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Some properties of analytic functions related with bounded positive real part

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Abstract

In this paper, we define new subclass of analytic functions related with bounded positive real part, and coefficients estimates, duality and neighborhood are considered.

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1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in Δ is denoted by $\mathcal{S}$. Let $f(z)$ and $g(z)$ be analytic in Δ. Then the function $f(z)$ is said to be subordinate to $g(z)$ in Δ, written by $f(z) \prec g(z)$, if there exists a function $w(z)$, $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ and $z \in \Delta$. Furthermore, if the function g is univalent in Δ, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow (f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta)).$$

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The convolution or Hadamard product of two power series $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is defined as the power series $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$. A function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*_\gamma(\alpha)$, $0 \le \alpha < 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$, if it satisfies

$$\mathfrak{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > \alpha \qquad z \in \Delta. \tag{1.2}$$

We remark that $\mathcal{S}^*_1(\alpha) \equiv \mathcal{S}^*(\alpha)$ is the class of starlike functions of order α and $\mathcal{S}^*(0) \equiv \mathcal{S}^*$, is the class of starlike functions. Also, a function $f \in \mathcal{A}$ belongs to the class $\mathcal{K}_\gamma(\alpha)$, $0 \le \alpha < 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$, if and only if $zf' \in \mathcal{S}^*_\gamma(\alpha)$. Note that $\mathcal{K}_1(\alpha)$ will be the class of convex functions of order α, i.e. $\mathcal{K}_1(\alpha) \equiv \mathcal{K}(\alpha)$ and $\mathcal{K}(0) \equiv \mathcal{K}$, is the class of convex functions. Recall that a set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if and only if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E, while a set E is said to be convex if and only if it is starlike with respect to each of its points, that is, if and only if the linear segment joining any two points of E lies entirely in E.

We denote by $\mathcal{M}_\gamma(\beta)$, the class of all functions $f(z)$ of the form (1.1) such that satisfy the following inequality:

$$\mathfrak{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} < \beta \qquad z \in \Delta, \tag{1.3}$$

where $\beta > 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$. We recall that the class $\mathcal{M}_1(\beta)$ was investigated earlier by Uralegaddi et al. (see [12]). Also a function $f \in \mathcal{A}$ is in the class $\mathcal{N}_\gamma(\beta)$ if and only if $zf'(z) \in \mathcal{M}_\gamma(\beta)$.

Now, motivated by the above classes, we define two new subclasses of analytic functions as follows:

Definition 1.1. Let α and β be real numbers such that $0 \le \alpha < 1$ and $\beta > 1$. Then the function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*_\gamma(\alpha, \beta)$ if f satisfies the two-sided inequality:

$$\alpha < \mathfrak{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} < \beta \qquad z \in \Delta, \tag{1.4}$$

where $\gamma \in \mathbb{C}\backslash\{0\}$. Also the function $f \in \mathcal{A}$ belongs to the class $\mathcal{K}_\gamma(\alpha, \beta)$ if f satisfies the following condition:

$$\alpha < \mathfrak{Re}\left\{1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\} < \beta \qquad z \in \Delta, \tag{1.5}$$

where $\gamma \in \mathbb{C}\backslash\{0\}$.

We remark that the class $\mathcal{S}^*_1(\alpha, \beta) \equiv \mathcal{S}^*(\alpha, \beta)$ was first investigated by Kuroki and Owa [2]. It is easy to see that $f \in \mathcal{K}_\gamma(\alpha, \beta)$ if and only if $zf' \in \mathcal{S}^*_\gamma(\alpha, \beta)$. Note that for given $0 \le \alpha < 1$ and $\beta > 1$, $f \in \mathcal{S}^*_\gamma(\alpha, \beta)$ if and only if f satisfies the following two subordination equations:

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1 + (1 - 2\alpha)z}{1 - z} \quad \text{and} \quad 1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1 - (1 - 2\beta)z}{1 + z}.$$

Because the functions $\frac{1+(1-2\alpha)z}{1-z}$ and $\frac{1-(1-2\beta)z}{1+z}$ map Δ onto the right half plane, having real part greater than α, and the left half plane, having real part smaller than β, respectively.

