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1. Introduction and statement of the results

Let $B(z, r)$ denote the open ball in C with centre z and radius r and $\overline{B}(z, r)$ denote its closure.The Gauss Lucas Theorem states that every critical point of a complex polynomial P of degree atmost n lies in the convex hull of its zeros.B. Sendove conjectured that if all the zeros of P lies in $\overline{B}(0, 1)$ then for any zero w of P the disk $\overline{B}(w, 1)$ contains at least one zero of P' see [[4], problem 4.1]. In connection with this conjecture Brown [3] posed the following problem.

Let Q_n denote the set of all complex polynomials of the form $P(z) = z\prod_{k=1}^{n-1}(z - z_k)$, where $|z_k| \geq 1, 1 \leq k \leq n-1$. Find the best constant C_n such that $P'(z) \neq 0$. in $B(0, C_n)$ for all P in Q_n. Brown conjectured that $C_n = \frac{1}{n}$.

Recently, the conjecture was settled by Aziz and Zargar [2]. In fact by proving the following:

Theorem 1.1. For all P in Q_n , $P'(z)$ does not vanish if $z \; in \left(0, \frac{1}{n}\right)$.

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Here, in this paper we shall present the following generalisation of Theorem 1.1.

Theorem 1.2. Let

$$P(z) = z^m \prod_{j=1}^{n-m} (z - z_j)$$

be a polynomial of degree n,with $|z_j| \geq 1, j = 1, 2, ...n - m$. Then for $1 \leq r \leq m$, the polynomial $P'(z)$, the rth derivative of $P(z)$ does not vanish in

$$0 < |z| < \frac{m(m-1)(m-2)...(m-r+1)}{n(n-1)(n-2)...(n-r+1)}.$$

Remark 1.3. Taking $r = 1, m = 1$, we get Theorem A (Browns Conjecture).

The following result immediately follows from the proof of Theorem 1.2.

Corollary 1.4. Let

$$P(z) = z^m \prod_{j=1}^{n-m} (z - z_j)$$

be a polynomial of degree n, with $|z_j| \geq 1, j = 1, 2, ...n - m$. Then the polynomial $P''(z)$ does not vanish in

$$0 < |z| < \frac{m(m-1)}{n(n-1)}.$$

Taking $m = 2$ in Corollary 1.4, we get the following result.

Corollary 1.5. Let

$$P(z) = z^2 \prod_{j=1}^{n-2} (z - z_j)$$

be a polynomial of degree n,with $|z_j| \geq 1, j = 1, 2, \ldots, n - 2$. Then the polynomial $P''(z)$ does not vanish in

$$0 < |z| < \frac{2}{n(n-1)}.$$

2. Lemmas

For the proof of Theorem 1.2, we need the following lemmas.the first lemma is walsh's Coincidance Theorem [[4], p 47] (see also [1]).

Lemma 2.1. If $G(z_1, z_2, \ldots, z_n)$ is a symmetric n-linear form of total degree n in $(z_1, z_2, \ldots, z_n$ and let C be a circular region containing the n points $\alpha_1, \alpha_2, , \alpha_n$ then there exists at least one point $\alpha\ inC$ such that

$$G(\alpha_1, \alpha_2, \ldots, \alpha_n) = G(w_1, w_2, \ldots, w_n).$$

Lemma 2.2. If

$$P(z) = z^m \prod_{k=1}^{n-m} (z - z_k)$$

be a polynomial of degree n,with $|z_k| \geq 1, 1 \leq k \leq n - m$. Then for the polynomial $P'(z)$ does not vanish in

$$0 < |z| < \frac{m}{n}.$$

Lemma 2.2 is due to Aziz and Zargar [2].

Lemma 2.3. *If $P(z)$ is a Polynomial of degree n such that $P(z)$ does not vanish in $\|z| < 1$, then the polynomial $zP'(z) + 2P(z)$ does not vanish in $|z| < \frac{2}{n+2}$.*

