Some properties of analytic functions related with bounded positive real part

Rahim Kargar\textsuperscript{a,}\textsuperscript{*}, Ali Ebadian\textsuperscript{b}, Janusz Sokół\textsuperscript{c}

\textsuperscript{a}Young Researchers and Elite Club, Urmia Branch, Islamic Azad University, Urmia, Iran
\textsuperscript{b}Department of Mathematics, Payame Noor University, Tehran, Iran
\textsuperscript{c}University of Rzeszów, Faculty of Mathematics and Natural Sciences, ul. Prof. Pigonia 1, 35-310 Rzeszów, Poland

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Abstract

In this paper, we define new subclass of analytic functions related with bounded positive real part, and coefficients estimates, duality and neighborhood are considered.

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1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  \hspace{1cm} (1.1)

which are analytic in the open unit disk $\Delta = \{ z \in \mathbb{C} : |z| < 1 \}$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\Delta$ is denoted by $\mathcal{S}$. Let $f(z)$ and $g(z)$ be analytic in $\Delta$. Then the function $f(z)$ is said to be subordinate to $g(z)$ in $\Delta$, written by $f(z) \prec g(z)$, if there exists a function $w(z)$, $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ and $z \in \Delta$. Furthermore, if the function $g$ is univalent in $\Delta$, then we have the following equivalence:

$$f(z) \prec g(z) \iff (f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta)).$$
The convolution or Hadamard product of two power series \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) and \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \) is defined as the power series \((f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n\). A function \( f \in \mathcal{A} \) belongs to the class \( S_*^\gamma(\alpha) \), \( 0 \leq \alpha < 1 \) and \( \gamma \in \mathbb{C} \setminus \{0\} \), if it satisfies

\[
\Re \left\{ 1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > \alpha \quad z \in \Delta. \tag{1.2}
\]

We remark that \( S_*^\gamma(\alpha) \equiv S^\gamma(\alpha) \) is the class of starlike functions of order \( \alpha \) and \( S^\gamma(0) \equiv S^* \), is the class of starlike functions. Also, a function \( f \in \mathcal{A} \) belongs to the class \( \mathcal{K}_\gamma(\alpha) \), \( 0 \leq \alpha < 1 \) and \( \gamma \in \mathbb{C} \setminus \{0\} \), if and only if \( zf' \in S_*^\gamma(\alpha) \). Note that \( \mathcal{K}_1(\alpha) \) will be the class of convex functions of order \( \alpha \), i.e. \( \mathcal{K}_1(\alpha) \equiv \mathcal{K}(\alpha) \) and \( \mathcal{K}(0) \equiv \mathcal{K} \), is the class of convex functions. Recall that a set \( E \subset \mathbb{C} \) is said to be starlike with respect to a point \( w_0 \in E \) if and only if the linear segment joining \( w_0 \) to every other point \( w \in E \) lies entirely in \( E \), while a set \( E \) is said to be convex if and only if it is starlike with respect to each of its points, that is, if and only if the linear segment joining any two points of \( E \) lies entirely in \( E \).

We denote by \( \mathcal{M}_\gamma(\beta) \), the class of all functions \( f(z) \) of the form \((1)\) such that satisfy the following inequality:

\[
\Re \left\{ 1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} < \beta \quad z \in \Delta, \tag{1.3}
\]

where \( \beta > 1 \) and \( \gamma \in \mathbb{C} \setminus \{0\} \). We recall that the class \( \mathcal{M}_1(\beta) \) was investigated earlier by Urallegadi et al. (see [12]). Also a function \( f \in \mathcal{A} \) is in the class \( \mathcal{N}_\gamma(\beta) \) if and only if \( zf'(z) \in \mathcal{M}_\gamma(\beta) \).