Recently, many authors (see [1, 2, 3, 4, 11]) have studied an analytic function $P_{\alpha,\beta} : \Delta \to \mathbb{C}$ as follows:

$$P_{\alpha,\beta}(z) = 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z}\right) \qquad (0 \le \alpha < 1 \text{ and } \beta > 1) \tag{1.6}$$

and have gained some interesting results. Note that the function $P_{\alpha,\beta}(z)$ defined by (1.6) is a convex univalent function in Δ and has the form:

$$P_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

where

$$B_n = \frac{\beta-\alpha}{n\pi} i \left(1 - e^{2n\pi i \frac{1-\alpha}{\beta-\alpha}}\right) \qquad (n = 1, 2, \ldots). \tag{1.7}$$

The function $P_{\alpha,\beta}(z)$ was first introduced by Kuroki and Owa [2] and they proved that $P_{\alpha,\beta}$ maps Δ onto a convex domain

$$\Omega_{\alpha,\beta} = \{w \in \mathbb{C} : \alpha < \mathfrak{Re}\{w\} < \beta\}, \tag{1.8}$$

conformally.

In sequel, we prove that the following lemma and by it we present an example for the class $\mathcal{S}^*_\gamma(\alpha, \beta)$.

Lemma 1.2. *The function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*_\gamma(\alpha, \beta)$, $0 \le \alpha < 1$ and $\beta > 1$, if and only if there exists an analytic function p, $p(0) = 1$ and $p(z) \prec P_{\alpha,\beta}(z)$ such that*

$$f(z) = z \exp\left(\gamma \int_0^z \frac{p(t)-1}{t} dt\right) \qquad z \in \Delta. \tag{1.9}$$

Proof . Assume that $f \in \mathcal{S}^*_\gamma(\alpha, \beta)$ and let $p(z) = 1 + \frac{1}{\gamma}([zf'(z)/f(z)] - 1)$. Then $p(z) \prec P_{\alpha,\beta}(z)$ and integrating this equation we obtain (1.9). If f is given by (1.9), with an analytic p, $p(0) = 1$, $p(z) \prec P_{\alpha,\beta}(z)$, then differentiating logarithmically (1.9) we obtain $[zf'(z)/f(z)] - 1 = p(z)$, and therefore $[zf'(z)/f(z)] - 1 \prec P_{\alpha,\beta}(z)$ and $f \in \mathcal{S}^*_\gamma(\alpha, \beta)$. $\square$

Example 1.3. Putting $p = P_{\alpha,\beta}$ in the above Lemma 1.2 we obtain the function

$$F(z) = z \exp\left(\gamma \int_0^z \frac{P_{\alpha,\beta}(t)-1}{t} dt\right) \qquad z \in \Delta \tag{1.10}$$

and so $F(z) \in \mathcal{S}^*_\gamma(\alpha, \beta)$.

After some calculations and by (1.7) and (1.10), we have

$$F(z) = z + \gamma B_1 z^2 + \frac{\gamma}{2}\left(B_2 + \gamma B_1^2\right) z^3 + \cdots \in \mathcal{S}^*_\gamma(\alpha, \beta). \tag{1.11}$$

Using (1.8) and by the definition of subordination, we can obtain Lemma 1.4 and Lemma 1.5, directly.

