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Dhage iteration method for PBVPs of nonlinear first order hybrid integro-differential equations

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Abstract

In this paper, author proves the algorithms for the existence as well as the approximation of solutions to a couple of periodic boundary value problems of nonlinear first order ordinary integro-differential equations using operator theoretic techniques in a partially ordered metric space. The main results rely on the Dhage iteration method embodied in the recent hybrid fixed point theorems of Dhage in a partially ordered normed linear space. The approximation of the solutions are obtained under weaker mixed partial continuity and partial Lipschitz conditions. Our hypotheses and abstract results are also illustrated by some numerical examples.

Keywords: Hybrid differential equation; Hybrid fixed point theorem; Dhage iteration method; Existence and approximation theorem.
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1. Introduction

Given a closed and bounded interval $J = [0, T]$ of the real line $\mathbb{R}$ consider the periodic boundary value problem (PBVP) for the first order ordinary nonlinear hybrid integrodifferential equation (HIDE),

$$\left.\begin{aligned} x'(t) + \lambda x(t) &= f\left(t, x(t), \int_0^t g(s, x(s))\, ds\right),\ t \in J, \\ x(0) &= x(T), \end{aligned}\right\} \tag{1.1}$$

for some $\lambda \in \mathbb{R}$, $\lambda > 0$, where $g : J \times \mathbb{R} \to \mathbb{R}$ and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

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By a *solution* of the HIDE (1.1), we mean a differentiable function $u \in C(J, \mathbb{R})$ that satisfies problem (1.1), where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

The HIDE (1.1) is well-known in the literature and includes

$$\left.\begin{aligned} x'(t) + \lambda x(t) = f(t, x(t)),\ t \in J, \\ x(0) = x(T), \end{aligned}\right\} \tag{1.2}$$

$$\left.\begin{aligned} x'(t) + \lambda x(t) = \int_0^t g(s, x(s))\, ds,\ t \in J, \\ x(0) = x(T), \end{aligned}\right\} \tag{1.3}$$

and

$$\left.\begin{aligned} x'(t) + \lambda x(t) = f\left(t, \int_0^t x(s)\, ds\right),\ t \in J, \\ x(0) = x(T), \end{aligned}\right\} \tag{1.4}$$

as special cases. That is, the results in this paper include results for the differential equations (1.2), (1.3), and (1.4) on J.

The existence and uniqueness of solutions of the nonlinear HIDE (1.1) under usual compactness and Lipschitz type conditions have been discussed at length in the literature. These conditions are considered to be very strong assumptions in the study of nonlinear differential and integral equations. Similarly, upper and lower solution method and monotone iterative technique also require the assumption that both the lower as well as upper solution exist and preserve the order relation. However, a recent trend for the existence of solution for such nonlinear problem is to assume only one of lower and upper solutions. In the present paper, we prove existence and uniqueness of solutions of the HDE (1.1) under the weaker partial compactness and partial Lipschitz type conditions via the Dhage iteration method by assuming one of lower and upper solutions to exist.

The remainder of this paper is organized as follows. In Section 2, we give some preliminary concepts and key fixed point theorems that will be used in subsequent parts of the paper. In Section 3, we present existence and uniqueness results for initial value problems, and in Section 4, we give an existence result for initial value problems for hybrid differential equations with linear perturbations of the first type.

2. Auxiliary Results

Unless otherwise mentioned, throughout this paper we let E denote a partially ordered real normed linear space with the order relation $\preceq$ and the norm $\|\cdot\|$ in which addition and scalar multiplication by positive real numbers are preserved by $\preceq$. A few details on such partially ordered normed linear spaces appear in Dhage [3] and the references therein.

Two elements x and y in E are said to be **comparable** if either the relation $x \preceq$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all elements of C are comparable. We say that E is *regular* if for any nondecreasing (resp. nonincreasing) sequence $\{x_n\}$ in E such that $x_n \to x^*$ as $n \to \infty$, we have that $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. Conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [19] and the references therein.

We will need the following definitions in the sequel.

Definition 2.1. A mapping $\mathcal{T} : E \to E$ is said to be **isotone** or **monotone nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is **monotone nonincreasing** if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is said to be **monotonic** or simply **monotone** if it is either monotone nondecreasing or monotone nonincreasing on E.

The following terminologies may be found in any book on nonlinear analysis and applications such as Kreyszig [20] or Granas and Dugundji [17].

Definition 2.2. An operator $\mathcal{T}$ from a normed linear space E into itself is **compact** if $\mathcal{T}(E)$ is a relatively compact subset of E. We say that $\mathcal{T}$ is **totally bounded** if for any bounded subset S of E, $\mathcal{T}(S)$ is a relatively compact subset of E. If $\mathcal{T}$ is continuous and totally bounded, then it is called **completely continuous** on E.

Definition 2.3. [Dhage [4]] A mapping $\mathcal{T} : E \to E$ is **partially continuous** at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. The mapping $\mathcal{T}$ is partially continuous on E if it is partially continuous at every point in E.

It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E and vice versa.

Definition 2.4. A non-empty subset S of the partially ordered Banach space E is called **partially bounded** if every chain C in S is bounded. An operator $\mathcal{T}$ on a partially normed linear space E into itself is called **partially bounded** if $\mathcal{T}(E)$ is a partially bounded subset of E. $\mathcal{T}$ is called **uniformly partially bounded** if all chains C in $\mathcal{T}(E)$ are bounded by a unique constant.

