On best proximity points for multivalued cyclic $F$-contraction mappings

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Abstract

In this paper, we establish and prove the existence of best proximity points for multivalued cyclic $F$-contraction mappings in complete metric spaces. Our results improve and extend various results in literature.

Keywords: best proximity point; cyclic contraction; $F$-contraction; multivalued mapping; metric space.

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1. Introduction

Throughout this paper, for metric space $(X, d)$, We denote $C_b(X)$ by the family of all non-empty closed bounded subsets of a metric space $(X, d)$. The Pompeiu-Hausdorff metric induced by $d$ on $C_b(X)$ is given by

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\},$$

for every $A, B \in C_b(X)$, where $d(a, B) = \inf \{d(a, b) : b \in B\}$ is the distance from $a$ to $B \subseteq X$.

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Remark 1.1. The following properties of the Pompeiu-Hausdorff metric induced by \( d \) are well-known:

1. \( H \) is a metric on \( C_b(X) \).
2. If \( A, B \in C_b(X) \) and \( h > 1 \) be given, then for every \( a \in A \) there exists \( b \in B \) such that \( d(a, b) \leq hH(A, B) \).

In 1992, Banach contraction principle was defined by Banach (see [1]). Let \( T : X \rightarrow X \) be a self mapping of a complete metric \((X, d)\), such that \( d(Tx, Ty) \leq Ld(x, y) \) for each \( x, y \in X \), where \( 0 \leq L < 1 \). Then, \( T \) has a unique fixed point. Further, since Banach’s fixed point theorem, because of its simplicity, usefulness and applications, it has become a very popular tools solving the existence problems in many branches of mathematics analysis. Several authors have improved, extended and generalized Banach’s fixed point theorem in many directions (see in [2, 3, 4, 5, 6] and references therein).

In a different way, if \( T \) is a non-self mapping then there is no fixed point from equation \( Tx = x \). The investigation of this case that there is an element \( x \) such that \( d(x, Tx) \) is minimum. This point becomes a concept of best proximity point theorem, so this theorem guarantees the existence of an element \( x \) such that \( d(x, Tx) = d(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\} \) then \( x \) is called a best proximity point of non-self mapping \( T \). Since a non-self mapping \( T \) has no fixed point, but this mapping gives a best proximity point so it is optimal approximate solution of the fixed point equation \( Tx = x \). If \( d(A, B) = 0 \), then a fixed point and a best proximity point are same point. A best proximity point is reduced to a fixed point if \( T \) is a self mapping.

In 1969, Fan [7] be the first who study in area of the best proximity point theorem. He established a classical best approximation theorem. After ward several researchers have been extended the best proximity theorem in many directions (see in [8, 9, 10, 11, 12, 13, 14, 15] and references therein).

In the same year, Nadler [16] given new idea of the Banach contraction principle. Researcher extended the theorem from single valued mapping to multivalued mapping.

Lemma 1.2. (16) Let \((X, d)\) be a metric space. If \( A, B \in C_b(X) \) and \( a \in A \), then for each \( \epsilon > 0 \), there exists \( b \in B \) such that \( d(a, b) \leq H(A, B) + \epsilon \).

Nadler [16] also combine the idea of Lipschitz mappings with multivalued mappings and fixed point theorems as follows:

Theorem 1.3. (16) Let \((X, d)\) be a complete metric space and \( T : X \rightarrow C_b(X) \). If there exists \( k \in [0, 1) \), such that

\[
H(Tx, Ty) \leq kd(x, y),
\]

for all \( x, y \in X \), then \( T \) has at least one fixed point, that is, there exists \( z \in X \) such that \( z \in Tz \).

In 2003, Kirk, Srinavasan and Veeramani [17] introduced a concept of cyclic contraction which generalized Banach’s contraction. They also proved fixed point theorems in complete metric spaces, as follows:

Definition 1.4. (17) Let \( A \) and \( B \) be non-empty closed subsets of a complete metric space \( X \) and \( T : A \cup B \rightarrow A \cup B \) be a mapping. Then \( T \) is called a cyclic mapping if and only if \( T(A) \subseteq B \) and \( T(B) \subseteq A \).