Lemma 1.4. Let $f \in \mathcal{A}$, $0 \le \alpha < 1$ and $\beta > 1$. Then $f \in \mathcal{S}^*_\gamma(\alpha, \beta)$ if and only if

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z}\right) \qquad z \in \Delta. \tag{1.12}$$

Lemma 1.5. Let $f \in \mathcal{A}$, and $0 \le \alpha < 1$ and $\beta > 1$. Then $f \in \mathcal{K}_\gamma(\alpha, \beta)$ if and only if

$$1 + \frac{1}{\gamma}\left(\frac{zf''(z)}{f'(z)}\right) \prec 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z}\right) \qquad z \in \Delta. \tag{1.13}$$

Lemma 1.6. (Rogosinski [5]) Let $q(z) = \sum_{n=1}^{\infty} C_n z^n$ be analytic and univalent in Δ, and suppose that $q(z)$ maps Δ onto a convex domain. If $p(z) = \sum_{n=1}^{\infty} A_n z^n$ is analytic in Δ and satisfies the following subordination

$$p(z) \prec q(z) \qquad (z \in \Delta),$$

then

$$|A_n| \leq |C_1| \qquad (n = 1, 2, 3, \ldots).$$

The structure of the paper is the following. In Section 2 we estimate the coefficients of functions belonging to the class $\mathcal{S}^*_\gamma(\alpha,\beta)$ and we obtain the dual of the class $\mathcal{S}^*_\gamma(\alpha,\beta)$. Also sufficient condition for functions belonging to this subclass is obtained. In Section 3 we define a new TN_λ-neighborhood and we show that $TN_\lambda(f) \subset \mathcal{S}^*_\gamma(\alpha,\beta)$, where f defined by (1.1). Finally, in Section 4 we give two open problems.

2. Coefficients and Duality

One of our main results is the following.

Theorem 2.1. *Assume that* $0 \leq \alpha < 1$, $\beta > 1$ *and* $\gamma \in \mathbb{C}\backslash\{0\}$. *Let the function* f *given by* (1.1) *be in the class* $\mathcal{S}^*_\gamma(\alpha,\beta)$. *Then*

$$|a_n| \leq \begin{cases} |\gamma B_1| & n = 2, \\ \frac{|\gamma B_1|}{n-1} \prod_{k=2}^{n-1} \left(1 + \frac{|\gamma B_1|}{k-1}\right) & n = 3, 4, \ldots, \end{cases} \tag{2.1}$$

where B_1 *is given by* (1.7) *and*

$$|B_1| = \frac{2(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}. \tag{2.2}$$

Proof . Define the function $p(z)$ by

$$zf'(z) = [\gamma(p(z)-1)+1]f(z) \qquad z \in \Delta. \tag{2.3}$$

Then according to the assertion of Lemma 1.4, we have

$$p(z) \prec P_{\alpha,\beta}(z) \qquad z \in \Delta, \tag{2.4}$$

where $P_{\alpha,\beta}(z)$ defined by (1.6). If we let

$$p(z) = 1 + \sum_{n=1}^{\infty} A_n z^n,$$

then by Lemma 1.6, we see that the subordination (2.4) implies that

$$|A_n| \leq |B_1| \qquad (n = 1, 2, 3, \ldots). \tag{2.5}$$

Now by equating the coefficients of z^n in both sides of equality (2.3) we have

$$na_n = \gamma(A_{n-1} + A_{n-2}a_2 + \cdots + A_1 a_{n-1}) + a_n \qquad (n = 2, 3, \ldots). \tag{2.6}$$

A simple calculation together with the inequality (2.5) yields that

$$\begin{aligned}|a_n| &= \frac{|\gamma|}{n-1} \times |A_{n-1} + A_{n-2}a_2 + \cdots + A_1 a_{n-1}| \\ &\le \frac{|\gamma|}{n-1} \times (|A_{n-1}| + |A_{n-2}||a_2| + \cdots + |A_1||a_{n-1}|) \\ &\le \frac{|\gamma B_1|}{n-1} \sum_{k=1}^{n-1} |a_k|,\end{aligned}$$

where $|B_1|$ is given by (2.2) and $|a_1| = 1$. Hence, we have $|a_2| \le |\gamma B_1|$. In order to prove the remaining part of the theorem, we need to show that

$$\frac{|\gamma B_1|}{n-1} \sum_{k=1}^{n-1} |a_k| \le \frac{|\gamma B_1|}{n-1} \prod_{k=2}^{n-1} \left(1 + \frac{|\gamma B_1|}{k-1}\right) \qquad (n = 3, 4, 5 \ldots). \tag{2.7}$$

Using induction and simple calculation, we could to prove the inequality (2.7). Hence, the desired estimate for $|a_n|\,(n = 3, 4, 5, \ldots)$ follows, as asserted in (2.1). This completes the proof of Theorem 2.1. □

If we take $\gamma = 1$ in the Theorem 2.1, then we have.