Now, motivated by the above classes, we define two new subclasses of analytic functions as follows:

**Definition 1.1.** Let \( \alpha \) and \( \beta \) be real numbers such that \( 0 \leq \alpha < 1 \) and \( \beta > 1 \). Then the function \( f \in \mathcal{A} \) belongs to the class \( S_*^\gamma(\alpha, \beta) \) if \( f \) satisfies the two-sided inequality:

\[
\alpha < \Re \left\{ 1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} < \beta \quad z \in \Delta, \tag{1.4}
\]

where \( \gamma \in \mathbb{C} \setminus \{0\} \). Also the function \( f \in \mathcal{A} \) belongs to the class \( \mathcal{K}_\gamma(\alpha, \beta) \) if \( f \) satisfies the following condition:

\[
\alpha < \Re \left\{ 1 + \frac{1}{\gamma} \frac{zf''(z)}{f'(z)} \right\} < \beta \quad z \in \Delta, \tag{1.5}
\]

where \( \gamma \in \mathbb{C} \setminus \{0\} \).

We remark that the class \( S_*^\gamma(\alpha, \beta) \equiv S^\gamma(\alpha, \beta) \) was first investigated by Kuroki and Owa [2]. It is easy to see that \( f \in \mathcal{K}_\gamma(\alpha, \beta) \) if and only if \( zf' \in S_*^\gamma(\alpha, \beta) \). Note that for given \( 0 \leq \alpha < 1 \) and \( \beta > 1 \), \( f \in S_*^\gamma(\alpha, \beta) \) if and only if \( f \) satisfies the following two subordination equations:

\[
1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) < \frac{1 + (1 - 2\alpha)z}{1 - z} \quad \text{and} \quad 1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) < \frac{1 - (1 - 2\beta)z}{1 + z}.
\]

Because the functions \( \frac{1 + (1 - 2\alpha)z}{1 - z} \) and \( \frac{1 - (1 - 2\beta)z}{1 + z} \) map \( \Delta \) onto the right half plane, having real part greater than \( \alpha \), and the left half plane, having real part smaller than \( \beta \), respectively.

Recently, many authors (see [11, 12, 13, 14, 15]) have studied an analytic function \( P_{\alpha, \beta} : \Delta \to \mathbb{C} \) as follows:

\[
P_{\alpha, \beta}(z) = 1 + \frac{\beta - \alpha}{\pi} i \log \left( \frac{1 - e^{2\pi i \frac{1 - \alpha}{\beta - \alpha}} z}{1 - z} \right) \quad (0 \leq \alpha < 1 \text{ and } \beta > 1) \tag{1.6}
\]
and have gained some interesting results. Note that the function $P_{\alpha,\beta}(z)$ defined by (1.6) is a convex univalent function in $\Delta$ and has the form:

$$P_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

where

$$B_n = \frac{\beta - \alpha}{n\pi} i \left(1 - e^{2n\pi i \frac{1-\alpha}{\beta-\alpha}}\right) \quad (n = 1, 2, \ldots).$$

(1.7)

The function $P_{\alpha,\beta}(z)$ was first introduced by Kuroki and Owa [2] and they proved that $P_{\alpha,\beta}$ maps $\Delta$ onto a convex domain

$$\Omega_{\alpha,\beta} = \{w \in \mathbb{C} : \alpha < \Re\{w\} < \beta\},$$

(1.8)

conformally.

In sequel, we prove that the following lemma and by it we present an example for the class $S^{*}_\gamma(\alpha, \beta)$.

**Lemma 1.2.** The function $f \in \mathcal{A}$ belongs to the class $S^{*}_\gamma(\alpha, \beta)$, $0 \leq \alpha < 1$ and $\beta > 1$, if and only if there exists an analytic function $p$, $p(0) = 1$ and $p(z) < P_{\alpha,\beta}(z)$ such that

$$f(z) = z \exp \left(\gamma \int_0^z \frac{p(t) - 1}{t} dt\right) \quad z \in \Delta.$$  

(1.9)

**Proof.** Assume that $f \in S^{*}_\gamma(\alpha, \beta)$ and let $p(z) = 1 + \frac{1}{\gamma} ([zf'(z)/f(z)] - 1)$. Then $p(z) < P_{\alpha,\beta}(z)$ and integrating this equation we obtain (1.9). If $f$ is given by (1.9), with an analytic $p$, $p(0) = 1$, $p(z) < P_{\alpha,\beta}(z)$, then differentiating logarithmically (1.9) we obtain $[zf'(z)/f(z)] - 1 = p(z)$, and therefore $[zf'(z)/f(z)] < 1 < P_{\alpha,\beta}(z)$ and $f \in S^{*}_\gamma(\alpha, \beta)$. □

**Example 1.3.** Putting $p = P_{\alpha,\beta}$ in the above Lemma 1.2 we obtain the function

$$F(z) = z \exp \left(\gamma \int_0^z \frac{P_{\alpha,\beta}(t) - 1}{t} dt\right) \quad z \in \Delta.$$  

(1.10)

and so $F(z) \in S^{*}_\gamma(\alpha, \beta)$.

