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# On best proximity points for multivalued cyclic $F$-contraction mappings

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**Abstract**

In this paper, we establish and prove the existence of best proximity points for multivalued cyclic $F$-contraction mappings in complete metric spaces. Our results improve and extend various results in literature.

*Keywords:* best proximity point; cyclic contraction; $F$-contraction; multivalued mapping; metric space.
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## 1. Introduction

Throughout this paper, for metric space $(X, d)$, We denote $C_b(X)$ by the family of all non-empty closed bounded subsets of a metric space $(X, d)$. The Pompeiu-Hausdorff metric induced by $d$ on $C_b(X)$ is given by

$$H(A, B) = \max\left\{ \sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A) \right\},$$

for every $A, B \in C_b(X)$, where $d(a, B) = \inf\{d(a, b) : b \in B\}$ is the distance from $a$ to $B \subseteq X$.

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**Remark 1.1.** The following properties of the Pompeiu-Hausdorff metric induced by $d$ are well-known:

1. $H$ is a metric on $C_b(X)$.
2. If $A, B \in C_b(X)$ and $h > 1$ be given, then for every $a \in A$ there exists $b \in B$ such that $d(a, b) \leq hH(A, B)$.

In 1992, Banach contraction principle was defined by Banach (see [1]). Let $T : X \to X$ be a self mapping of a complete metric $(X, d)$, such that $d(Tx, Ty) \leq Ld(x, y)$ for each $x, y \in X$, where $0 \leq L < 1$. Then, $T$ has a unique fixed point. Further, since Banach's fixed point theorem, because of its simplicity, usefulness and applications, it has become a very popular tools solving the existence problems in many branches of mathematics analysis. Several authors have improved, extended and generalized Banach's fixed point theorem in many directions (see in [2, 3, 4, 5, 6] and references therein).

In a different way, if $T$ is a non-self mapping then there is no fixed point from equation $Tx = x$. The investigation of this case that there is an element $x$ such that $d(x, Tx)$ is minimum. This point becomes a concept of best proximity point theorem, so this theorem guarantees the existence of an element $x$ such that $d(x, Tx) = d(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\}$ then $x$ is called a best proximity point of non-self mapping $T$. Since a non-self mapping $T$ has no fixed point, but this mapping gives a best proximity point so it is optimal approximate solution of the fixed point equation $Tx = x$. If $d(A, B) = 0$, then a fixed point and a best proximity point are same point. A best proximity point is reduced to a fixed point if $T$ is a self mapping.

In 1969, Fan [7] be the first who study in area of the best proximity point theorem. He established a classical best approximation theorem. After ward several researchers have been extended the best proximity theorem in many directions (see in [8, 9, 10, 11, 13, 12, 14, 15] and references therein).

In the same year, Nadler [16] given new idea of the Banach contraction principle. Researcher extended the theorem from single valued mapping to multivalued mapping.

**Lemma 1.2.** ([16]) Let $(X, d)$ be a metric space. If $A, B \in C_b(X)$ and $a \in A$, then for each $\epsilon > 0$, there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \epsilon$.

Nadler [16] also combine the idea of Lipschitz mappings with multivalued mappings and fixed point theorems as follows:

**Theorem 1.3.** ([16]) Let $(X, d)$ be a complete metric space and $T : X \to C_b(X)$. If there exists $k \in [0, 1)$, such that

$$H(Tx, Ty) \leq kd(x, y), \tag{1.1}$$

for all $x, y \in X$, then $T$ has at least one fixed point, that is, there exists $z \in X$ such that $z \in Tz$.

In 2003, Kirk, Srinavasan and Veeramani [17] introduced a concept of cyclic contraction which generalized Banach's contraction. They also proved fixed point theorems in complete metric spaces, as follows:

**Definition 1.4.** ([17]) Let $A$ and $B$ be non-empty closed subsets of a complete metric space $X$ and $T : A \cup B \to A \cup B$ be a mapping. Then $T$ is called a *cyclic mapping* if and only if $T(A) \subseteq B$ and $T(B) \subseteq A$ .

**Theorem 1.5.** ([17]) Let $A$ and $B$ be non-empty closed subsets of a complete metric space $X$ and $T : A \cup B \to A \cup B$ be a mapping.Then $T$ is called a *cyclic contraction* if and only if $T$ satisfies this condition.