Corollary 2.2. (see [2]) Let $0 \le \alpha < 1$, $\beta > 1$ and let the function f be given by (1.1) be in the class $\mathcal{S}^*(\alpha, \beta)$. Then

$$|a_n| \le \prod_{k=2}^{n} \left(\frac{k-2+|B_1|}{k-1}\right) \qquad (n = 2, 3, \ldots),$$

where $|B_1|$ is given by (2.2).

Now, we recall the definition of duality. For a set $\mathcal{V} \subset \mathcal{A}$, the dual set $\mathcal{V}$, denoted by $\mathcal{V}^*$ is defined as

$$\mathcal{V}^* = \left\{ g \in \mathcal{A} : \frac{1}{z}(f * g)(z) \neq 0 \quad \text{for all} \quad f \in \mathcal{V}\, \text{and}\, z \in \Delta \right\}.$$

The standard reference to duality theory for convolutions is the monograph by Ruscheweyh [6], and his paper [7]. First, we get the duality for the class $\mathcal{S}^*_\gamma(\alpha)$.

Theorem 2.3. *Assume that $f \in \mathcal{A}$, $0 \le \alpha < 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$. The function $f \in \mathcal{S}^*_\gamma(\alpha)$, if and only if*

$$\frac{f(z)}{z} * \frac{h_\alpha(z)}{z} \neq 0 \qquad z \in \Delta,$$

where

$$h_\alpha(z) := h(\alpha, \gamma, x)(z) = \frac{z\left(1 + \frac{x+1-2\gamma(1-\alpha)}{2\gamma(1-\alpha)} z\right)}{(1-z)^2} \qquad (|x| = 1). \tag{2.8}$$

Proof . According to the definition of $\mathcal{S}^*_\gamma(\alpha)$ and since $1 + \frac{1}{\gamma}[(zf'(z)/f(z)) - 1] = 1$ at $z = 0$, we have

$$\frac{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right) - \alpha}{1 - \alpha} \neq \frac{x-1}{x+1} \qquad (|x| = 1, x \neq -1, z \in \Delta),$$

which leads to

$$(x+1)zf'(z) + f(z)[2\gamma(1-\alpha) - (x+1)] \neq 0. \tag{2.9}$$

Where as $f(z) * l(z) = f(z)$ and $zf' = f * zl'(z)$, where $l(z) = z/(1-z)$, hence the relation (2.9) leads to

$$\frac{f(z)}{z} * \left[\frac{x+1}{(1-z)^2} + \frac{2\gamma(1-\alpha) - (x+1)}{1-z}\right] \neq 0,$$

or

$$\frac{f(z)}{z} * \left[\frac{2\gamma(1-\alpha) + [x+1-2\gamma(1-\alpha)]z}{(1-z)^2}\right] \neq 0.$$

With a little calculation we have

$$\frac{f(z)}{z} * \left[\frac{1 + \frac{x+1-2\gamma(1-\alpha)}{2\gamma(1-\alpha)} z}{(1-z)^2}\right] \neq 0,$$

which here ends the proof of Theorem 2.3. □

Remark 2.4. The special cases of $\gamma = 1, \alpha = 1/2$ and $\gamma = 1$ in Theorem 2.3 are contained in [8] and [10], respectively.