After some calculations and by (1.7) and (1.10), we have

$$F(z) = z + \gamma B_1 z^2 + \frac{\gamma}{2} (B_2 + \gamma B_1^2) z^3 + \cdots \in S^{*}_\gamma(\alpha, \beta).$$

(1.11)

Using (1.8) and by the definition of subordination, we can obtain Lemma 1.4 and Lemma 1.5, directly.

**Lemma 1.4.** Let $f \in \mathcal{A}$, $0 \leq \alpha < 1$ and $\beta > 1$. Then $f \in S^{*}_\gamma(\alpha, \beta)$ if and only if

$$1 + \frac{1}{\gamma} \left(\frac{zf'(z)}{f(z)} - 1\right) \prec 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha} z}}{1 - z}\right) \quad z \in \Delta.$$  

(1.12)

**Lemma 1.5.** Let $f \in \mathcal{A}$, and $0 \leq \alpha < 1$ and $\beta > 1$. Then $f \in K^{*}_\gamma(\alpha, \beta)$ if and only if

$$1 + \frac{1}{\gamma} \left(\frac{zf''(z)}{f'(z)}\right) \prec 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha} z}}{1 - z}\right) \quad z \in \Delta.$$  

(1.13)
Lemma 1.6. (Rogosinski [5]) Let \( q(z) = \sum_{n=1}^\infty C_n z^n \) be analytic and univalent in \( \Delta \), and suppose that \( q(z) \) maps \( \Delta \) onto a convex domain. If \( p(z) = \sum_{n=1}^\infty A_n z^n \) is analytic in \( \Delta \) and satisfies the following subordination
\[
p(z) \prec q(z) \quad (z \in \Delta),
\]
then
\[
|A_n| \leq |C_1| \quad (n = 1, 2, 3, \ldots).
\]

The structure of the paper is the following. In Section 2 we estimate the coefficients of functions belonging to the class \( S^*_\gamma(\alpha, \beta) \) and we obtain the dual of the class \( S^*_\gamma(\alpha, \beta) \). Also sufficient condition for functions belonging to this subclass is obtained. In Section 3 we define a new \( TN_\lambda \)-neighborhood and we show that \( TN_\lambda(f) \subset S^*_\gamma(\alpha, \beta) \), where \( f \) defined by \(1.1\). Finally, in Section 4 we give two open problems.

2. Coefficients and Duality

One of our main results is the following.

Theorem 2.1. Assume that \( 0 \leq \alpha < 1, \beta > 1 \) and \( \gamma \in \mathbb{C}\setminus\{0\} \). Let the function \( f \) given by \(1.1\) be in the class \( S^*_\gamma(\alpha, \beta) \). Then
\[
|a_n| \leq \begin{cases} 
|\gamma B_1| & n = 2, \\
\frac{|\gamma B_1|}{n-1} \prod_{k=2}^{n-1} \left( 1 + \frac{|\gamma B_1|}{k-1} \right) & n = 3, 4, \ldots,
\end{cases}
\] (2.1)
where \( B_1 \) is given by \(1.7\) and
\[
|B_1| = \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}.
\] (2.2)

Proof. Define the function \( p(z) \) by
\[
z f'(z) = [\gamma(p(z) - 1) + 1]f(z) \quad z \in \Delta.
\] (2.3)
Then according to the assertion of Lemma 1.4, we have
\[
p(z) \prec P_{\alpha, \beta}(z) \quad z \in \Delta,
\] (2.4)
where \( P_{\alpha, \beta}(z) \) defined by \(1.6\). If we let
\[
p(z) = 1 + \sum_{n=1}^\infty A_n z^n,
\]
then by Lemma 1.6 we see that the subordination \(2.4\) implies that
\[
|A_n| \leq |B_1| \quad (n = 1, 2, 3, \ldots).
\] (2.5)
Now by equating the coefficients of \( z^n \) in both sides of equality \(2.3\) we have
\[
n a_n = \gamma(A_{n-1} + A_{n-2}a_2 + \cdots + A_1a_{n-1}) + a_n \quad (n = 2, 3, \ldots).
\] (2.6)
A simple calculation together with the inequality (2.5) yields that