Definition 2.5. A non-empty subset S of the partially ordered Banach space E is called **partially compact** if every chain C in S is a relatively compact subset of E. A mapping $\mathcal{T} : E \to E$ is called **partially compact** if $\mathcal{T}(E)$ is a partially relatively compact subset of E. $\mathcal{T}$ is called **uniformly partially compact** if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called **partially totally bounded** if for any bounded subset S of E, $\mathcal{T}(S)$ is a partially relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

Remark 2.6. Suppose that $\mathcal{T}$ is a nondecreasing operator on E into itself. Then $\mathcal{T}$ is a partially bounded or partially compact if $\mathcal{T}(C)$ is a bounded or relatively compact subset of E for each chain C in E.

Definition 2.7. The operator $\mathcal{T}$ is said to be **partially totally bounded** on E if $\mathcal{T}(E)$ is a partially relatively compact subset of E. If the operator $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

Remark 2.8. Note that every compact mapping on a partially normed linear space is partially compact, and every partially compact mapping is partially totally bounded. However, the reverse implications do not hold. Every completely continuous mapping is partially completely continuous. Every partially completely continuous mapping is partially continuous and partially totally bounded, but the converse may not be true.

Definition 2.9. [Dhage [4, 5]] The order relation $\preceq$ and the metric d on a non-empty set E are said to be $\mathcal{D}$**-compatible** if for any monotone sequence $\{x_n\}$ in E with a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to x^*, the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be $\mathcal{D}$-compatible if $\preceq$ and the metric d defined by the norm $\|\cdot\|$ are $\mathcal{D}$-compatible. A subset S of E is called **Janhavi** if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are $\mathcal{D}$-compatible in it. In particular, if $S = E$, then E is called a **Janhavi metric** or **Janhavi Banach space**.

Clearly, the set $\mathbb{R}$ of real numbers with the usual order relation $\leq$ and the norm defined by the absolute value function has this property. Similarly, every finite dimensional Euclidean space $\mathbb{R}^n$ possesses the compatibility property with respect to the usual component-wise order relation $\leq$ and the standard norm $\|\cdot\|$ in $\mathbb{R}^n$ and so is a **Janhavi Banach space**.

The **Dhage iteration principle** developed in a series of papers [4, 5, 6] is embodied in the following hybrid fixed point theorems, which are the main tools used in obtaining the results in this paper. The central idea of Dhage iteration principle may be described as "**the monotonic convergence of a sequence of successive approximations to the solution of a nonlinear equation beginning with a lower or an upper solution of the equation as its initial approximation**." The aforesaid convergence principle forms a very useful tool in the existence theory of nonlinear analysis and is called **Dhage iteration method** for nonlinear equations. As will be seen, the Dhage iteration method is different from the usual Picard's successive iterations. The details of Dhage iteration method along with its applications appear in Dhage [9, 10, 11], Dhage and Dhage [14, 15], Dhage *al.* [16] and references therein.

Theorem 2.10. [Dhage [5]] Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain C of E. Let $\mathcal{T} : E \to E$ be a partially continuous, nondecreasing, and partially compact operator. If there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $\mathcal{T}x_0 \preceq x_0$, then the operator equation $\mathcal{T}x = x$ has a solution x^* in E, and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.

Remark 2.11. The regularity of E in Theorem 2.10 above may be replaced with a stronger continuity condition of the operator $\mathcal{T}$ on E (see Dhage [5]).

The following hybrid fixed point theorems will be used to prove some of our existence and uniqueness results for the solutions of the HIDE (1.1). We need the following notion of a $\mathcal{D}$-function in these theorems.

Definition 2.12. An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$.

Definition 2.13. An operator $\mathcal{T} : E \to E$ is a partially nonlinear $\mathcal{D}$-contraction if there exists a $\mathcal{D}$-function ψ such that

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi\big(\|x - y\|\big) \tag{2.1}$$

for all comparable elements $x, y \in E$, where $0 < \psi(r) < r$ for $r > 0$. In particular, if $\psi(r) = k\,r$, $k > 0$, then $\mathcal{T}$ is a partially linear contraction on E with a contraction constant k.

Theorem 2.14. [Dhage [4]] Let $(E, \preceq, \|\cdot\|)$ be a partially ordered Banach space and let $\mathcal{T}: E \to E$ be a nondecreasing and partially nonlinear $\mathcal{D}$-contraction. Suppose that there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$. If $\mathcal{T}$ is continuous or E is regular, then $\mathcal{T}$ has a fixed point x^*, and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*. Moreover, the fixed point x^* is unique if every pair of elements in E has a lower and an upper bound.

Theorem 2.15. [Dhage [6, 7]] Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular, partially ordered, complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain C of E. Let $\mathcal{A}$, $\mathcal{B}: E \to E$ be two nondecreasing operators such that:

(a) $\mathcal{A}$ is a partially bounded and partially nonlinear $\mathcal{D}$-contraction;

(b) $\mathcal{B}$ is partially continuous and partially compact;

(c) there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0 + \mathcal{B}x_0$.

Then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^* in E, and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, $n = 0, 1, \ldots$, converges monotonically to x^*.

Remark 2.16. We remark that hypothesis (a) of Theorem 2.10 implies that operator $\mathcal{A}$ is partially continuous on E. The regularity of E in above Theorem 2.10 may be replaced with a stronger continuity condition of the operators $\mathcal{A}$ and $\mathcal{B}$ on E which is a result proved in Dhage [4]. Again, the compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E holds if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$.

Notice that the Dhage iteration method presented in the above hybrid fixed point theorems has been employed in Dhage [8, 9], Dhage and Dhage [14] and Dhage *et al.* [16] to approximate solutions of initial value problems for nonlinear first order ordinary differential equation under some natural hybrid conditions. In the following section, we approximate the solutions of certain IVPs for nonlinear integro-differential equations via successive approximations beginning with a lower or upper solution.