Theorem 2.5. *Assume that $f \in \mathcal{A}$, $0 \leq \alpha < 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$. The function $f \in \mathcal{K}_\gamma(\alpha)$, if and only if*

$$\frac{f(z)}{z} * \frac{l_\alpha(z)}{z} \neq 0 \qquad z \in \Delta,$$

where

$$l_\alpha(z) := l(\alpha, \gamma, x)(z) = \frac{z\left(1 + \frac{x+1-\gamma(1-\alpha)}{\gamma(1-\alpha)} z\right)}{(1-z)^3} \qquad (|x| = 1). \tag{2.10}$$

Proof . Using the relations $f \in \mathcal{K}_\gamma(\alpha) \Leftrightarrow zf'(z) \in \mathcal{S}^*_\gamma(\alpha)$, $(zf')' = f' * \frac{1}{(1-z)^2}$, $f'(z) * [z/(1-z)] = f'(z)$ and $zf' * g = f * zg'$ we can easily prove this theorem. So we omit the details. □

Remark 2.6. If we take $\gamma = 1$ in Theorem 2.5, then leads to a result which is obtained by H. Silverman et al. [10]. Also, the case $\gamma = 1, \alpha = 0$ was obtained by Ruscheweyh [8].

Definition 2.7. We define $\mathcal{V}^*_\alpha$ and $\mathcal{V}^*_\beta$ as follows:

$$\mathcal{V}^*_\alpha = (\mathcal{S}^*_\gamma(\alpha))^* := \left\{h_\alpha(z) \in \mathcal{A} : \frac{f(z)}{z} * \frac{h_\alpha(z)}{z} \neq 0, f \in \mathcal{S}^*_\gamma(\alpha), 0 \leq \alpha < 1, z \in \Delta\right\}$$

and

$$\mathcal{V}^*_\beta = (\mathcal{M}_\gamma(\beta))^* := \left\{h_\beta(z) \in \mathcal{A} : \frac{f(z)}{z} * \frac{h_\beta(z)}{z} \neq 0, f \in \mathcal{M}_\gamma(\beta), \beta > 1, z \in \Delta\right\},$$

where $h_\alpha(z)$ is given by (2.8) and

$$h_\beta(z) := h(\beta, \gamma, x)(z) = \frac{z\left(1 + \frac{x+1-2\gamma(1-\beta)}{2\gamma(1-\beta)} z\right)}{(1-z)^2} \qquad (|x| = 1). \tag{2.11}$$

Definition 2.8. We define by $\mathcal{W}_\delta^*$ the dual of the class $\mathcal{S}_\gamma^*(\alpha,\beta)$ as follows:

$$\mathcal{W}_\delta^* := \begin{cases} h_\delta(z) \in \mathcal{V}_\alpha^* & 0 \le \delta < 1, \\ h_\delta(z) \in \mathcal{V}_\beta^* & \delta > 1, \end{cases}$$

where

$$h_\delta(z) := h(\delta,\gamma,x)(z) = \frac{z\left(1 + \frac{x+1-2\gamma(1-\delta)}{2\gamma(1-\delta)} z\right)}{(1-z)^2} \qquad (|x|=1). \tag{2.12}$$

Using the definition of duality, we can obtain the following lemma directly.

Lemma 2.9. A function $f \in \mathcal{A}$ is in the class $\mathcal{S}_\gamma^*(\alpha,\beta)$ if and only if for all $h_\delta \in \mathcal{W}_\delta^*$

$$\frac{f(z)}{z} * \frac{h_\delta(z)}{z} \neq 0 \qquad z \in \Delta,$$

where h_δ is given by (2.12).