Theorem 1.5. (17) Let \( A \) and \( B \) be non-empty closed subsets of a complete metric space \( X \) and \( T : A \cup B \rightarrow A \cup B \) be a mapping. Then \( T \) is called a cyclic contraction if and only if \( T \) satisfies this condition.
1. $T$ is cyclic mapping.
2. For some $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$, for all $x \in A, y \in B$.

Then, $T$ has a fixed point in $A \cap B$.

After that in 2006, Eldred and Veeramani [18] gave sufficient condition for guarantee the existence of a best proximity point for a cyclic contraction mapping $T$.

**Definition 1.6.** ([18]) Let $A$ and $B$ be non-empty closed subsets of a complete metric space $(X, d)$. Let $T : A \cup B \rightarrow A \cup B$ be a cyclic contraction mapping and there exists $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) + (1 - k)d(A, B)$$

for all $x \in A$ and $y \in B$,

where $d(A, B) = \inf \{d(x, y) : x \in A, y \in B\}$. A point $x \in A \cup B$ is said to be best proximity point for $T$ if $d(x, Tx) = d(A, B)$.

Recently Wardowski [19] proved one of interesting in fixed point theorem which is $F-$ contraction mapping on complete metric spaces.

The aim of this paper, we introduce the notation and concept of multivalued cyclic $F$-contraction pair and prove a best proximity point such a mappings in a complete metric space via property UC* due to Sintunavarat and Kumam [20].

2. Preliminaries

Now, recall elementary results and some basic definitions in the literature. In this paper, we denote $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ by the set of positive integers, the set of real numbers and the set of non-negative real numbers, respectively.

**Definition 2.1.** Let $A$ and $B$ be non-empty subsets of a metric space $X$. A point $x \in A$ is said to be a best proximity point if it satisfies the following condition

$$d(x, Tx) = d(A, B).$$

We have that a best proximity point reduces to a fixed point for a multivalued mapping if the underlying mapping is a self-mapping.

**Definition 2.2.** A Banach space $(X, \| \cdot \|)$ is said to be

1. strictly convex if the following condition holds for all $x, y \in X$:

$$\|x\| = \|y\| = 1 \text{ and } x \neq y \implies \frac{\|x + y\|}{2} < 1;$$

2. uniformly convex if for each $\epsilon$ with $0 < \epsilon \leq 2$, there exists $\delta > 0$ such that the following condition holds for all $x, y \in X$:

$$\|x\| \leq 1, \|y\| \leq 1 \text{ and } \|x - y\| \geq \epsilon \implies \frac{\|x + y\|}{2} < 1 - \delta.$$ 

**Remark 2.3.** It is easy to see that a uniformly convexity implies strictly convexity but the converse is not true.
Definition 2.4. ([21]) Let $A$ and $B$ be nonempty subsets of a metric space $X$. The ordered pair $(A, B)$ is said to satisfy the property $UC$ if the following holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in $A$ and $\{y_n\}$ be a sequence in $B$ such that $d(x_n, y_n) \to d(A, B)$ and $d(z_n, y_n) \to d(A, B)$, then $d(x_n, z_n) \to 0$.

Example 2.5. ([21]) The following are some examples of a pair of nonempty subsets $(A, B)$ satisfying the property UC.

1. Every pair of nonempty subsets $A,B$ of a metric space $(X,d)$ such that $d(A,B) = 0$.
2. Every pair of nonempty subsets $A,B$ of a uniformly convex Banach space $X$ such that $A$ is convex.
3. Every pair of nonempty subsets $A,B$ of a strictly convex Banach space where $A$ is convex and relatively compact and the closure of $B$ is weakly compact.

Definition 2.6. ([20]) Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. The ordered pair $(A,B)$ satisfies the property $UC^*$ if $(A,B)$ has property UC and the following condition holds:

1. $d(z_n, y_n) \to d(A,B)$ as $n \to \infty$.
2. For each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_m, y_n) \leq d(A,B) + \epsilon$ for all $m > n \geq N$, then $d(x_n, z_n) \to 0$ as $n \to \infty$.

Example 2.7. The following are some examples of a pair of nonempty subsets $(A,B)$ satisfying the property $UC^*$.

1. Every pair of nonempty subsets $A$ and $B$ of a metric space $(X,d)$ such that $d(A,B) = 0$.
2. Every pair of nonempty closed subsets $A$ and $B$ of a uniformly convex Banach space $X$ such that $A$ is convex (see Lemma 3.7 in [18]).

