Int. J. Nonlinear Anal. Appl. 8 (2017) No. 1, 11-21
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.11099.1542

The operators over the generalized intuitionistic fuzzy sets

Ezzatallah Baloui Jamkhaneh

Department of Statistics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran

(Communicated by M. Eshaghi)

Abstract

In this paper, newly defined level operators and modal-like operators over extensional generalized intuitionistic fuzzy sets ($GIFS_B$) are proposed. Some of the basic properties of the new operators are discussed.

Keywords: Generalized intuitionistic fuzzy sets; intuitionistic fuzzy sets; modal-like operators; level operators.
2010 MSC: 03E72.

1. Introduction

Modal operators, topological operators, level operators, negation operators are different groups of operators over the intuitionistic fuzzy sets (IFS) due to Atanassov [2]. Atanassov [3] defined intuitionistic fuzzy modal operator of $\boxplus_\alpha A$ and $\boxtimes_\alpha A$. Then Dencheva [10] defined an extension of these operators as $\boxplus_{\alpha,\beta} A$ and $\boxtimes_{\alpha,\beta} A$. Atanassov [4] defined another extension of these operators as $\boxplus_{\alpha,\beta,\gamma} A$ and $\boxtimes_{\alpha,\beta,\gamma} A$. Atanassov [5] defined an operator as $\boxdot_{\alpha,\beta,\gamma,\eta}$ which is an extension of all the operators defined. In 2007, Cuvaleglu defined operator of $E_{\alpha,\beta}$ over $IFSs$ and studied some of its properties. $E_{\alpha,\beta}$ is a generalization of Atanassov's operator $\boxplus_\alpha A$. Cuvalcoglu et al. [9] examined intuitionistic fuzzy operators with matrices and they examined algebraic structures of them. Atanassov [1] defined level operators $P_{\alpha,\beta}$, $Q_{\alpha,\beta}$. In 2008, he studied some relations between intuitionistic fuzzy negations and intuitionistic fuzzy level operators $P_{\alpha,\beta}$, $Q_{\alpha,\beta}$. In 2009, Parvathi and Geetha defined some level operators, max-min implication operators and $P_{\alpha,\beta}$, $Q_{\alpha,\beta}$ operators on temporal intuitionistic fuzzy sets. Lupianez [11] show relations between topological operators and intuitionistic fuzzy topology. The intuitionistic fuzzy operators are important from the point of view application. The

Email address: e_baloui2008@yahoo.com (Ezzatallah Baloui Jamkhaneh)

Received: February 2016 *Revised:* October 2016

intuitionistic fuzzy operators applied in contracting a classifier recognizing imbalanced classes, image recognition, image processing, multi-criteria decision making, deriving the similarity measure, sales analysis, new product marketing, medical diagnosis, financial services, solving optimization problems and etc. Baloui Jamkhaneh and Nadarajah [7] considered a new generalized intuitionistic fuzzy sets ($GIFS_B$) and introduced some operators over $GIFS_B$. By analogy we shall introduce the some of operators over $GIFS_B$ and we will discuss their properties.

2. Preliminaries

In this section we recall some of the basic definition of IFS which will be helpful in further study of the paper. Let X be a non empty set.

Definition 2.1. (Atanassov, [1]) An IFS A in X is defined as an object of the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership and non-membership functions of A respectively and $0 \le \mu_A(x) + \nu_A(x) \le 1$ for each $x \in X$.

Definition 2.2. (Baloui Jamkhaneh and Nadarajah, [7]) Generalized intuitionistic fuzzy sets ($GIFS_B$) A in X, is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership and degree of non-membership functions of A respectively, and $0 \le \mu_A(x)^\delta + \nu_A(x)^\delta \le 1$ for each $x \in X$ where $\delta = n or \frac{1}{n}, n = 1, 2, \dots, N$. The collection of all our generalized intuitionistic fuzzy sets is denoted by $GIFS_B(\delta, X)$.

Definition 2.3. (Baloui Jamkhaneh and Nadarajah, [7]) The degree of non-determinacy (uncertainty) of an element $x \in X$ to the $GIFS_B$ A is defined by

$$\pi_A(x) = (1 - \mu_A(x)^\delta - \nu_A(x)^\delta)^{\frac{1}{\delta}}$$

Definition 2.4. (Baloui Jamkhaneh and Nadarajah, [7]) Let A and B be two $GIFS_B$s such that

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}, \quad B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\},$$

define the following relations and operations on A and B

i. $A \subset B$ if and only if $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x), \forall x \in X$,

ii. $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x), \forall x \in X$,

iii. $A \bigcup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle : x \in X\}$,

iv. $A \bigcap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle : x \in X\}$,

v. $A + B = \{\langle x, \mu_A(x)^\delta + \mu_B(x)^\delta - \mu_A(x)^\delta \mu_B(x)^\delta, \nu_A(x)^\delta \nu_B(x)^\delta\rangle : x \in X\}$,

vi. $A.B = \{\langle x, \mu_A(x)^\delta . \mu_B(x)^\delta, \nu_A(x)^\delta + \nu_B(x)^\delta - \nu_A(x)^\delta \nu_B(x)^\delta\rangle : x \in X\}$,

vii. $\bar{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$.