Lemma 2.10. *Let $h_\delta(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{W}_\delta^*$. Then*

$$|c_n| \le n + (n-1)|u(x,\gamma,\delta)| \qquad (n = 2,3,\ldots),$$

where

$$u(x,\gamma,\delta) := \frac{x+1-2\gamma(1-\delta)}{2\gamma(1-\delta)}. \tag{2.13}$$

Proof . Let $h_\delta \in \mathcal{W}_\delta^*$. Then for any $\gamma \in \mathbb{C}\backslash\{0\}$, $\delta \ge 0$ and $|x| = 1$, we have

$$\begin{aligned} h_\delta(z) &= \frac{z}{(1-z)^2} + u(x,\gamma,\delta)\frac{z^2}{(1-z)^2} \\ &= (z + 2z^2 + 3z^3 + \cdots) + u(x,\gamma,\delta)(z^2 + 2z^3 + 3z^4 + \cdots) \\ &= z + \sum_{n=2}^{\infty} c_n z^n, \end{aligned}$$

where

$$c_n = n + (n-1)u(x,\gamma,\delta), \tag{2.14}$$

and so $|c_n| \le n + (n-1)|u(x,\gamma,\delta)|$ and $n = 2,3,\ldots$. □

Corollary 2.11. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$. If*

$$\sum_{n=2}^{\infty} [n + (n-1)|u(x,\gamma,\delta)|]|a_n| < 1,$$

then $f \in \mathcal{S}_\gamma^*(\alpha,\beta)$, where $u(x,\gamma,\alpha)$ defined by (2.13).

Proof . Let $h_\delta(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{W}_\delta^*$. Then we have

$$\begin{aligned}\left|\frac{(f * h_\delta)(z)}{z}\right| &= \left|1 + \sum_{n=2}^{\infty} a_n c_n z^{n-1}\right| \\ &\geq 1 - \sum_{n=2}^{\infty} |a_n||c_n||z| > 1 - \sum_{n=2}^{\infty} |a_n||c_n| > 0,\end{aligned}$$

Thus $\frac{(f*h_\delta)(z)}{z} \neq 0$ and now from Lemma 2.9 we have $f \in \mathcal{S}_\gamma^*(\alpha, \beta)$. $\square$

For each complex number $\varepsilon \neq -1$ and $f \in \mathcal{A}$, define the function $F_\varepsilon(z)$ by

$$F_\varepsilon(z) = \frac{f(z) + \varepsilon z}{1 + \varepsilon} \qquad z \in \Delta. \tag{2.15}$$

Lemma 2.12. *Assume that $h_\delta \in \mathcal{W}_\delta^*$ and $\lambda > 0$. Let $F_\varepsilon(z)$ be defined by (2.15) belongs to the class $\mathcal{S}_\gamma^*(\alpha, \beta)$ for $|\varepsilon| < \lambda$. Then*

$$\left|\frac{1}{z}\left(f * h_\delta\right)(z)\right| > \lambda \qquad z \in \Delta.$$

Proof . If $F_\varepsilon(z) \in \mathcal{S}_\gamma^*(\alpha, \beta)$ for $|\varepsilon| < \lambda$, it follows from Lemma 2.9 that

$$\frac{1}{z}\left(F_\varepsilon * h_\delta\right)(z) \neq 0 \qquad z \in \Delta,$$

where h_δ defined by (2.12). The above relation is equivalent to

$$\frac{1}{z}\left(\frac{(f * h_\delta)(z) + \varepsilon z}{1 + \varepsilon}\right) \neq 0 \qquad z \in \Delta,$$

and hence

$$\frac{1}{z}\left(f * h_\delta\right)(z) \neq -\varepsilon \qquad z \in \Delta,$$

for every $|\varepsilon| < \lambda$. Therefore

$$\left|\frac{1}{z}\left(f * h_\delta\right)(z)\right| > \lambda \qquad z \in \Delta$$

and concluding the proof. $\square$

3. *T*-Neighborhood

Let $f(z)$ be defined by (1.1), λ is a positive number and $T = \{T_n\}_{n=2}^{\infty}$ is a sequence of positive numbers. Then the TN_λ-neighborhood of the function f is defined as

$$TN_\lambda(f) = \left\{g \in \mathcal{A} : g(z) = z + \sum_{n=2}^{\infty} b_n z^n : \sum_{n=2}^{\infty} T_n |b_n - a_n| \leq \lambda\right\}, \tag{3.1}$$

where we define T_n $(n = 2, 3, \ldots)$, as follows:

$$T_n = n + (n-1)|u(x, \gamma, \delta)| \quad \text{and} \quad u(x, \gamma, \delta) = \frac{x + 1 - 2\gamma(1-\delta)}{2\gamma(1-\delta)}. \tag{3.2}$$

St. Ruscheweyh in [9] considered $T_n = \{n\}_{n=2}^{\infty}$ and showed that if $f \in \mathcal{K}$, then $TN_{1/4}(f) \subset \mathcal{S}^*$.