Wardowski [19] defined the following contraction which was called $F$-contraction as follows:

Definition 2.8. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping which is satisfying the following conditions:

$(F_1)$ $F$ is strictly increasing, i.e. for all $\alpha, \beta \in \mathbb{R}^+$, $F(\alpha) < F(\beta)$ whenever $\alpha < \beta$.

$(F_2)$ For each sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of positive real numbers $\lim_{n \to \infty} \alpha_n = 0$ iff $\lim_{n \to \infty} F(\alpha_n) = -\infty$.

$(F_3)$ There exists $k \in (0,1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

We denote by $\mathcal{F}$ the family of all functions $F$ that satisfy the conditions $(F_1) - (F_3)$. For examples of the function $F$ the reader is referred to [19] and [22].

Definition 2.9. Let $(X, d)$ be a metric space. A self-mapping $T$ on $X$ is called an $F$-contraction mapping if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that

$$\forall x,y \in X, \ d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)). \quad (2.1)$$

Remark 2.10. From $(F_1)$ and (2.1) it easy to see that every $F$-contraction is necessarily continuous.
3. The main results

**Definition 3.1.** Let $A$ and $B$ be non-empty subsets of a metric space $X$. Let $T : A \to 2^B$ and $S : B \to 2^A$ be multivalued mappings. The ordered pair $(T, S)$ is said to be a multivalued cyclic $F$-contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that

$$H(Tx, Sy) > 0 \Rightarrow 2\tau + F(H(Tx, Sy)) \leq F(kd(x, y)) + (1 - k)d(A, B),$$

(3.1)

for all $x, y \in X$, where $k \in (0, 1)$.

**Theorem 3.2.** Let $A$ and $B$ be non-empty closed subsets of a complete metric space $X$ such that $(A, B)$ and $(B, A)$ satisfy the property $UC^*$. Let $T : A \to C_b(B)$ and $S : B \to C_b(A)$. If $(T, S)$ is a multivalued cyclic $F$-contraction pair, then $T$ has a best proximity point in $A$ or $S$ has a best proximity point in $B$.

**Proof.** We divide the case into two.

**Case 1:** Assume that $d(A, B) = 0$.

Now, we will construct the sequence $\{x_n\}$ in $X$ as follows. Let $x_0 \in A$ be arbitrary point. Since $Tx_0 \in C_b(B)$, we can choose $x_1 \in Tx_0$. If $Tx_0 \neq Sx_1$, since $F$ is continuous from the right then there exists a real number $h > 1$ and $\tau > 0$ such that

$$F(hH(Tx_0, Sx_1)) < F(H(Tx_0, Sx_1)) + \tau.$$

From $d(x_1, Sx_1) < hH(Tx_0, Sx_1)$, we deduce that there exists $x_2 \in Sx_1$ such that

$$d(x_1, x_2) \leq hH(Tx_0, Sx_1).$$

It follows from definition of $F$, we have

$$F(d(x_1, x_2)) \leq F(hH(Tx_0, Sx_1)) < F(H(Tx_0, Sx_1)) + \tau$$

which implies

$$F(d(x_1, x_2)) \leq F(H(Tx_0, Sx_1)) + \tau$$
$$\leq F(kd(x_0, x_1)) + \tau - 2\tau$$
$$\leq F(kd(x_0, x_1)) - \tau$$
$$\leq F(d(x_0, x_1)) - \tau.$$

Otherwise, if $Tx_2 \neq Sx_1$, since $F$ is continuous from the right then there exists a real number $h > 1$ and $\tau > 0$ such that

$$F(hH(Sx_1, Tx_2)) < F(H(Sx_1, Tx_2)) + \tau.$$

Now from $d(x_2, Tx_2) < hH(Sx_1, Tx_2)$, we obtain that there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq hH(Sx_1, Tx_2).$$

Consequently, we get

$$F(d(x_2, x_3)) \leq F(hH(Sx_1, Tx_2)) < F(H(Sx_1, Tx_2)) + \tau.$$
which implies
\[ F(d(x_2, x_3)) \leq F(H(Sx_1, Tx_2)) + \tau \]
\[ \leq F(kd(x_1, x_2)) + \tau - 2\tau \]
\[ \leq F(kd(x_1, x_2)) - \tau \]
\[ \leq F(d(x_1, x_2)) - \tau. \]