1. $T$ is cyclic mapping.
2. For some $k \in (0,1)$ such that $d(Tx,Ty) \leq kd(x,y)$, for all $x \in A$, $y \in B$.

Then, $T$ has a fixed point in $A \cap B$.

After that in 2006, Eldred and Veeramani [18] gave sufficient condition for guarantee the existence of a best proximity point for a cyclic contraction mapping $T$.

**Definition 1.6.** ([18]) Let $A$ and $B$ be non-empty closed subsets of a complete metric space $(X,d)$. Let $T : A \cup B \to A \cup B$ be a cyclic contraction mapping and there exists $k \in (0,1)$ such that

$$d(Tx,Ty) \leq kd(x,y) + (1-k)d(A,B) \text{ } for \text{ } all \text{ } x \in A \text{ } and \text{ } y \in B,$$

where $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$. A point $x \in A \cup B$ is said to be best proximity point for $T$ if $d(x,Tx) = d(A,B)$.

Recently Wardowski [19] proved one of interesting in fixed point theorem which is $F-$ contraction mapping on complete metric spaces.

The aim of this paper, we introduce the notation and concept of multivalued cyclic $F$-contraction pair and prove a best proximity point such a mappings in a complete metric space via property UC* due to Sintunavarat and Kumam [20].

## 2. Preliminaries

Now, recall elementary results and some basic definitions in the literature. In this paper, we denote $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ by the set of positive integers, the set of real numbers and the set of non-negative real numbers, respectively.

**Definition 2.1.** Let $A$ and $B$ be non-empty subsets of a metric space $X$ and $T : A \to 2^B$ be a multivalued mapping. A point $x \in A$ is said to be a *best proximity point* of a multivalued mapping $T$ if it satisfies the following condition

$$d(x,Tx) = d(A,B).$$

We have that a best proximity point reduces to a fixed point for a multivalued mapping if the underlying mapping is a self-mapping.

**Definition 2.2.** A Banach space $(X, \|\cdot\|)$ is said to be

1. *strictly convex* if the following condition holds for all $x, y \in X$:

$$\|x\| = \|y\| = 1 \text{ and } x \neq y \Longrightarrow \left\| \frac{x+y}{2} \right\| < 1;$$

2. *uniformly convex* if for each $\epsilon$ with $0 < \epsilon \leq 2$, there exists $\delta > 0$ such that the following condition holds for all $x, y \in X$:

$$\|x\| \leq 1, \|y\| \leq 1 \text{ and } \|x-y\| \geq \epsilon \Longrightarrow \left\| \frac{x+y}{2} \right\| < 1-\delta.$$

**Remark 2.3.** It is easy to see that a uniformly convexity implies strictly convexity but the converse is not true.

**Definition 2.4.** ([21]) Let $A$ and $B$ be nonempty subsets of a metric space $X$. The ordered pair $(A, B)$ is said to satisfy the *property UC* if the following holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in $A$ and $\{y_n\}$ be a sequence in $B$ such that $d(x_n, y_n) \to d(A, B)$ and $d(z_n, y_n) \to d(A, B)$, then $d(x_n, z_n) \to 0$.

**Example 2.5.** ([21]) The following are some examples of a pair of nonempty subsets $(A, B)$ satisfying the property UC.

1. Every pair of nonempty subsets $A, B$ of a metric space $(X, d)$ such that $d(A, B) = 0$.
2. Every pair of nonempty subsets $A, B$ of a uniformly convex Banach space $X$ such that $A$ is convex.
3. Every pair of nonempty subsets $A, B$ of a strictly convex Banach space where $A$ is convex and relatively compact and the closure of $B$ is weakly compact.

**Definition 2.6.** ([20]) Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. The ordered pair $(A, B)$ satisfies the *property* $UC^*$ if $(A, B)$ has property UC and the following condition holds: If $\{x_n\}$ and $\{z_n\}$ are sequences in $A$ and $\{y_n\}$ is a sequence in $B$ satisfying:

1. $d(z_n, y_n) \to d(A, B)$ as $n \to \infty$.
2. For each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_m, y_n) \leq d(A, B) + \epsilon$$

   for all $m > n \geq N$,

then $d(x_n, z_n) \to 0$ as $n \to \infty$.