3. Existence and Uniqueness Theorems

The equivalent integral form of the HIDE (1.1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.1}$$

and

$$x \leq y \quad \text{if and only if} \quad x(t) \leq y(t) \quad \text{for all} \quad t \in J. \tag{3.2}$$

Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and is also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ is regular and is a lattice, so every pair of elements in the space has an upper and a lower bound in the space. The next lemma concerning the compatibility of sets in $C(J, \mathbb{R})$ follows by an application of the Arzellá-Ascoli theorem.

Lemma 3.1. *Let $\big(C(J, \mathbb{R}), \leq, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by* (3.1) *and* (3.2) *respectively. Then, every partially compact subset of $C(J, \mathbb{R})$ possesses compatibility property w.r.t. $\|\cdot\|$ and $\leq$ and so is Janhavi.*

Proof . Let S be a partially compact subset of $C(J,\mathbb{R})$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \le x_2(t) \le \cdots \le x_n(t) \le \cdots$$

for each $t \in J$. Suppose that a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}_{k\in\mathbb{N}}$ of the monotone sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ converges at $t \in J$. By the monotonicity, the sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ itself is convergent and converges to a point $x(t)$ in $\mathbb{R}$ for each $t \in J$, i.e., the sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ converges point-wise in S. To show the convergence is uniform, it suffices to show that the sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set, and consequently $\{x_n\}_{n\in\mathbb{N}}$, is an equicontinuous sequence by the Arzelá-Ascoli theorem. Hence, $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges uniformly to x. Therefore, $\|\cdot\|$ and $\le$ are compatible in S, and this proves the lemma. □

We need the following definition in the sequel.

Definition 3.2. A differentiable function $u \in C(J,\mathbb{R})$ is a lower solution of the HIDE (1.1) if it satisfies

$$\left.\begin{aligned} u'(t) + \lambda u(t) \le f\left(t, u(t), \int_0^t g(s,u(s))\,ds\right), \\ u(0) \le x(T), \end{aligned}\right\}$$

for all $t \in J$. Similarly, an upper solution $v \in C^1(J,\mathbb{R})$ to the HIDE (3.1) is defined on J by reversing the above inequalities.

3.1. Existence theorem

We will make use of the following assumptions:

(H_1) There exists a constant $M_f > 0$ such that $|f(t,x,y)| \le M_f$ for all $t \in J$ and $x \in \mathbb{R}$.

(H_2) The function $f(t,x,y)$ is monotone nondecreasing in x and y for each $t \in J$.

(H_3) The function $g(t,x)$ is monotone nondecreasing in x for each $t \in J$.

(H_4) The HIDE (3.1) has a lower solution $u \in C^1(J,\mathbb{R})$.

The following useful lemma is obvious and may be found in Dhage [2] and Nieto [22].

Lemma 3.3. For any function $\sigma \in L^1(J,\mathbb{R})$, x is a solution to the differential equation

$$\left.\begin{aligned} x'(t) + \lambda x(t) = \sigma(t), \quad t \in J, \\ x(0) = x(T), \end{aligned}\right\} \tag{3.3}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T G_\lambda(t,s)\,\sigma(s)\,ds \tag{3.4}$$

where,

$$G_\lambda(t,s) = \begin{cases} \dfrac{e^{\lambda s - \lambda t + \lambda T}}{e^{\lambda T} - 1}, & \text{if} \quad 0 \le s \le t \le T, \\ \dfrac{e^{\lambda s - \lambda t}}{e^{\lambda T} - 1}, & \text{if} \quad 0 \le t < s \le T. \end{cases} \tag{3.5}$$

Notice that the Green's function G_λ is continuous and nonnegative on $J \times J$ and therefore, the number

$$K_\lambda := \max\{\, |G_\lambda(t,s)| \,:\, t,s \in [0,T]\,\}$$

exists for all $\lambda \in \mathbb{R}^+$. For the sake of convenience, we write $G_\lambda(t,s) = G(t,s)$ and $K_\lambda = K$.

Lemma 3.4. *If there exists a function $u \in C(J,\mathbb{R})$ such that*

$$\left.\begin{aligned} u'(t) + \lambda u(t) &\le \sigma(t), \quad t \in J, \\ u(0) &\le u(T), \end{aligned}\right\} \tag{3.6}$$

then

$$u(t) \le \int_0^T G(t,s)\,\sigma(s)\,ds \tag{3.7}$$

for all $t \in J$, where $G(t,s)$ is a Green's function given by (3.5).

Proof . Suppose that the function $u \in C(J,\mathbb{R})$ satisfies the inequalities given in (3.14). Multiplying the first inequality in (3.14) by $e^{\lambda t}$,

$$\left(e^{\lambda t}u(t)\right)' \le e^{\lambda t}\sigma(t).$$

A direct integration of above inequality from 0 to t yields

$$e^{\lambda t}u(t) \le u(0) + \int_0^t e^{\lambda s}\sigma(s)\,ds, \tag{3.8}$$

for all $t \in J$. Therefore, in particular,

$$e^{\lambda T}u(T) \le u(0) + \int_0^T e^{\lambda s}\sigma(s)\,ds. \tag{3.9}$$

Now $u(0) \le u(T)$, so one has

$$u(0)e^{\lambda T} \le u(T)e^{\lambda T}. \tag{3.10}$$

From (3.18) and (3.19) it follows that

$$e^{\lambda T}u(0) \le u(0) + \int_0^T e^{\lambda s}\sigma(s)\,ds \tag{3.11}$$

which further yields

$$u(0) \le \int_0^T \frac{e^{\lambda s}}{(e^{\lambda T}-1)}\sigma(s)\,ds. \tag{3.12}$$

Substituting (3.12) in (3.7) we obtain

$$u(t) \le \int_0^T G(t,s)\sigma(s)\,ds,$$

for all $t \in J$. This completes the proof. □

Similarly, we have:

Lemma 3.5. If there exists a function $v \in C(J,\mathbb{R})$ such that

$$\left.\begin{aligned} v'(t) + \lambda v(t) \geq \sigma(t), \ \ t \in J, \\ v(0) \geq v(T), \end{aligned}\right\}$$

then

$$v(t) \geq \int_0^T G(t,s)\,\sigma(s)\,ds$$

for all $t \in J$, where $G(t,s)$ is a Green's function given by (3.5).