By induction, we can find \( \{x_n\} \) such that
\[ F(d(x_n, x_{n+1})) \leq F(kd(x_{n-1}, x_n)) - \tau \]
\[ \leq F(d(x_{n-1}, x_n)) - \tau \]
\[ \vdots \]
\[ \leq F(kd(x_0, x_1)) - n\tau \]
\[ \leq F(d(x_0, x_1)) - n\tau. \]

Let \( \beta_n := d(x_n, x_{n+1}) \). From above, we receive \( \lim_{n \to \infty} F(\beta_n) = -\infty \) that together with \( (F_2) \) gives
\[ \lim_{n \to \infty} \beta_n = 0. \]

Also from \( (F_3) \), we have
\[ \exists l \in (0, 1) \text{ such that } \lim_{n \to \infty} \beta_n^l F(\beta_n) = 0. \]

Now, it follows that
\[ \beta_n^l F(\beta_n) - \beta_n^l F(\beta_0) \leq \beta_n^l (F(\beta_0) - n\tau) - \beta_n^l F(\beta_0) \]
\[ \leq \beta_n^l F(\beta_0) - \beta_n^l n\tau - \beta_n^l F(\beta_0) \]
\[ \leq -\beta_n^l n\tau \]
\[ \leq 0, \text{ for all } n \in \mathbb{N}. \]

Letting \( n \) as \( n \to \infty \), so, we obtain
\[ n\beta_n^l = 0 \text{ for all } n \in \mathbb{N}. \]

From above, \( \lim_{n \to \infty} n\beta_n^l = 0 \) there exist \( n_1 \in \mathbb{N} \) such that \( n\beta_n^l \leq 1 \) for all \( n \geq n_1 \).

Therefore, \( \beta_n \leq \frac{1}{n^{\frac{1}{l}}} \), for all \( n \geq n_1 \).

Let \( m, n \in \mathbb{N} \) such that \( m > n \geq n_1 \). We compute that
\[ d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \]
\[ = \beta_n + \beta_{n+1} + \ldots + \beta_{m-1} \]
\[ = \sum_{i=n}^{m-1} \beta_i \]
\[ \leq \sum_{i=n}^{\infty} \beta_i \]
\[ \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{l}}}. \]
By the convergence of the P series $\sum_{i=n}^{\infty} \frac{1}{i^2}$, so as $n \to \infty$, we obtain $d(x_n, x_m) \to 0$ as $n \to \infty$. Hence \{x_n\} is a Cauchy sequence. Since completeness of $X$, then \{x_n\} converges to some point $z \in X$. Clearly, the subsequence $\{x_{2n}\}$ and $\{x_{2n-1}\}$ converge to same point $z$. Since $A$ and $B$ are closed, we obtain that $z \in A \cap B$.

From (3.1), for all $x, y \in X$ and $k \in (0, 1)$ with $H(Tx, Sy) > 0$ and $d(A, B) = 0$, we get

$$2\tau + F(H(Tx, Sy)) \leq F(kd(x, y)) \leq F(d(x, y)).$$

Since $F$ is strictly increasing, we get $H(Tx, Sy) < d(x, y)$ and so $H(Tx, Sy) \leq d(x, y)$ for all $x, y \in X$. Then

$$d(x_{2n+1}, Tz) \leq H(Sx_{2n}, Tz) \leq d(x_{2n}, z).$$

Passing to limit $n \to \infty$, we obtain $d(z, Tz) = d(A, B)$. Similarity, we also derive $d(Sz, z) = d(A, B)$.

**Case 2:** We will show that $T$ or $S$ have best proximity points in $A$ and $B$, respectively. Under the assumption of $d(A, B) > 0$, suppose to the contrary, that is for all $a \in A$, $d(a, Ta) > d(A, B)$ and for all $b' \in B$, $d(Sb', b') > d(A, B)$.

For each $a \in A$ and $b \in Ta$, we have

$$d(A, B) < d(a, Ta) \leq d(a, b). \quad (3.2)$$

Since $(T, S)$ is a multivalued cyclic $F$-contraction pair, such that

$$F(H(Ta, Sb)) \leq F(kd(a, b) + (1 - k)d(A, B)) - 2\tau \leq F(kd(a, b) + (1 - k)d(A, B)) \quad (3.3)$$

for all $a \in A$ and $b \in Ta$. Since $F$ is strictly increasing, we get

$$H(Ta, Sb) < kd(a, b) + (1 - k)d(A, B) \quad (3.5)$$

for all $a \in A$ and $b \in Ta$.