**Example 2.7.** The following are some examples of a pair of nonempty subsets $(A, B)$ satisfying the property UC*.

1. Every pair of nonempty subsets $A$ and $B$ of a metric space $(X, d)$ such that $d(A, B) = 0$.
2. Every pair of nonempty closed subsets $A$ and $B$ of a uniformly convex Banach space $X$ such that $A$ is convex (see Lemma 3.7 in [18]).

Wardowski [19] defined the following contraction which was called $F$-contraction as follows:

**Definition 2.8.** Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping which is satisfying the following conditions:

$(F_1)$ $F$ is strictly increasing, i.e. for all $\alpha, \beta \in \mathbb{R}^+$, $F(\alpha) < F(\beta)$ whenever $\alpha < \beta$.

$(F_2)$ For each sequence $\{\alpha_n\}_{n \in N}$ of positive real numbers $\lim\limits_{n\to\infty} \alpha_n = 0$ iff $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$.

$(F_3)$ There exists $k \in (0, 1)$ such that $\lim\limits_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

We denote by $\mathcal{F}$ the family of all functions $F$ that satisfy the conditions $(F_1) - (F_3)$. For examples of the function $F$ the reader is referred to [19] and [22].

**Definition 2.9.** Let $(X, d)$ be a metric space. A self-mapping $T$ on $X$ is called an $F$-contraction mapping if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that

$$\forall x, y \in X, \ [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))]. \tag{2.1}$$

**Remark 2.10.** Form $(F_1)$ and (2.1) it easy to see that every $F$-contraction is necessarily continuous.

## 3. The main results

**Definition 3.1.** Let $A$ and $B$ be non-empty subsets of a metric space $X$. Let $T : A \to 2^B$ and $S : B \to 2^A$ be multivalued mappings. The ordered pair $(T, S)$ is said to be a multivalued cyclic $F-$contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that

$$H(Tx, Sy) > 0 \Rightarrow 2\tau + F(H(Tx, Sy)) \leq F(kd(x, y) + (1 - k)d(A, B)), \tag{3.1}$$

for all $x, y \in X$, where $k \in (0, 1)$.

**Theorem 3.2.** *Let $A$ and $B$ be non-empty closed subsets of a complete metric space $X$ such that $(A, B)$ and $(B, A)$ satisfy the property $UC^*$. Let $T : A \to C_b(B)$ and $S : B \to C_b(A)$. If $(T, S)$ is a multivalued cyclic $F-$contraction pair, then $T$ has a best proximity point in $A$ or $S$ has a best proximity point in $B$.*

**Proof .** We divide the case into two.
**Case 1:** Assume that $d(A, B) = 0$.
Now, we will construct the sequence $\{x_n\}$ in $X$ as follows. Let $x_0 \in A$ be arbitrary point. Since $Tx_0 \in C_b(B)$, we can choose $x_1 \in Tx_0$. If $Tx_0 \neq Sx_1$, since $F$ is continuous from the right then there exists a real number $h > 1$ and $\tau > 0$ such that

$$F(hH(Tx_0, Sx_1)) < F(H(Tx_0, Sx_1)) + \tau.$$

From $d(x_1, Sx_1) < hH(Tx_0, Sx_1)$, we deduce that there exists $x_2 \in Sx_1$ such that

$$d(x_1, x_2) \leq hH(Tx_0, Sx_1).$$

It follows from definition of $F$, we have

$$F(d(x_1, x_2)) \leq F(hH(Tx_0, Sx_1)) < F(H(Tx_0, Sx_1)) + \tau$$

which implies

$$\begin{aligned} F(d(x_1, x_2)) &\leq F(H(Tx_0, Sx_1)) + \tau \\ &\leq F(kd(x_0, x_1)) + \tau - 2\tau \\ &\leq F(kd(x_0, x_1)) - \tau \\ &\leq F(d(x_0, x_1)) - \tau. \end{aligned}$$

Otherwise, if $Tx_2 \neq Sx_1$, since $F$ is continuous from the right then there exists a real number $h > 1$ and $\tau > 0$ such that

$$F(hH(Sx_1, Tx_2)) < F(H(Sx_1, Tx_2)) + \tau.$$

Now from $d(x_2, Tx_2) < hH(Sx_1, Tx_2)$, we obtain that there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq hH(Sx_1, Tx_2).$$