Our main existence result in this section is contained in the following theorem.

Theorem 3.6. *Assume that conditions* (H_1)*–*(H_4) *hold. Then the HIDE* (1.1) *has a solution* x^* *defined on* J *and the sequence* $\{x_n\}_{n=1}^{\infty}$ *of successive approximations defined by*

$$\begin{aligned} x_1(t) &= u(t), \\ x_{n+1}(t) &= \int_0^T G(t,s) f\left(s, x_n(s), \int_0^s g(\tau, x_n(\tau))\,d\tau\right) ds, \end{aligned} \tag{3.13}$$

for all $t \in J$*, converges monotonically to* x^**.*

Proof . By Lemma 3.3, the HIDE (1.1) is equivalent to the nonlinear integral equation

$$x(t) = \int_0^T G(t,s) f\left(s, x(s), \int_0^s g(\tau, x(\tau))\,d\tau\right) ds, \ \ t \in J. \tag{3.14}$$

Set $E = C(J,\mathbb{R})$. Then, from Lemma 3.1 it follows that every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E. Define the operator $\mathcal{T}$ by

$$\mathcal{T}x(t) = \int_0^T G(t,s) f\left(s, x(s), \int_0^s g(\tau, x(\tau))\,d\tau\right) ds, \ \ t \in J. \tag{3.15}$$

From the continuity of the integral, it follows that $\mathcal{T}$ maps E into itself. The HIDE (3.1) is then equivalent to the operator equation

$$\mathcal{T}x(t) = x(t), \ \ t \in J. \tag{3.16}$$

Through a series of steps, we shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.10.

Step I: $\mathcal{T}$ *is a nondecreasing operator on* E*.*

Let x, $y \in E$ with $x \leq y$. Then, from (H_2), we obtain

$$\begin{aligned} \mathcal{T}x(t) &= \int_0^T G(t,s) f\left(s, x(s), \int_0^s g(\tau, x(\tau))\,d\tau\right) ds \\ &\leq \int_0^T G(t,s) f\left(s, y(s), \int_0^s g(\tau, y(\tau))\,d\tau\right) ds \\ &= \mathcal{T}y(t), \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{T}$ is a nondecreasing operator on E.

Step II: $\mathcal{T}$ *is partially continuous operator on* E.

Let $\{x_n\}$ be a sequence of points of a chain C in E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by the dominated convergence theorem,

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{T}x_n(t) &= \lim_{n\to\infty} \left[\int_0^T G(t,s) f\left(s, x_n(s), \int_0^s g(\tau, x_n(\tau))\, d\tau\right) ds\right] \\
&= \int_0^T G(t,s) \left[\lim_{n\to\infty} f\left(s, x_n(s), \int_0^s g(\tau, x_n(\tau))\, d\tau\right)\right] ds \\
&= \int_0^T G(t,s) f\left(s, x(s), \int_0^s g(\tau, x(\tau))\, d\tau\right) ds \\
&= \mathcal{T}x(t),
\end{aligned}$$

for all $t \in J$. This shows that $\{\mathcal{T}x_n\}$ converges to $\mathcal{T}x$ pointwise on J.

Next, we show that $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1,\, t_2 \in J$ with $t_1 < t_2$. Then

$$\begin{aligned}
&|\mathcal{T}x_n(t_2) - \mathcal{T}x_n(t_1)| \\
&\quad \leq \left|\int_0^T G(t_1,s) f\left(s, x_n(s), \int_0^s g(\tau, x_n(\tau))\, d\tau\right) ds \right.\\
&\qquad \left. - \int_0^T G(t_1,s) f\left(s, x_n(s), \int_0^s g(\tau, x_n(\tau))\, d\tau\right) ds\right| \\
&\quad \leq \left|\int_0^T \big[G(t_2,s) - G(t_1,s)\big] f\left(s, x_n(s), \int_0^s g(\tau, x_n(\tau))\, d\tau\right) ds\right| \\
&\quad \leq \left|\int_0^T \big|G(t_2,s) - G(t_1,s)\big| \left| f\left(s, x_n(s), \int_0^s g(\tau, x_n(\tau))\, d\tau\right)\right| ds\right| \\
&\quad \leq M_f \int_0^T \big|G(t_2,s) - G(t_1,s)\big|\, ds \\
&\quad \to 0 \quad as \quad t_1 \to t_2,
\end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{T}x_n \to \mathcal{T}x$ uniformly and therefore, $\mathcal{T}$ is a partially continuous operator on E.

Step III: $\mathcal{T}$ *is partially compact operator on* E.

Let C be an arbitrary chain in E. We will show that $\mathcal{T}(C)$ is a uniformly bounded and equicontinuous set in E. To show that $\mathcal{T}(C)$ is uniformly bounded, let $x \in C$. Then,

$$\begin{aligned}
|\mathcal{T}x(t)| &\leq \left|\int_0^T G(t,s) f\left(s, x(s), \int_0^s g(\tau, x(\tau))\, d\tau\right) ds\right| \\
&\leq \int_0^T G(t,s) \left| f\left(s, x(s), \int_0^s g(\tau, x(\tau))\, d\tau\right)\right| ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \; K\, M_f\, T \\
&= \; r,
\end{aligned}
$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{T}x\| \leq r$ for all $x \in C$. Hence, $\mathcal{T}(C)$ is a uniformly bounded subset of E.