Similarly, we have that for each $b' \in B$ and $a' \in Sb'$, we get

$$F(H(Ta', Sb')) < F(kd(a', b') + (1 - k)d(A, B)) \quad (3.6)$$

and

$$H(Ta', Sb') < kd(a', b') + (1 - k)d(A, B). \quad (3.7)$$

Next we will construct the sequence $\{x_n\}$ in $A \cup B$. Let $x_0$ be arbitrary point in $A$ and $x_1 \in Tx_0 \subseteq B$.

From (3.3), there exists $x_2 \in Sx_1$ such that

$$F(d(x_1, x_2)) \leq F(H(Tx_0, Sx_1)) + \tau \leq F(kd(x_0, x_1) + (1 - k)d(A, B)) - 2\tau + \tau \leq F(kd(x_0, x_1) + (1 - k)d(A, B)) - \tau < F(kd(x_0, x_1) + (1 - k)d(A, B))$$

and since $F$ is strictly increasing, we get

$$d(x_1, x_2) < kd(x_0, x_1) + (1 - k)d(A, B). \quad (3.8)$$
Since \( x_1 \in B \) and \( x_2 \in Sx_1 \) from \((3.6)\), we can find \( x_3 \in Tx_2 \) such that
\[
d(x_2, x_3) < kd(x_1, x_2) + (1 - k)d(A, B).
\]
(3.9)
Consequently, we can define the sequence \( \{x_n\} \) in \( A \cup B \) such that
\[
x_{2n-1} \in Tx_{2n-2}, \ x_{2n} \in Sx_{2n-1}
\]
and
\[
d(x_n, x_{n+1}) < kd(x_{n-1}, x_n) + (1 - k)d(A, B)
\]
for all \( n \in \mathbb{N} \). Since \( d(A, B) \leq d(x_{n-1}, x_n) \) for all \( n \in \mathbb{N} \), we get
\[
d(x_n, x_{n+1}) < kd(x_{n-1}, x_n) + (1 - k)d(A, B)
\]
\[
\leq kd(x_{n-1}, x_n) + (1 - k)d(x_{n-1}, x_n)
\]
\[
\leq d(x_{n-1}, x_n)
\]
(3.11)
and
\[
d(x_n, x_{n+1}) < kd(x_{n-1}, x_n) + (1 - k)d(A, B)
\]
\[
< k(kd(x_{n-2}, x_{n-1}) + (1 - k)d(A, B)) + (1 - k)d(A, B)
\]
\[
< k^2d(x_{n-2}, x_{n-1}) + (k - k^2)d(A, B) + (1 - k)d(A, B)
\]
\[
< k^2d(x_{n-2}, x_{n-1}) + (1 - k^2)d(A, B)
\]
\[
\vdots
\]
\[
< k^ad(x_0, x_1) + (1 - k^a)d(A, B).
\]
(3.12)
Hence \( d(A, B) \leq d(x_n, x_{n+1}) < k^a d(x_0, x_1) + (1 - k^a)d(A, B) \) for all \( n \in \mathbb{N} \).

Since \( k \in (0, 1) \), we obtain
\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = d(A, B).
\]
(3.13)
From equation \((3.13)\), we get
\[
\lim_{n \to \infty} d(x_{2n}, x_{2n+1}) = d(A, B).
\]
(3.14)
and
\[
\lim_{n \to \infty} d(x_{2n+2}, x_{2n+1}) = d(A, B).
\]
(3.15)
Since \( \{x_{2n}\} \) and \( \{x_{2n+2}\} \) are two sequences in \( A \) and \( \{x_{2n+1}\} \) is sequence \( B \) with \( (A, B) \) which satisfies the property UC*, we derive that
\[
\lim_{n \to \infty} d(x_{2n}, x_{2n+2}) = 0.
\]
(3.16)
Since \( (B, A) \) satisfies the property UC* and by \((3.13)\), we have
\[
\lim_{n \to \infty} d(x_{2n-1}, x_{2n+1}) = 0.
\]
(3.17)
Next, we will show that for each \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that for all \( m > n \geq N \), we have
\[
\lim_{n \to \infty} d(x_{2m}, x_{2m+1}) \leq d(A, B) + \epsilon.
\]
(3.18)
Suppose the contrary, that is there exists \( \epsilon_0 > 0 \) such that for each \( k \geq 1 \) there is \( m_k > n_k \geq k \) such that
\[
d(x_{2m_k}, x_{2n_k+1}) > d(A, B) + \epsilon_0.
\]
(3.19)
Moreover, corresponding to $n_k$, we can choose $m_k$ in such a way that it is the smallest integer with