Remark 3.1. Note that if we take $x = \gamma = 1$ and $\delta = 0$ in the sequence of T_n, then we have $T_n \equiv \{n\}_{n=2}^{\infty}$.

We have the following result on the TN_λ-neighborhoods for the class $\mathcal{S}_\gamma^*(\alpha, \beta)$.

Theorem 3.2. *Assume that* $0 \le \alpha < 1$, $\beta > 1$ *and* $\gamma \in \mathbb{C}\backslash\{0\}$. *Let the function* $f \in \mathcal{A}$ *and for any* $\varepsilon \in \mathbb{C}$, $\lambda > 0$ *and* $|\varepsilon| < \lambda$, $F_\varepsilon(z)$ *given by* (2.15) *be in the class* $\mathcal{S}_\gamma^*(\alpha, \beta)$. *Then* $TN_\lambda(f) \subset \mathcal{S}_\gamma^*(\alpha, \beta)$.

Proof . Let $g \in \mathcal{A}$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be in $TN_\lambda(f)$. Then

$$\begin{aligned}\left|\frac{1}{z}\left(g * h_\delta\right)(z)\right| &= \left|\frac{1}{z}\left(f * h_\delta\right)(z) + \frac{1}{z}[(g-f) * h_\delta](z)\right| \\ &\ge \left|\frac{1}{z}\left(f * h_\delta\right)(z)\right| - \left|\frac{1}{z}[(g-f) * h_\delta](z)\right|,\end{aligned}$$

where h_δ defined by (2.12). Making use of Lemma 2.12, we obtain

$$\begin{aligned}\left|\frac{1}{z}\left(g * h_\delta\right)(z)\right| &\ge \lambda - \left|\sum_{n=2}^{\infty} \frac{(b_n - a_n) c_n z^n}{z}\right| \\ &> \lambda - \sum_{n=2}^{\infty} |b_n - a_n||c_n|,\end{aligned}$$

where c_n is given by (2.14). Since $g \in TN_\lambda(f)$, it follows that

$$\sum_{n=2}^{\infty} |b_n - a_n||c_n| \le \sum_{n=2}^{\infty} [n + (n-1)|u(x, \gamma, \delta)|]|b_n - a_n| \le \lambda$$

and thus

$$\left|\frac{1}{z}\left(g * h_\delta\right)(z)\right| > \lambda - \lambda = 0.$$

Therefore $\left|\frac{1}{z}\left(g * h_\delta\right)(z)\right| \neq 0$. Now, from Lemma 2.9, we get $g \in \mathcal{S}_\gamma^*(\alpha, \beta)$. This completes the proof. □

4. Open problems

In this section, we give two open problems.

Open Problem 1. In [3], the authors proved that if $f \in \mathcal{S}^*(\alpha, \beta)$, then

$$\frac{1}{3 - 2\alpha} < \mathfrak{Re}\left\{\frac{f(z)}{z}\right\} < \frac{1}{3 - 2\beta} \qquad (1/2 \le \alpha < 1, 1 < \beta < 3/2).$$

What the bounds for $\mathfrak{Re}\{f(z)/z\}$ are, when $f \in \mathcal{S}_\gamma^*(\alpha, \beta)$?

Open Problem 2. Let T_n be defined by (3.2). Then find the greatest $\lambda := \lambda(x, \gamma, \delta)$ so that if $f \in \mathcal{K}$, then $TN_\lambda(f) \subset \mathcal{S}^*$.

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