To show that $\mathcal{T}(C)$ is an equicontinuous set in E, let t_1, $t_2 \in J$ with $t_1 < t_2$. Then,

$$
\begin{aligned}
&|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)| \\
&\quad = \left| \int_0^T G(t_1,s) f\left(s, x(s), \int_0^s g(\tau, x(\tau))\, d\tau\right) ds \right. \\
&\qquad \left. - \int_0^T G(t_1,s) f\left(s, x(s), \int_0^s g(\tau, x(\tau))\, d\tau\right) ds \right| \\
&\quad \leq \left| \int_0^T \big[G(t_2,s) - G(t_1,s)\big] f\left(s, x(s), \int_0^s g(\tau, x(\tau))\, d\tau\right) ds \right| \\
&\quad \leq \left| \int_0^T \big|G(t_2,s) - G(t_1,s)\big| \left| f\left(s, x(s), \int_0^s g(\tau, x(\tau))\, d\tau\right)\right| ds \right| \\
&\quad \leq M_f \int_0^T \big|G(t_2,s) - G(t_1,s)\big|\, ds \\
&\quad \to 0 \quad as \quad t_1 \to t_2,
\end{aligned}
$$

uniformly for all $x \in C$. Hence $\mathcal{T}(C)$ is compact subset of E and consequently $\mathcal{T}$ is a partially compact operator on E into itself.

Step IV: *u satisfies the operator inequality $u \leq \mathcal{T}u$.*

Since condition (H_4) holds, u is a lower solution of (3.1) defined on J so that

$$
\left.
\begin{aligned}
u'(t) + \lambda u(t) \leq f\left(t, x(t), \int_0^t g(s, x(s))\, ds\right) \\
u(0) \leq x(T)
\end{aligned}
\right\} \tag{3.17}
$$

for all $t \in J$. Applying Lemma 3.4 to the inequality (3.17), we obtain

$$
u(t) \leq \int_{t_0}^t G(t,s) f\left(s, x(s), \int_0^s g(\tau, x(\tau))\, d\tau\right) ds, \tag{3.18}
$$

for all $t \in J$. This shows that u is a lower solution of the operator equation $x = \mathcal{T}x$.

Thus, $\mathcal{T}$ satisfies all the conditions of Theorem 2.10, and in view of Remark 2.11, we can conclude that the operator equation $\mathcal{T}x = x$ has a solution. Thus, the integral equation and the HIDE (1.1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (3.13) converges monotonically to x^*. This completes the proof of the theorem. □

Remark 3.7. The conclusion of Theorem 3.6 also remains true if we replace the condition (H_4) with the following one.

(H_4') The HIDE (3.1) has an upper solution $v \in C^1(J, \mathbb{R})$.

We illustrate our result with the following example.

Example 3.8. Let $J = [0, 1]$ and consider the HIDE

$$\left.\begin{aligned} x'(t) + x(t) = \tanh x(t) + \tanh\Big(\int_0^t g(s, x(s))\,ds\Big),\ t \in J,\\ x(0) = x(1). \end{aligned}\right\} \tag{3.19}$$

where $g : J \times \mathbb{R} \to \mathbb{R}$ is the function defined by

$$g(t, x) = \begin{cases} x + 1, & \text{if} \quad x \le 0,\\ 1 + \log(x + 1), & \text{if} \quad x > 0. \end{cases}$$

Here, $\lambda = 1$, $c = 1$, and $f(t, x, y) = \tanh x + \tanh y$. Clearly, the functions f and g are continuous on $J \times \mathbb{R}$, and f satisfies (H_1) with $M_f = 2$. Moreover, $g(t, x)$ is nondecreasing in x for each $t \in J$, and $f(t, x, y)$ is nondecreasing in x and y for each $t \in J$, so conditions (H_2) and (H_3) are satisfied.

Finally, the HIDE (3.19) has a lower solution u defined by $u(t) = -2e^t$ on J. Thus, all the hypotheses of Theorem 3.6 are satisfied, and so (3.19) has a solution x^* defined on J, and the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1(t) &= u(t),\\ x_{n+1}(t) &= \int_0^1 G(t, s)\ \tanh x_n(s)\,ds\\ &\quad + \int_0^1 G(t, s) \tanh\left(\int_0^s g(\tau, x_n(\tau))\,d\tau\right)\,ds \end{aligned}$$

for all $t \in J$, converges monotonically to x^*, where $G(t, s)$ is a Green's function associated with the homogeneous PBVP

$$\left.\begin{aligned} x'(t) + x(t) = 0,\ \ t \in J,\\ x(0) = x(1), \end{aligned}\right\} \tag{3.20}$$

given by

$$G(t, s) = \begin{cases} \dfrac{e^{s-t+1}}{e - 1}, & \text{if} \quad 0 \le s \le t \le 1,\\[2mm] \dfrac{e^{s-t}}{e - 1}, & \text{if} \quad 0 \le t < s \le 1. \end{cases} \tag{3.21}$$

Again, a similar conclusion holds if we replace the lower solution u with the upper solution $v(t) = 2e^t$, $t \in [0, 1]$ in view of Remark 3.14.

3.2. Uniqueness theorem

Next, we prove a uniqueness theorem for the HIDE (1.1) under the weaker partially Lipschitz condition. We will need the following conditions.

(H_5) There exists a constant $L > 0$ such that

$$0 \le g(t,x) - g(t,y) \le L(x-y)$$

for all $t \in J$ and $x, y \in \mathbb{R}$ with $x \ge y$.