\[ m_k > n_k \geq k \] satisfying (3.19). Then we obtain

\[ d(x_{2m_k}, x_{2n_k+1}) > d(A, B) + \epsilon_0 \] (3.20)

and

\[ d(x_{2(m_k-1)}, x_{2n_k+1}) \leq d(A, B) + \epsilon_0. \] (3.21)

From (3.20), (3.21) and the triangle inequality, we obtain

\[
\begin{align*}
d(A, B) + \epsilon_0 &< d(x_{2m_k}, x_{2n_k+1}) \\
&\leq d(x_{2m_k}, x_{2(m_k-1)}) + d(x_{2(m_k-1)}, x_{2n_k+1}) \\
&\leq d(x_{2m_k}, x_{2(m_k-1)}) + d(A, B) + \epsilon_0.
\end{align*}
\]

Using the fact that \( \lim_{k \to \infty} d(x_{2m_k}, x_{2(m_k-1)}) = 0 \). Letting \( k \to \infty \) in (3.22), we get

\[ \lim_{k \to \infty} d(x_{2m_k}, x_{2n_k+1}) = d(A, B) + \epsilon_0. \] (3.23)

From (3.10), (3.11) and \((T, S)\) is a multivalued cyclic \( F\)-contraction pair, we obtain

\[
\begin{align*}
d(x_{2m_k}, x_{2n_k+1}) &\leq d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+2}, x_{2n_k+3}) + d(x_{2n_k+3}, x_{2n_k+1}) \\
&\leq d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+1}, x_{2n_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \\
&< d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) + (kd(x_{2m_k}, x_{2n_k+1}) \\
&\quad + (1 - k)d(A, B)).
\end{align*}
\]

(3.24)

Letting \( k \to \infty \) in (3.24) and using (3.16), (3.17) and (3.23), we have

\[ d(A, B) + \epsilon_0 < k(d(A, B) + \epsilon_0) + (1 - k)d(A, B) = d(A, B) + k\epsilon_0 \]

which is a contradiction. Therefore, (3.18) holds. Since (3.14) and (3.18) hold, by using property \( UC^* \) of \((A, B)\), we obtain \( d(x_{2n}, x_{2m}) \to 0 \) as \( n \to \infty \). Therefore \( \{x_{2n}\} \) is a Cauchy sequence. Since \( X \) is complete and \( A \) is closed, we have

\[ \lim_{n \to \infty} x_{2n} = p \] (3.25)

for some \( p \in \overline{A} = A \). But

\[
\begin{align*}
d(A, B) &\leq d(p, x_{2n-1}) \\
&\leq d(p, x_{2n}) + d(x_{2n}, x_{2n-1})
\end{align*}
\]

for all \( n \in \mathbb{N} \). From (3.13) and (3.25),

\[ \lim_{n \to \infty} d(p, x_{2n-1}) = d(A, B). \] (3.26)

Since

\[
\begin{align*}
d(A, B) &< d(x_{2n}, Tp) \\
&\leq H(S_{2n-1}, Tp) \\
&= H(Tp, Sx_{2n-1}) \\
&< kd(p, x_{2n-1}) + (1 - k)d(A, B) \\
&\leq d(p, x_{2n-1})
\end{align*}
\]

(3.27)
for all \( n \in \mathbb{N} \). By (3.25) and (3.26), we get
\[
d(p, Tp) = d(A, B).
\] (3.28)

In a similar mode, we can conclude that the sequence \( \{x_{2n-1}\} \) is a Cauchy sequence in \( B \). Since \( X \) is complete and \( B \) is closed, we obtain
\[
\lim_{n \to \infty} x_{2n-1} = q
\] (3.29)
for some \( q \in \overline{B} = B \). Since
\[
d(A, B) \leq d(x_{2n}, q) \\
\leq d(x_{2n}, x_{2n-1}) + d(x_{2n-1}, q)
\]
for all \( n \in \mathbb{N} \). It follows from (3.13) and (3.29) that
\[
\lim_{n \to \infty} d(x_{2n}, q) = d(A, B).
\] (3.30)