Consequently, we get

$$F(d(x_2, x_3)) \leq F(hH(Sx_1, Tx_2)) < F(H(Sx_1, Tx_2)) + \tau$$

which implies

$$\begin{aligned}
F(d(x_2,x_3)) &\leq F(H(Sx_1,Tx_2))+\tau \\
&\leq F(kd(x_1,x_2))+\tau-2\tau \\
&\leq F(kd(x_1,x_2))-\tau \\
&\leq F(d(x_1,x_2))-\tau.
\end{aligned}$$

By induction, we can find $\{x_n\}$ such that

$$\begin{aligned}
F(d(x_n,x_{n+1})) &\leq F(kd(x_{n-1},x_n))-\tau \\
&\leq F(d(x_{n-1},x_n))-\tau \\
&\vdots \\
&\leq F(kd(x_0,x_1))-n\tau \\
&\leq F(d(x_0,x_1))-n\tau.
\end{aligned}$$

Let $\beta_n := d(x_n,x_{n+1})$. From above, we receive $\lim\limits_{n\to\infty} F(\beta_n) = -\infty$ that together with $(F_2)$ gives

$$\lim_{n\to\infty} \beta_n = 0.$$

Also from $(F_3)$, we have

$$\exists l \in (0,1) \text{ such that } \lim_{n\to\infty} \beta_n^l F(\beta_n) = 0.$$

Now, it follows that

$$\begin{aligned}
\beta_n^l F(\beta_n) - \beta_n^l F(\beta_0) &\leq \beta_n^l (F(\beta_0)-n\tau) - \beta_n^l F(\beta_0) \\
&\leq \beta_n^l F(\beta_0) - \beta_n^l n\tau - \beta_n^l F(\beta_0) \\
&\leq -\beta_n^l n\tau \\
&\leq 0, \text{ for all } n \in \mathbb{N}.
\end{aligned}$$

Letting $n$ as $n\to\infty$, so, we obtain

$$n\beta_n^l = 0 \text{ for all } n \in \mathbb{N}.$$

From above, $\lim\limits_{n\to\infty} n\beta_n^l = 0$ there exist $n_1 \in \mathbb{N}$ such that $n\beta_n^l \leq 1$ for all $n \geq n_1$.
Therefore, $\beta_n \leq \frac{1}{n^{\frac{1}{l}}}$, for all $n \geq n_1$.
Let $m,n \in \mathbb{N}$ such that $m > n \geq n_1$. We compute that

$$\begin{aligned}
d(x_n,x_m) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \ldots + d(x_{m-1},x_m) \\
&= \beta_n + \beta_{n+1} + \ldots + \beta_{m-1} \\
&= \sum_{i=n}^{m-1} \beta_i \\
&\leq \sum_{i=n}^{\infty} \beta_i \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{l}}}.
\end{aligned}$$

By the convergence of the P series $\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{l}}}$, so as $n\to\infty$, we obtain $d(x_n,x_m)\to 0$ as $n\to\infty$. Hence $\{x_n\}$ is a Cauchy sequence. Since completeness of $X$, then $\{x_n\}$ converges to some point $z\in X$. Clearly, the subsequence $\{x_{2n}\}$ and $\{x_{2n-1}\}$ converge to same point $z$. Since $A$ and $B$ are closed, we obtain that $z\in A\cap B$.
From (3.1), for all $x,y\in X$ and $k\in(0,1)$ with $H(Tx,Sy)>0$ and $d(A,B)=0$, we get

$$\begin{aligned}2\tau+F(H(Tx,Sy)) &\leq F(kd(x,y))\\ &\leq F(d(x,y)).\end{aligned}$$

Since $F$ is strictly increasing, we get $H(Tx,Sy)<d(x,y)$ and so $H(Tx,Sy)\leq d(x,y)$ for all $x,y\in X$. Then

$$d(x_{2n+1},Tz)\leq H(Sx_{2n},Tz)\leq d(x_{2n},z).$$

Passing to limit $n\to\infty$, we obtain $d(z,Tz)=d(A,B)$. Similarity, we also derive $d(Sz,z)=d(A,B)$.
**Case 2:** We will show that $T$ or $S$ have best proximity points in $A$ and $B$, respectively. Under the assumption of $d(A,B)>0$, suppose to the contrary, that is for all $a\in A$, $d(a,Ta)>d(A,B)$ and for all $b^{'}\in B$, $d(Sb^{'},b^{'})>d(A,B)$.
For each $a\in A$ and $b\in Ta$, we have