(H_6) There exists $\mathcal{D}$-functions ψ_1 and ψ_2 such that

$$0 \le f(t,x_1,x_2) - f(t,y_1,y_2) \le \psi_1(x_1-y_1) + \psi_2(x_2-y_2)$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \ge y_1$ and $x_2 \ge y_2$. Moreover, $\psi(r) = KT\big[\psi_1(r)+\psi_2(LTr)\big] < r$ for each $r > 0$.

Theorem 3.9. *Assume that conditions (H_4)–(H_6) hold. Then the HIDE* (1.1) *has a unique solution x^* defined on J, and the sequence $\{x_n\}$ of successive approximations defined by* (3.13) *converges monotonically to x^*.*

Proof . Set $E = C(J,\mathbb{R})$. Clearly, E is a lattice w.r.t. the order relation $\le$ and so lower and upper bounds exist for every pair of elements in E. Define the operator $\mathcal{T}$ by (3.15). Then, the HIDE (1.1) is equivalent to the operator equation (3.16). We shall show that $\mathcal{T}$ satisfies all the conditions of Theorem 2.14.

Clearly, $\mathcal{T}$ is a nondecreasing operator from E into itself. We wish to show that the operator $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-contraction on E, so let $x, y \in E$ with $x \ge y$. Then, by (H_5) and (H_6),

$$\begin{aligned}
&|\mathcal{T}x(t) - \mathcal{T}y(t)| \\
&\quad \le \left| \int_0^T G(t,s) f\left(s, x(s), \int_0^s g(\tau, x(\tau))\,d\tau\right) ds \right. \\
&\qquad\qquad \left. - \int_0^T G(t,s) f\left(s, y(s), \int_0^s g(\tau, y(\tau))\,d\tau\right) ds \right| \\
&\quad \le \int_0^T G(t,s) \left| f\left(s, x(s), \int_0^s g(\tau, x(\tau))\,d\tau\right) \right. \\
&\qquad\qquad \left. - f\left(s, y(s), \int_0^s g(\tau, y(\tau))\,d\tau\right) \right| ds \\
&\quad \le \int_0^T G(t,s) \left[\psi_1(x(s)-y(s)) + \psi_2\left(\int_0^t [g(\tau,x(\tau)) - g(\tau,x(\tau))]\tau \right) \right] ds \\
&\quad \le \int_0^T G(t,s) \left[\psi_1(x(s)-y(s)) + \psi_2\left(\int_0^t L(x(\tau)-y(\tau))\,d\tau \right) \right] ds \\
&\quad \le \int_0^T G(t,s) \left[\psi_1(|x(s)-y(s)|) + \psi_2\left(\int_0^t L|x(\tau)-y(\tau)|\,d\tau \right) \right] ds
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^T K\left[\psi_1(\|x-y\|)+\psi_2\Big(\int_0^s L\|x-y\|\,d\tau\Big)\right] ds \\
&\leq \int_0^T K\left[\psi_1(\|x-y\|)+\psi_2(LT\|x-y\|)\right] ds \\
&\leq \psi(\|x-y\|)
\end{aligned} \tag{3.22}$$

for all $t \in J$, where $\psi(r) = KT\big[\psi_1(r)+\psi_2(LTr)\big] < r$, $r > 0$.

Taking the supremum over t, we obtain

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x-y\|)$$

for all $x, y \in E$ with $x \geq y$. As a result, $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-contraction on E. Furthermore, as in the proof of Theorem 3.6, it can be shown that the function u given in condition (H_4) satisfies the operator inequality $u \leq \mathcal{T}u$ on J. Now a direct application of Theorem 2.14 yields that the HIDE (1.1) has a unique solution x^*, and the sequence $\{x_n\}$ of successive approximations defined by (3.13) converges monotonically to x^*. □

Remark 3.10. The conclusion of Theorem 3.9 also remains true if we replace the hypothesis (H_4) with (H_4').

To illustrate this theorem, we present the following example.

Example 3.11. Let $J = [0, 1]$ and consider the HIDE

$$\left.\begin{aligned}
x'(t) + x(t) = \frac{1}{2}\left[\tan^{-1} x(t) + \tan^{-1}\Big(\int_0^t g(s, x(s))\,ds\Big)\right],\ t \in J, \\
x(0) = x(1),
\end{aligned}\right\} \tag{3.23}$$

where $g : J \times \mathbb{R} \to \mathbb{R}$ is the function defined by

$$g(t,x) = \begin{cases} 1, & \text{if} \quad x \leq 0, \\ 1 + \dfrac{x}{1+x}, & \text{if} \quad x > 0. \end{cases}$$

Here, $\lambda = 1$, $c = 1$, $f(t, x, y) = \frac{1}{2}[\tan^{-1} x + \tan^{-1} y]$. Clearly, the functions f and g are continuous on $J \times \mathbb{R} \times \mathbb{R}$ and $J \times \mathbb{R}$, respectively. The function f satisfies (H_1) with $M_f = \frac{\pi}{2}$ and it is easy to show that g satisfies (H_5) with $L = 1$. Moreover, $f(t, x, y)$ is nondecreasing in x and y for each $t \in J$. To show that f satisfies (H_6) on $J \times \mathbb{R} \times \mathbb{R}$, let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ be such that $x_1 \geq y_1$ and $x_2 \geq y_2$. Then,