Since
\[
d(A, B) < d(Sq, x_{2n+1}) \\
\leq H(Sq, Tx_{2n}) \\
= H(Tx_{2n}, Sq) \\
< kd(x_{2n}, q) + (1 - k)d(A, B) \\
\leq d(x_{2n}, q)
\]
for all \( n \in \mathbb{N} \), then by (3.29) and (3.30), we have
\[
d(q, Sq) = d(A, B).
\] (3.32)

From (3.28) and (3.32), we get a contradiction. Therefore, \( T \) has a best proximity point in \( A \) or \( S \) has a best proximity point in \( B \). This completes the proof. □

**Remark 3.3.** If \( d(A, B) = 0 \), then Theorem 3.2 yields existence of a fixed point in \( A \cap B \) of two multivalued non-self mapping \( S \) and \( T \). Furthermore, if \( A = B = X \) and \( T = S \), then Theorem 3.2 reduces to multivalued \( F \)-contractions on metric spaces [23].

**Corollary 3.4.** Let \( A \) and \( B \) be non-empty closed convex subsets of a uniformly convex Banach space \( X \), \( T : A \to C_b(B) \) and \( S : B \to C_b(A) \). If \( (T, S) \) is a multivalued cyclic \( F \)-contraction pair, then \( T \) has a best proximity in \( A \) or \( S \) has a best proximity point in \( B \).

Now, we give some example for support our results.

**Example 3.5.** Consider the uniformly convex Banach space \( X = \mathbb{R} \) with Euclidean norm. Let \( A = [3,4] \) and \( B = [-4,-3] \). Then \( A \) and \( B \) are non-empty closed and convex subsets of \( X \) and \( d(A, B) = 6 \). Since \( (A, B) \) and \( (B, A) \) satisfy the property UC*. Let \( T : A \to C_b(B) \) and \( S : B \to C_b(A) \) be defined as
\[
Tx = \left[ \frac{-x - 3}{2}, -3 \right], \ x \in [3,4];
\]
and

\[ Sy = \left[ 3, -\frac{y + 3}{2} \right], \quad y \in [-4, -3]. \]

Let \( k \in (0, 1) \) and \( F : \mathbb{R}^+ \to \mathbb{R} \) is satisfy Definition 2.8 be defined by \( F(t) = \ln(t) \) for all \( t \in \mathbb{R}^+ \) and \( \tau > 0 \). Next, we show that \((T, S)\) is a multivalued cyclic \( F \)– contraction pair. For each \( x \in A \) and \( y \in B \), we have

\[
H(Tx, Sy) = H\left( \left[ -\frac{x - 3}{2}, -3 \right], \left[ 3, -\frac{y + 3}{2} \right] \right)
\leq \left| \left( -\frac{x - 3}{2} \right) - \left( -\frac{y + 3}{2} \right) \right|
= \left| \frac{-x + y - 6}{2} \right|
\leq \frac{1}{2}|x - y| + 3
= \frac{1}{2}d(x, y) + \frac{1}{2}d(A, B)
= kd(x, y) + (1 - k)d(A, B).
\]

Since \( \tau > 0 \), we get \( 0 < e^{-2\tau} < 1 \). Hence \( H(Tx, Sy) \leq e^{-2\tau}kd(x, y) + e^{-2\tau}(1 - k)d(A, B) \).

Since \( F \) strictly increasing, we get

\[
F(H(Tx, Sy)) \leq F(e^{-2\tau}kd(x, y) + (1 - k)d(A, B))
= \ln(e^{-2\tau}kd(x, y) + (1 - k)d(A, B))
= -2\tau + \ln(kd(x, y) + (1 - k)d(A, B)).
\]

It follows that \( F(H(Tx, Sy)) + 2\tau \leq F(kd(x, y) + (1 - k)d(A, B)) \). Therefore, all assumptions of Corollary 3.4 are satisfied and then \( T \) has a best proximity point in \( A \), that is a point \( x = 3 \).
Moreover, \( S \) also has a best proximity point in \( B \), that is a point \( y = -3 \).

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**References**