$$d(A,B)<d(a,Ta)\leq d(a,b). \tag{3.2}$$

Since $(T,S)$ is a multivalued cyclic $F$-contraction pair, such that

$$\begin{aligned}F(H(Ta,Sb)) &\leq F(kd(a,b)+(1-k)d(A,B))-2\tau && (3.3)\\ &< F(kd(a,b)+(1-k)d(A,B)) && (3.4)\end{aligned}$$

for all $a\in A$ and $b\in Ta$. Since $F$ is strictly increasing, we get

$$H(Ta,Sb)<kd(a,b)+(1-k)d(A,B) \tag{3.5}$$

for all $a\in A$ and $b\in Ta$.
Similarly, we have that for each $b^{'}\in B$ and $a^{'}\in Sb^{'}$, we get

$$F(H(Ta^{'},Sb^{'}))<F(kd(a^{'},b^{'})+(1-k)d(A,B)) \tag{3.6}$$

and

$$H(Ta^{'},Sb^{'})<kd(a^{'},b^{'})+(1-k)d(A,B). \tag{3.7}$$

Next we will construct the sequence $\{x_n\}$ in $A\cup B$. Let $x_0$ be arbitrary point in $A$ and $x_1\in Tx_0\subseteq B$. From (3.3), there exists $x_2\in Sx_1$ such that

$$\begin{aligned}F(d(x_1,x_2)) &\leq F(H(Tx_0,Sx_1))+\tau\\ &\leq F(kd(x_0,x_1)+(1-k)d(A,B))-2\tau+\tau\\ &\leq F(kd(x_0,x_1)+(1-k)d(A,B))-\tau\\ &< F(kd(x_0,x_1)+(1-k)d(A,B))\end{aligned}$$

and since $F$ is strictly increasing, we get

$$d(x_1,x_2)<kd(x_0,x_1)+(1-k)d(A,B). \tag{3.8}$$

Since $x_1 \in B$ and $x_2 \in Sx_1$ from (3.6), we can find $x_3 \in Tx_2$ such that

$$d(x_2, x_3) < kd(x_1, x_2) + (1-k)d(A, B). \tag{3.9}$$

Consequently, we can define the sequence $\{x_n\}$ in $A \cup B$ such that

$$x_{2n-1} \in Tx_{2n-2},\ x_{2n} \in Sx_{2n-1}$$

and

$$d(x_n, x_{n+1})) < kd(x_{n-1}, x_n) + (1-k)d(A, B) \tag{3.10}$$

for all $n \in \mathbb{N}$. Since $d(A, B) \leq d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, we get

$$\begin{aligned} d(x_n, x_{n+1}) &< kd(x_{n-1}, x_n) + (1-k)d(A, B) \\ &\leq kd(x_{n-1}, x_n) + (1-k)d(x_{n-1}, x_n) \\ &\leq d(x_{n-1}, x_n) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} d(x_n, x_{n+1}) &< kd(x_{n-1}, x_n) + (1-k)d(A, B) \\ &< k(kd(x_{n-2}, x_{n-1}) + (1-k)d(A, B)) + (1-k)d(A, B) \\ &< k^2 d(x_{n-2}, x_{n-1}) + (k-k^2)d(A, B) + (1-k)d(A, B) \\ &< k^2 d(x_{n-2}, x_{n-1}) + (1-k^2)d(A, B) \\ &\ \vdots \\ &< k^n d(x_0, x_1) + (1-k^n)d(A, B). \end{aligned} \tag{3.12}$$

Hence $d(A, B) \leq d(x_n, x_{n+1}) < k^n d(x_0, x_1) + (1-k^n)d(A, B)$ for all $n \in \mathbb{N}$.
Since $k \in (0, 1)$, we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B). \tag{3.13}$$

From equation (3.13), we get

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = d(A, B). \tag{3.14}$$

and

$$\lim_{n\to\infty} d(x_{2n+2}, x_{2n+1}) = d(A, B). \tag{3.15}$$

Since $\{x_{2n}\}$ and $\{x_{2n+2}\}$ are two sequences in $A$ and $\{x_{2n+1}\}$ is sequence $B$ with $(A, B)$ which satisfies the property UC*, we derive that