$$\begin{aligned}
0 &\leq f(t, x_1, x_2) - f(t, y_1, y_2) \\
&\leq \frac{1}{2}\Big[\tan^{-1} x_1 - \tan^{-1} y_1 + \tan^{-1} x_2 - \tan^{-1} y_2\Big] \\
&\leq \frac{1}{2}\cdot\frac{x_1 - y_1}{1+\xi_1^2} + \frac{1}{2}\cdot\frac{x_2 - y_2}{1+\xi_2^2} \\
&\leq \psi_1(x_1 - y_1) + \psi_2(x_2 - y_2)
\end{aligned}$$

for all $t \in J$ and for some $x_1 > \xi_1 > y_1$ and $x_2 > \xi_2 > y_2$, where ψ_1 and ψ_2 are $\mathcal{D}$-functions defined by $\psi_1(r) = \dfrac{1}{2}\dfrac{r}{1+\xi_1^2}$ and $\psi_2(r) = \dfrac{1}{2}\dfrac{r}{1+\xi_2^2}$ for $0 < \xi_1,\, \xi_2 < r$. Furthermore,

$$KT\big[\psi_1(r) + \psi_2(LTr)\big] \leq \frac{1}{2} \cdot \big[\psi_1(r) + \psi_2(r)\big] = \frac{r}{1+\xi^2} < r,$$

where $\xi = \min\{\xi_1, \xi_2\}$.

Finally, the HIDE (3.23) has a lower solution $u(t) = -4e^t$ defined on J. Thus, all the hypotheses of Theorem 3.9 are satisfied and so we conclude that the HIDE (3.23) has a unique solution x^* defined on J. In addition, the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_0(t) &= u(t), \\ x_{n+1}(t) &= \frac{1}{2}\int_0^1 G(t,s)\, \tan^{-1} x_n(s)\, ds \\ &\quad + \frac{1}{2}\int_0^1 G(t,s)\, \tan^{-1}\left(\int_0^s g(\tau, x_n(\tau))\, d\tau\right) ds \end{aligned}$$

for all $t \in J$, converges monotonically to x^*, where $G(t,s)$ is a Green's function associated with the homogeneous PBVP (3.20) given by (3.21).

4. Linear Perturbations of the First Type

Sometimes it is possible that the nonlinearity f involved in the HIDE (1.1) satisfies neither the hypotheses of Theorem 3.6 nor the hypotheses of Theorem 3.9. However, by splitting the function f into the form $f = f_1 + f_2$, the functions f_1 and f_2 may satisfy the conditions of Theorems 3.6 and 3.9, respectively. In the terminology of Dhage [3], the resulting equation is called a hybrid integro-differential equation with a linear perturbation of the first type. The problems of this kind may be tackled with the hybrid fixed point theorem involving the sum of two operators in a Banach space. See Dhage [1] and the references therein. The purpose of this section is to obtain an existence result for an equation of this type.

Given the notations in the previous sections, we consider the nonlinear hybrid HIDE

$$\left.\begin{aligned} x'(t) + \lambda x(t) &= f_1\left(t, x(t), \int_0^t g(s, x(s))\, ds\right) \\ &\quad + f_2\left(t, x(t), \int_0^t g(s, x(s))\, ds\right), \\ x(0) &= x(T), \end{aligned}\right\} \tag{4.1}$$

for all $t \in J$, where $f_1,\, f_2 : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

By a *solution* of the HIDE (4.1) we mean a function $x \in C^1(J, \mathbb{R})$ that satisfies equation (4.1), where $C^1(J, \mathbb{R})$ is the usual Banach space of continuously differentiable real-valued functions defined on J.

The HIDE (4.1) is a hybrid differential equation with a linear perturbation of the first type (see Dhage [4] and the references therein). The HIDE (4.1) is well-known and existence and other properties of its solutions have been discussed at length in the literature. Here, we show that existence of solutions can be proved under mixed partial Lipschitz and partial compactness type conditions. We will need the following definition.

Definition 4.1. A differentiable function $u \in C(J, \mathbb{R})$ is said to be a lower solution of the HIDE (4.1) if it satisfies

$$\left.\begin{aligned} u'(t) + \lambda u(t) &\le f_1\left(t, u(t), \int_0^t g(s, u(s))\, ds\right) \\ &\qquad + f_2\left(t, u(t), \int_0^t g(s, u(s))\, ds\right), \\ u(0) &\le u(T), \end{aligned}\right\}$$

for all $t \in J$. Similarly, an upper solution $v \in C^1(J, \mathbb{R})$ to the HIDE (4.1) is defined on J by reversing the above inequalities.

We will also need the following condition.

(H_7) The HIDE (4.1) has a lower solution $u \in C^1(J, \mathbb{R})$.

Theorem 4.2. *Assume that (H_1)–(H_3) hold with f replaced by f_2, and let (H_1) and (H_5)–(H_6) hold with f replaced by f_1. If (H_7) holds, then the HIDE* (4.1) *has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$\begin{aligned} x_1(t) &= u(t), \\ x_{n+1}(t) &= \int_0^T G(t,s) f_1\left((s, x_n(s), \int_0^s g(\tau, x_n(\tau))\, d\tau\right) ds \\ &\quad + \int_0^T G(t,s) f_2\left((s, x_n(s), \int_0^s g(\tau, x_n(\tau))\, d\tau\right) ds, \end{aligned} \tag{4.2}$$

for $t \in J$, converges monotonically to x^*, where $G(t,s)$ is a Green's function defined by (3.5) *on $J \times J$.*

Proof . Set $E = C(J, \mathbb{R})$. Then, from Lemma 3.1 it follows that every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\le$ in E. By Lemma 3.3, the HIDE (4.1) is equivalent to the nonlinear integral equation

$$\begin{aligned} x(t) &= \int_0^T G(t,s) f_1\left(s, x(s), \int_0^s g(\tau, x(\tau))\, d\tau\right) ds \\ &\quad + \int_0^T G(t,s) f_2\left(s, x(s), \int_0^s g(\tau, x(\tau))\, d\tau\right) ds, \ t \in J, \end{aligned} \tag{4.3}$$

where $G(t,s)$ is a Green's function defined by (3.5) on $J \times J$.