$$|a_n| = \frac{|\gamma|}{n-1} \times |A_{n-1} + A_{n-2}a_2 + \cdots + A_1a_{n-1}|$$

$$\leq \frac{|\gamma|}{n-1} \times (|A_{n-1}| + |A_{n-2}||a_2| + \cdots + |A_1||a_{n-1}|)$$

$$\leq \frac{|\gamma B_1|}{n-1} \sum_{k=1}^{n-1} |a_k|,$$

where $|B_1|$ is given by (2.2) and $|a_1| = 1$. Hence, we have $|a_2| \leq |\gamma B_1|$. In order to prove the remaining part of the theorem, we need to show that

$$\frac{|\gamma B_1|}{n-1} \sum_{k=1}^{n-1} |a_k| \leq |\gamma B_1| \prod_{k=2}^{n} \left( 1 + \frac{|\gamma B_1|}{k-1} \right) \quad (n = 3, 4, 5 \ldots).$$

Using induction and simple calculation, we could prove the inequality (2.7). Hence, the desired estimate for $|a_n| (n = 3, 4, 5, \ldots)$ follows, as asserted in (2.1). This completes the proof of Theorem 2.1.

If we take $\gamma = 1$ in the Theorem 2.1, then we have.

**Corollary 2.2.** (see [2]) Let $0 \leq \alpha < 1$, $\beta > 1$ and let the function $f$ be given by (1.1) be in the class $S^*(\alpha, \beta)$. Then

$$|a_n| \leq \prod_{k=2}^{n} \left( 1 + \frac{1}{k-1} |\gamma B_1| \right) \quad (n = 2, 3, \ldots),$$

where $|B_1|$ is given by (2.2).

Now, we recall the definition of duality. For a set $\mathcal{V} \subset \mathcal{A}$, the dual set $\mathcal{V}$, denoted by $\mathcal{V}^*$ is defined as

$$\mathcal{V}^* = \left\{ g \in \mathcal{A} : \frac{1}{z} (f * g)(z) \neq 0 \quad \text{for all} \quad f \in \mathcal{V} \text{ and } z \in \Delta \right\}.$$

The standard reference to duality theory for convolutions is the monograph by Ruscheweyh [6], and his paper [7]. First, we get the duality for the class $S^*_\gamma(\alpha)$.

**Theorem 2.3.** Assume that $f \in \mathcal{A}$, $0 \leq \alpha < 1$ and $\gamma \in \mathbb{C} \setminus \{0\}$. The function $f \in S^*_{\gamma}(\alpha)$, if and only if

$$\frac{f(z)}{z} * \frac{h_\alpha(z)}{z} \neq 0 \quad z \in \Delta,$$

where

$$h_\alpha(z) := h(\alpha, \gamma, x)(z) = \frac{z \left( 1 + \frac{z+1-2\gamma(1-\alpha)}{2\gamma(1-\alpha)} \right)}{(1-z)^2} \quad (|x| = 1).$$

**Proof.** According to the definition of $S^*_{\gamma}(\alpha)$ and since $1 + \frac{1}{\gamma}[(zf'(z)/f(z)) - 1] = 1$ at $z = 0$, we have

$$1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) - \frac{x}{1 - \alpha} \neq x - 1 \quad (|x| = 1, x \neq -1, z \in \Delta).$$
which leads to
\[(x + 1)zf'(z) + f(z)[2\gamma(1 - \alpha) - (x + 1)] \neq 0.\]  
(2.9)

Where as \(f(z) \ast l(z) = f(z)\) and \(zf' = f \ast zl'(z)\), where \(l(z) = z/(1 - z)\), hence the relation (2.9) leads to
\[\frac{f(z)}{z} \ast \left[\frac{x + 1}{(1 - z)^2} + \frac{2\gamma(1 - \alpha) - (x + 1)}{1 - z}\right] \neq 0,
\]
or
\[\frac{f(z)}{z} \ast \left[\frac{2\gamma(1 - \alpha) + [x + 1 - 2\gamma(1 - \alpha)]z}{(1 - z)^2}\right] \neq 0.
\]