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+2}) = 0. \tag{3.16}$$

Since $(B, A)$ satisfies the property UC* and by (3.13), we have

$$\lim_{n\to\infty} d(x_{2n-1}, x_{2n+1}) = 0. \tag{3.17}$$

Next, we will show that for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m > n \geq N$, we have

$$\lim_{n\to\infty} d(x_{2m}, x_{2n+1}) \leq d(A, B) + \epsilon. \tag{3.18}$$

Suppose the contrary, that is there exists $\epsilon_0 > 0$ such that for each $k \geq 1$ there is $m_k > n_k \geq k$ such that

$$d(x_{2m_k}, x_{2n_k+1}) > d(A, B) + \epsilon_0. \tag{3.19}$$

Moreover, corresponding to $n_k$, we can choose $m_k$ in such a way that it is the smallest integer with $m_k > n_k \geq k$ satisfying (3.19). Then we obtain

$$d(x_{2m_k}, x_{2n_k+1}) > d(A,B) + \epsilon_0 \tag{3.20}$$

and

$$d(x_{2(m_k-1)}, x_{2n_k+1}) \leq d(A,B) + \epsilon_0. \tag{3.21}$$

From (3.20), (3.21) and the triangle inequality, we obtain

$$\begin{aligned} d(A,B) + \epsilon_0 &< d(x_{2m_k}, x_{2n_k+1}) \\ &\leq d(x_{2m_k}, x_{2(m_k-1)}) + d(x_{2(m_k-1)}, x_{2n_k+1}) \\ &\leq d(x_{2m_k}, x_{2(m_k-1)}) + d(A,B) + \epsilon_0. \end{aligned} \tag{3.22}$$

Using the fact that $\lim_{k\to\infty} d(x_{2m_k}, x_{2(m_k-1)}) = 0$. Letting $k \to \infty$ in (3.22), we get

$$\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) = d(A,B) + \epsilon_0. \tag{3.23}$$

From (3.10), (3.11) and $(T,S)$ is a multivalued cyclic $F$- contraction pair, we obtain

$$\begin{aligned} d(x_{2m_k}, x_{2n_k+1}) &\leq d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+2}, x_{2n_k+3}) + d(x_{2n_k+3}, x_{2n_k+1}) \\ &\leq d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+1}, x_{2n_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \\ &< d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) + (kd(x_{2m_k}, x_{2n_k+1}) \\ &\quad +(1-k)d(A,B)). \end{aligned} \tag{3.24}$$

Letting $k \to \infty$ in (3.24) and using (3.16), (3.17) and (3.23), we have

$$d(A,B) + \epsilon_0 < k(d(A,B) + \epsilon_0) + (1-k)d(A,B) = d(A,B) + k\epsilon_0$$

which is a contradiction. Therefore, (3.18) holds. Since (3.14) and (3.18) hold, by using property UC* of $(A,B)$, we obtain $d(x_{2n}, x_{2m}) \to 0$ as $n \to \infty$. Therefore $\{x_{2n}\}$ is a Cauchy sequence. Since $X$ is complete and A is closed, we have

$$\lim_{n\to\infty} x_{2n} = p \tag{3.25}$$

for some $p \in \overline{A} = A$. But

$$\begin{aligned} d(A,B) &\leq d(p, x_{2n-1}) \\ &\leq d(p, x_{2n}) + d(x_{2n}, x_{2n-1}) \end{aligned}$$

for all $n \in \mathbb{N}$. From (3.13) and (3.25),

$$\lim_{n\to\infty} d(p, x_{2n-1}) = d(A,B). \tag{3.26}$$

Since

$$\begin{aligned} d(A,B) &< d(x_{2n}, Tp) \\ &\leq H(S_{2n-1}, Tp) \\ &= H(Tp, Sx_{2n-1}) \\ &< kd(p, x_{2n-1}) + (1-k)d(A,B) \\ &\leq d(p, x_{2n-1}) \end{aligned} \tag{3.27}$$

for all $n \in \mathbb{N}$. By (3.25) and (3.26), we get

$$d(p, Tp) = d(A, B). \tag{3.28}$$

In a similar mode, we can conclude that the sequence $\{x_{2n-1}\}$ is a Cauchy sequence in $B$. Since $X$ is complete and $B$ is closed, we obtain

$$\lim_{n\to\infty} x_{2n-1} = q \tag{3.29}$$

for some $q \in \overline{B} = B$. Since

$$\begin{aligned} d(A, B) &\leq d(x_{2n}, q) \\ &\leq d(x_{2n}, x_{2n-1}) + d(x_{2n-1}, q) \end{aligned}$$