Define the operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = \int_0^T G(t,s) f_1\left(s, x(s), \int_0^s g(\tau, x(\tau))\, d\tau\right) ds, \ t \in J, \tag{4.4}$$

and

$$\mathcal{B}x(t) = \int_0^T G(t,s) f_2\left(s, x(s), \int_0^s g(\tau, x(\tau))\, d\tau\right) ds, \ \ t \in J. \tag{4.5}$$

Clearly, $\mathcal{A}$, $\mathcal{B} : E \to E$. Also, the HIDE (4.1) is equivalent to the operator equation

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \quad t \in J. \tag{4.6}$$

Following arguments similar to those used in the proofs of Theorems 3.6 and 3.9, it can be shown that the operator $\mathcal{A}$ is a partially bounded and nonlinear $\mathcal{D}$-contraction and $\mathcal{B}$ is a partially continuous and partially compact operator on E. Furthermore, as in the proof of Theorem 3.6, it can be shown that the function u given in condition (H_4) satisfies the operator inequality $u \leq \mathcal{A}u + \mathcal{B}u$ on J. A direct application of Theorem 2.15 yields that the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^*. Consequently, the HIDE (4.1) has a solution x^*, and the sequence $\{x_n\}_{n=1}^{\infty}$ defined by (4.2) converges monotonically to x^*. This completes the proof of the theorem. □

The conclusion of Theorem 4.2 remains true if we replace the (H_7) by

(H_7') The HIDE (4.1) has an upper solution $v \in C^1(J, \mathbb{R})$.

Example 4.3. Let $J = [0, 1]$ and consider the HIDE

$$\left.\begin{aligned} x'(t) + x(t) = \tan^{-1} x(t) + \tanh\left(\int_0^t g(s, x(s))\, ds\right), \ \ t \in J, \\ x(0) = x(T). \end{aligned}\right\} \tag{4.7}$$

where $g : J \times \mathbb{R} \to \mathbb{R}$ is the function defined by

$$g(t,x) = \begin{cases} x + 1, & \text{if} \quad x \leq 0, \\ x^2 + 1, & \text{if} \quad x > 0. \end{cases}$$

Here, $\lambda = 1$, $c = 1$, $f_1(t, x, y) = \tan^{-1} x$ and $f_2(t, x, y) = \tanh y$. Then the function f_1 satisfies (H_1) with $M_{f_1} = \dfrac{\pi}{2}$ and satisfies (H_6) with $\psi_1(r) = \dfrac{r}{1+\xi^2}$, $0 < \xi < r$, and $\psi_2(r) = 0$. Now f_2 satisfies (H_1) with $M_{f_2} = 1$ and is nondecreasing in y, so (H_2) holds. Similarly, g satisfies (H_3). Finally, $u(t) = -3e^t$ for all $t \in J$ is a lower solution of the HIDE (4.7) on J, and so (H_7) is satisfied. Therefore, by Theorem 4.2, the HIDE (4.7) has a solution x^* on J, and the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_1(t) = -3e^{-t},$$

$$\begin{aligned} x_{n+1}(t) = &\int_0^1 G(t,s) \tan^{-1} x_n(s)\, ds \\ &+ \int_0^1 G(t,s) \tanh\left(\int_0^s g(\tau, x_n(\tau))\, d\tau\right) ds \end{aligned}$$

for each $t \in J$, converges monotonically to x^*, where $G(t, s)$ is a Green's function associated with the homogeneous PBVP (3.20) given by (3.21).

Remark 4.4. We note that if the HIDE (1.1) or (4.1) has a lower solution u as well as an upper solution v such that $u \leq v$, then the corresponding solutions x_* and x^* of the HIDE (1.1) or (4.1) satisfy $x_* \leq x^*$ and they are the minimal and maximal solutions in the vector segment $[u, v]$ of the Banach space $E = C(J, \mathbb{R})$. This is because the order relation $\leq$ defined by (3.2) is equivalent to the order relation defined by the order cone $\mathcal{K} = \{x \in C(J, \mathbb{R}) \mid x \geq \theta\}$ which is a closed set in $C(J, \mathbb{R})$. Thus, Dhage iteration method is also useful for proving the maximal and minimal solutions in a vector segment of the partially ordered Banach space E.

5. Concluding Remarks

From the foregoing discussion it should be clear that the Dhage iteration method is a powerful tool for proving existence results for certain PBVPs of nonlinear hybrid integro-differential equations. However, it has some limitations in that unlike Picard's method, it does not give the rate of convergence of the sequence of successive approximations. However, if we consider linear partial contraction instead of nonlinear partial contraction in Theorem 2.14, then it also yields the rate of convergence of the sequence of successive approximations to the solution of the problem in question. Again as mentioned in Dhage [12], the Dhage iteration method is different from that of generalized iteration mehod of Heikkilá [19] and the monotone iterative technique presented in Ladde *et al.* [21] and the references therein. Further we conjecture that other qualitative aspects of the solutions such as maximal and minimal solutions, differential inequalities and comparison principle and stability etc. could also be proved using Dhage iteration method. See Dhage [9] and the references therein. The PBVP of integro-differential equations considered here for which we have illustrated the Dhage iteration method to obtain algorithms for the solutions under weaker partially Lipschitz and compactness conditions is of fairly simple nature. An analogous study could also be made for more complicated integro-differential equations using a similar approach with appropriate modifications. Results along these lines will be left to future work.

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