With a little calculation we have
\[\frac{f(z)}{z} \ast \left[\frac{1 + x + 1 - 2\gamma(1 - \alpha)}{2\gamma(1 - \alpha)}\right] \neq 0,
\]
which here ends the proof of Theorem 2.3. □

**Remark 2.4.** The special cases of \(\gamma = 1, \alpha = 1/2\) and \(\gamma = 1\) in Theorem 2.3 are contained in [8] and [10], respectively.

**Theorem 2.5.** Assume that \(f \in A, 0 \leq \alpha < 1\) and \(\gamma \in \mathbb{C}\setminus\{0\}\). The function \(f \in K_\gamma(\alpha)\), if and only if
\[\frac{f(z)}{z} \ast l_\alpha(z) \neq 0, \quad z \in \Delta,
\]
where
\[l_\alpha(z) := l(\alpha, \gamma, x)(z) = \frac{z(1 + x + 1 - \gamma(1 - \alpha))}{(1 - z)^3} (|x| = 1).\]  
(2.10)

**Proof.** Using the relations \(f \in K_\gamma(\alpha) \iff zf'(z) \in S_\gamma^*(\alpha)\), \((zf')' = f' \ast \left[\frac{1}{(1 - z)^2}\right], f'(z) \ast \left[z/(1 - z)\right] = f'(z)\) and \(zf' \ast g = f \ast zg'\) we can easily prove this theorem. So we omit the details. □

**Remark 2.6.** If we take \(\gamma = 1\) in Theorem 2.5 then leads to a result which is obtained by H. Silverman et al. [10]. Also, the case \(\gamma = 1, \alpha = 0\) was obtained by Ruscheweyh [8].

**Definition 2.7.** We define \(V_\alpha^*\) and \(V_\beta^*\) as follows:

\[V_\alpha^* = (S_\gamma^*(\alpha))^*: = \left\{h_\alpha(z) \in A: \frac{f(z)}{z} \ast \frac{h_\alpha(z)}{z} \neq 0, f \in S_\gamma^*(\alpha), 0 \leq \alpha < 1, z \in \Delta\right\}
\]
and
\[V_\beta^* = (M_\gamma(\beta))^*: = \left\{h_\beta(z) \in A: \frac{f(z)}{z} \ast \frac{h_\beta(z)}{z} \neq 0, f \in M_\gamma(\beta), \beta > 1, z \in \Delta\right\},
\]
where \(h_\alpha(z)\) is given by (2.8) and
\[h_\beta(z) := h(\beta, \gamma, x)(z) = \frac{z(1 + x + 1 - 2\gamma(1 - \beta))}{(1 - z)^2} (|x| = 1).\]  
(2.11)
Definition 2.8. We define by $W^*_\delta$ the dual of the class $S^*_\gamma(\alpha, \beta)$ as follows:

$$W^*_\delta := \left\{ \begin{array}{ll} h_\delta(z) \in \mathcal{V}^*_\alpha & 0 \leq \delta < 1, \\
                             h_\delta(z) \in \mathcal{V}^*_\beta & \delta > 1, \end{array} \right.$$ 

where

$$h_\delta(z) := h(\delta, \gamma, x)(z) = \frac{z(1 + \frac{x+1-2\gamma(1-\delta)}{2\gamma(1-\delta)}z)}{(1-z)^2} \quad (|x| = 1). \quad (2.12)$$

Using the definition of duality, we can obtain the following lemma directly.

Lemma 2.9. A function $f \in A$ is in the class $S^*_\gamma(\alpha, \beta)$ if and only if for all $h_\delta \in W^*_\delta$

$$\frac{f(z)}{z} * \frac{h_\delta(z)}{z} \neq 0 \quad z \in \Delta,$$

where $h_\delta$ is given by (2.12).