for all $n \in \mathbb{N}$. It follows from (3.13) and (3.29) that

$$\lim_{n\to\infty} d(x_{2n}, q) = d(A, B). \tag{3.30}$$

Since

$$\begin{aligned} d(A, B) &< d(Sq, x_{2n+1}) \\ &\leq H(Sq, Tx_{2n}) \\ &= H(Tx_{2n}, Sq) \\ &< kd(x_{2n}, q) + (1-k)d(A, B) \\ &\leq d(x_{2n}, q) \end{aligned} \tag{3.31}$$

for all $n \in \mathbb{N}$, then by (3.29) and (3.30), we have

$$d(q, Sq) = d(A, B). \tag{3.32}$$

From (3.28) and (3.32), we get a contradiction. Therefore, $T$ has a best proximity point in $A$ or $S$ has a best proximity point in $B$. This completes the proof. □

**Remark 3.3.** If $d(A, B) = 0$, then Theorem 3.2 yields existence of a fixed point in $A \cap B$ of two multivalued non-self mapping $S$ and $T$. Furthermore, if $A = B = X$ and $T = S$, then Theorem 3.2 reduces to multivalued $F$- contractions on metric spaces [23].

**Corollary 3.4.** Let $A$ and $B$ be non-empty closed convex subsets of a uniformly convex Banach space $X$, $T : A \to C_b(B)$ and $S : B \to C_b(A)$. If $(T, S)$ is a multivalued cyclic $F$-contraction pair, then $T$ has a best proximity in $A$ or $S$ has a best proximity point in $B$.

Now, we give some example for support our results.

**Example 3.5.** Consider the uniformly convex Banach space $X = \mathbb{R}$ with Euclidean norm. Let $A =$ [3,4] and $B =$ [-4,-3]. Then $A$ and $B$ are non-empty closed and convex subsets of $X$ and $d(A, B) = 6$. Since $(A, B)$ and $(B, A)$ satisfy the property UC*. Let $T : A \to C_b(B)$ and $S : B \to C_b(A)$ be defined as

$$Tx = \left[\frac{-x-3}{2}, -3\right], \ x \in [3, 4];$$

and

$$Sy = \left[3, \frac{-y+3}{2}\right],\ y \in [-4,-3].$$

Let $k \in (0,1)$ and $F : \mathbb{R}^+ \to \mathbb{R}$ is satisfy Definition 2.8 be defined by $F(t) = \ln(t)$ for all $t \in \mathbb{R}^+$ and $\tau > 0$. Next, we show that $(T, S)$ is a multivalued cyclic $F-$ contraction pair. For each $x \in A$ and $y \in B$, we have

$$\begin{aligned} H(Tx, Sy) &= H\left(\left[\frac{-x-3}{2}, -3\right], \left[3, \frac{-y+3}{2}\right]\right) \\ &\le \left|\left(\frac{-x-3}{2}\right) - \left(\frac{-y+3}{2}\right)\right| \\ &= \left|\frac{-x+y-6}{2}\right| \\ &\le \frac{1}{2}|x-y| + 3 \\ &= \frac{1}{2}d(x,y) + \frac{1}{2}d(A,B) \\ &= kd(x,y) + (1-k)d(A,B). \end{aligned}$$

Since $\tau > 0$, we get $0 < e^{-2\tau} < 1$. Hence $H(Tx, Sy) \le e^{-2\tau}kd(x,y) + e^{-2\tau}(1-k)d(A,B))$.
Since $F$ strictly increasing, we get

$$\begin{aligned} F(H(Tx, Sy)) &\le F(e^{-2\tau}(kd(x,y) + (1-k)d(A,B))) \\ &= \ln(e^{-2\tau}(kd(x,y) + (1-k)d(A,B))) \\ &= \ln(e^{-2\tau}) + \ln(kd(x,y) + (1-k)d(A,B)) \\ &= -2\tau + \ln(kd(x,y) + (1-k)d(A,B)). \end{aligned}$$

It follows that $F(H(Tx, Sy)) + 2\tau \le F(kd(x,y) + (1-k)d(A,B))$. Therefore, all assumptions of Corollary 3.4 are satisfied and then $T$ has a best proximity point in $A$, that is a point $x = 3$. Moreover, $S$ also has a best proximity point in $B$, that is a point $y = -3$.

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