Proof . By hypothesis

$$P(z) = \prod_{k=1}^{n}(z - z_k)$$

is a polynomial of degree n having all its zeros in $|z| \geq 1$, so that $|z_k| \geq 1, k = 1, 2, \ldots, n$. We prove all the zeros of

$$H(z) = zP'(z) + 2P(z)$$

lie in

$$|z| \geq \frac{2}{n+2}.$$

To prove this let w be any zero of $P(z)$ then

$$H(w) = zP'(w) + 2P(w) = 0.$$

Clearly $H(z)$ is linear symmetric in the zeros $z_1, z_2, \ldots, z_n$ of $P(z)$. Therefore by Lemma 2.1, we can find atleast one point β with $|\beta| \geq 1$, such that

$$P(z) = (z - \beta)^n.$$

which gives

$$H(w) = wnP'(w - \beta)^{n-1} + 2P(w - \beta)^n = 0$$

which implies

$$(w - \beta)^{n-1}[nw + 2(w - \beta)] = 0$$

which gives,

$$(w - \beta) = 0, or nw + (w - \beta) = 0.$$

If $w - \beta = 0$, then clearly $|w| = |\beta| \geq 1$. Now if,

$$nw + (w - \beta) = 0$$

then

$$w = \frac{2\beta}{n+2}$$

which gives,

$$|w| = \frac{2}{n+2}|\beta| \geq \frac{2}{n+2}.$$

Since w is any zero of

$$H(z) = zP'(z) + 2P(z)$$

therefore, it follows that

$$zP'(z) + 2P(z)$$

does not vanish in

$$|z| < \frac{2}{n+2},$$

which completes the proof of Lemma (2.3). □

3. Proof of Theorem

Proof . We have

$$P(z) = z^m Q(z)$$

where

$$Q(z) = \prod_{j=1}^{n-m} (z - z_j), |z_j| \geq 1, j = 1, 2, ..., n - m.$$

So, it follows by Lemma 2.2 that $P'(z)$ does not vanish in the disk

$$0 < |z| < \frac{m}{n}.$$

That is

$$\begin{aligned} P'(z) &= z^m Q'(z) + m z^{m-1} Q(z) \\ &= z^{m-1}(zQ'(z) + mzQ(z)) \\ & z^{m-1} T(z) \end{aligned}$$

where

$$T(z) = (zQ'(z) + mzQ(z))$$

does not vanish in

$$0 < |z| < \frac{m}{n}.$$

Replacing z by $\frac{mz}{n}$, it follows that

$$H(z) = P'\{\frac{mz}{n}\}$$

does not vanish in

$$0 < |z| < 1.$$

Now

$$H(z) = P'(\frac{mz}{n}) = (\frac{m}{n})^{m-1} z^{m-1} T(\frac{mz}{n}).$$

Applying Lemma 2.2 to the polynomial $H(z)$, it follows that $H'(z)$ does not vanish in the disk

$$0 < |z| < \frac{m-1}{n-1}.$$

.

Replacing z by $\frac{nz}{m}$, we get $P''(z)$ does not vanish in $0 < |z| < \frac{m(m-1)}{n(n-1)}$.. $n \geq 2$ which yields that,

$$\begin{aligned} P''(z) &= z^{m-1} T'(z) + (m-1) z^{m-2} T(z) \\ &= z^{m-1}(zT'(z) + (m-1)T(z)) \\ &= z^{m-1} R(z) \end{aligned}$$

does not vanish in $0 < |z| < \dfrac{m(m-1)}{n(n-1)}$. Thus, it follows by Lemma 2.3 that

$$R(z) = (zT'(z) + (m-1)T(z))$$

does not vanish in $0 < |z| < \dfrac{m(m-1)}{n(n-1)}$. Replacing z by $\dfrac{m(m-1)z}{n(n-1)}$, we have

$$R\left(\frac{m(m-1)}{n(n-1)}z\right) = (m-1)T\left(\frac{m(m-1)}{n(n-1)}z\right) + \left(\frac{m(m-1)}{n(n-1)}z\right)T\left(\frac{m(m-1)}{n(n-1)}z\right)$$

does not vanish in $0 < |z| < 1$. Therefore, it follows that

$$S(z) = P''\left(\frac{m(m-1)}{n(n-1)}z\right)$$

,

$$= \left(\frac{m(m-1)}{n(n-1)}z\right)^{m-1} z^{m-1}R\left(\frac{m(m-1)}{n(n-1)}z\right)$$

does not vanish in $0 < |z| < 1$. Applying Lemma 2.2, we get

$$S'(z) = P''\left(\frac{m(m-1)}{n(n-1)}z\right)$$

does not vanish in $0 < |z| < \dfrac{(m-2)}{(n-2)}$ and this completes the proof of Theorem 1.2. □

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