Lemma 2.10. Let $h_\delta(z) = z + \sum_{n=2}^{\infty} c_n z^n \in W^*_\delta$. Then

$$|c_n| \leq n + (n-1)|u(x, \gamma, \delta)| \quad (n = 2, 3, \ldots),$$

where

$$u(x, \gamma, \delta) := \frac{x+1-2\gamma(1-\delta)}{2\gamma(1-\delta)}. \quad (2.13)$$

Proof. Let $h_\delta \in W^*_\delta$. Then for any $\gamma \in \mathbb{C}\{0\}$, $\delta \geq 0$ and $|x| = 1$, we have

$$h_\delta(z) = \frac{z}{(1-z)^2} + u(x, \gamma, \delta) \frac{z^2}{(1-z)^2}$$

$$= (z + 2z^2 + 3z^3 + \cdots) + u(x, \gamma, \delta)(z^2 + 2z^3 + 3z^4 + \cdots)$$

$$= z + \sum_{n=2}^{\infty} c_n z^n,$$

where

$$c_n = n + (n-1)u(x, \gamma, \delta), \quad (2.14)$$

and so $|c_n| \leq n + (n-1)|u(x, \gamma, \delta)|$ and $n = 2, 3, \ldots$. □

Corollary 2.11. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A$. If

$$\sum_{n=2}^{\infty} |n + (n-1)|u(x, \gamma, \delta)||a_n| < 1,$$

then $f \in S^*_\gamma(\alpha, \beta)$, where $u(x, \gamma, \alpha)$ defined by (2.13).
Proof. Let $h_{\delta}(z) = z + \sum_{n=2}^{\infty} c_n z^n \in W_{\delta}^*$. Then we have

$$
\left| \frac{(f \ast h_{\delta})(z)}{z} \right| = \left| 1 + \sum_{n=2}^{\infty} a_n c_n z^{n-1} \right|
\geq 1 - \sum_{n=2}^{\infty} |a_n||c_n||z| > 1 - \sum_{n=2}^{\infty} |a_n||c_n| > 0,
$$

Thus $\frac{(f \circ h_{\delta})(z)}{z} \neq 0$ and now from Lemma 2.9 we have $f \in S_{\gamma}^*(\alpha, \beta)$. □

For each complex number $\varepsilon \neq -1$ and $f \in \mathcal{A}$, define the function $F_{\varepsilon}(z)$ by

$$
F_{\varepsilon}(z) = \frac{f(z) + \varepsilon z}{1 + \varepsilon} \quad z \in \Delta.
$$

(2.15)

Lemma 2.12. Assume that $h_{\delta} \in W_{\delta}^*$ and $\lambda > 0$. Let $F_{\varepsilon}(z)$ be defined by (2.15) belongs to the class $S_{\gamma}^*(\alpha, \beta)$ for $|\varepsilon| < \lambda$. Then

$$
\left| \frac{1}{z} (f \ast h_{\delta})(z) \right| > \lambda \quad z \in \Delta.
$$

Proof. If $F_{\varepsilon}(z) \in S_{\gamma}^*(\alpha, \beta)$ for $|\varepsilon| < \lambda$, it follows from Lemma 2.9 that

$$
\frac{1}{z} (F_{\varepsilon} \ast h_{\delta})(z) \neq 0 \quad z \in \Delta,
$$

where $h_{\delta}$ defined by (2.12). The above relation is equivalent to

$$
\frac{1}{z} \left( \frac{(f \ast h_{\delta})(z) + \varepsilon z}{1 + \varepsilon} \right) \neq 0 \quad z \in \Delta,
$$

and hence

$$
\frac{1}{z} (f \ast h_{\delta})(z) \neq -\varepsilon \quad z \in \Delta,
$$

for every $|\varepsilon| < \lambda$. Therefore

$$
\left| \frac{1}{z} (f \ast h_{\delta})(z) \right| > \lambda \quad z \in \Delta
$$

and concluding the proof. □

3. $T$-Neighborhood

Let $f(z)$ be defined by (1.1), $\lambda$ is a positive number and $T = \{T_n\}_{n=2}^{\infty}$ is a sequence of positive numbers. Then the $TN_{\lambda}$-neighborhood of the function $f$ is defined as

$$
TN_{\lambda}(f) = \left\{ g \in \mathcal{A} : g(z) = z + \sum_{n=2}^{\infty} b_n z^n : \sum_{n=2}^{\infty} T_n |b_n - a_n| \leq \lambda \right\},
$$

(3.1)

where we define $T_n$ ($n = 2, 3, \ldots$), as follows:

$$
T_n = n + (n - 1)|u(x, \gamma, \delta)| \quad \text{and} \quad u(x, \gamma, \delta) = \frac{x + 1 - 2\gamma(1 - \delta)}{2\gamma(1 - \delta)}.
$$

(3.2)

St. Ruscheweyh in [9] considered $T_n = \{n\}_{n=2}^{\infty}$ and showed that if $f \in \mathcal{K}$, then $TN_{1/4}(f) \subset S^*$. 

For each complex number $\varepsilon \neq -1$ and $f \in \mathcal{A}$, define the function $F_{\varepsilon}(z)$ by

$$
F_{\varepsilon}(z) = \frac{f(z) + \varepsilon z}{1 + \varepsilon} \quad z \in \Delta.
$$

(2.15)
Remark 3.1. Note that if we take \( x = \gamma = 1 \) and \( \delta = 0 \) in the sequence of \( T_n \), then we have \( T_n \equiv \{ n \}_{n=2}^{\infty} \).

We have the following result on the \( TN_\lambda \)-neighborhoods for the class \( S^*_\gamma(\alpha, \beta) \).

**Theorem 3.2.** Assume that \( 0 \leq \alpha < 1, \beta > 1 \) and \( \gamma \in \mathbb{C} \setminus \{0\} \). Let the function \( f \in \mathcal{A} \) and for any \( \varepsilon \in \mathbb{C}, \lambda > 0 \) and \( |\varepsilon| < \lambda \), \( F_\varepsilon(z) \) given by (2.15) be in the class \( S^*_\gamma(\alpha, \beta) \). Then \( TN_\lambda(f) \subset S^*_\gamma(\alpha, \beta) \).

**Proof.** Let \( g \in \mathcal{A} \) and \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \) be in \( TN_\lambda(f) \). Then

\[
\left| \frac{1}{z} (g * h_\delta)(z) \right| = \left| \frac{1}{z} (f * h_\delta)(z) + \frac{1}{z} [(g - f) * h_\delta](z) \right| \\
\geq \left| \frac{1}{z} (f * h_\delta)(z) \right| - \left| \frac{1}{z} [(g - f) * h_\delta](z) \right|
\]

where \( h_\delta \) defined by (2.12). Making use of Lemma 2.12 we obtain

\[
\left| \frac{1}{z} (g * h_\delta)(z) \right| \geq \lambda - \left( \sum_{n=2}^{\infty} \frac{(b_n - a_n)c_n z^n}{z} \right) \\
\geq \lambda - \sum_{n=2}^{\infty} |b_n - a_n| |c_n|,
\]

where \( c_n \) is given by (2.14). Since \( g \in TN_\lambda(f) \), it follows that

\[
\sum_{n=2}^{\infty} |b_n - a_n| |c_n| \leq \sum_{n=2}^{\infty} [n + (n - 1)|u(x, \gamma, \delta)|] |b_n - a_n| \leq \lambda
\]

and thus

\[
\left| \frac{1}{z} (g * h_\delta)(z) \right| > \lambda - \lambda = 0.
\]

Therefore \( \left| \frac{1}{z} (g * h_\delta)(z) \right| \neq 0 \). Now, from Lemma 2.9 we get \( g \in S^*_\gamma(\alpha, \beta) \). This completes the proof. \( \square \)

4. Open problems

In this section, we give two open problems.

**Open Problem 1.** In [3], the authors proved that if \( f \in S^*(\alpha, \beta) \), then

\[
\frac{1}{3 - 2\alpha} < \Re \left\{ \frac{f(z)}{z} \right\} < \frac{1}{3 - 2\beta} \quad (1/2 \leq \alpha < 1, 1 < \beta < 3/2).
\]

What the bounds for \( \Re \{ f(z)/z \} \) are, when \( f \in S^*_\gamma(\alpha, \beta) \)?

**Open Problem 2.** Let \( T_n \) be defined by (3.2). Then find the greatest \( \lambda := \lambda(x, \gamma, \delta) \) so that if \( f \in \mathcal{K} \), then \( TN_\lambda(f) \subset S^* \).
References