Let X is a non-empty finite set, and $A \in GIFS_B$, as $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$. Baloui Jamkhaneh and Nadarajah [7] introduced following operators of $GIFS_B$ and investigated some their properties

viii. $\Box A = \{\langle x, \mu_A(x), (1 - \mu_A(x)^\delta)^{\frac{1}{\delta}}\rangle : x \in X\}$, (modal logic: the necessity measure).

ix. $\Diamond A = \{\langle x, (1 - \nu_A(x)^\delta)^{\frac{1}{\delta}}, \nu_A(x)\rangle : x \in X\}$, (modal logic: the possibility measure).

3. Main results

Here, we will introduce new operators over the $GIFS_B$ which extend some operators in the research literature related to IFS. Let X is a non-empty finite set.

Definition 3.1. Letting $\alpha, \beta \in [0,1]$. For every $GIFS_B$ as $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, we define the level operators as follows

i. $P_{\alpha,\beta}(A) = \{\langle x, \max(\alpha^{\frac{1}{\delta}}, \mu_A(x)), \min(\beta^{\frac{1}{\delta}}, \nu_A(x)) : x \in X\}$, where $\alpha + \beta \leq 1$,

ii. $Q_{\alpha,\beta}(A) = \{\langle x, \min(\alpha^{\frac{1}{\delta}}, \mu_A(x)), \max(\beta^{\frac{1}{\delta}}, \nu_A(x)) : x \in X\}$, where $\alpha + \beta \leq 1$.

Theorem 3.2. *For every $A \in GIFS_B$ and $\alpha, \beta, \gamma, \eta \in [0,1]$, where $\alpha + \beta \leq 1$, $\gamma + \eta \leq 1$, we have*

i. $P_{\alpha,\beta}(A) \in GIFS_B$,

ii. $Q_{\alpha,\beta}(A) \in GIFS_B$,

iii. $\overline{P_{\alpha,\beta}(\bar{A})} = Q_{\beta,\alpha}(A)$,

iv. $Q_{\alpha,\beta}(P_{\gamma,\eta}(A)) = P_{\min(\alpha,\gamma),\max(\beta,\eta)}(Q_{\alpha,\beta}(A))$,

v. $P_{\alpha,\beta}(Q_{\gamma,\eta}(A)) = Q_{\max(\alpha,\gamma),\min(\beta,\eta)}(P_{\alpha,\beta}(A))$,

vi. $P_{\alpha,\beta}(P_{\gamma,\eta}(A)) = P_{\max(\alpha,\gamma),\min(\beta,\eta)}(P_{\alpha,\beta}(A))$,

vii. $Q_{\alpha,\beta}(Q_{\gamma,\eta}(A)) = Q_{\min(\alpha,\gamma),\max(\beta,\eta)}(Q_{\alpha,\beta}(A))$,

viii. $P_{\alpha,\beta}(Q_{\alpha,\beta}(A)) = Q_{\alpha,\beta}(P_{\alpha,\beta}(A)) = \{\langle x, \alpha^{\frac{1}{\delta}}, \beta^{\frac{1}{\delta}}\rangle : x \in X\}$,

ix. $Q_{\alpha,\beta}(A) \subset A \subset P_{\alpha,\beta}(A)$.

Proof .

(i)

$$\begin{aligned}\mu_{P_{\alpha,\beta}(A)}(x)^{\delta} + \nu_{P_{\alpha,\beta}(A)}(x)^{\delta} &= (\max(\alpha^{\frac{1}{\delta}}, \mu_A(x)))^{\delta} + (\min(\beta^{\frac{1}{\delta}}, \nu_A(x)))^{\delta},\\ &= \max(\alpha, \mu_A(x)^{\delta}) + \min(\beta, \nu_A(x)^{\delta}) = I,\end{aligned}$$

(1) if $\max(\alpha, \mu_A(x)^{\delta}) = \alpha$ and $\min(\beta, \nu_A(x)^{\delta}) = \beta$ then $I = \alpha + \beta \leq 1$.

(2) if $\max(\alpha, \mu_A(x)^{\delta}) = \alpha$ and $\min(\beta, \nu_A(x)^{\delta}) = \nu_A(x)^{\delta}$ then $I = \alpha + \nu_A(x)^{\delta} \leq \alpha + \beta \leq 1$.

(3) if $\max(\alpha, \mu_A(x)^{\delta}) = \mu_A(x)^{\delta}$ and $\min(\beta, \nu_A(x)^{\delta}) = \beta$ then $I = \mu_A(x)^{\delta} + \beta \leq \mu_A(x)^{\delta} + \nu_A(x)^{\delta} \leq 1$.

(4) if $\max(\alpha, \mu_A(x)^{\delta}) = \mu_A(x)^{\delta}$ and $\min(\beta, \nu_A(x)^{\delta}) = \nu_A(x)^{\delta}$ then $I = \mu_A(x)^{\delta} + \nu_A(x)^{\delta} \leq 1$.

The proof is completed. The Proof of (ii) is similar to that of (i). Proof of (iii) is obvious.

(iv)

$$\begin{aligned}
Q_{\alpha,\beta}(P_{\gamma,\eta}(A)) &= Q_{\alpha,\beta}\left(\left\{\langle x, \max\left(\gamma^{\frac{1}{\delta}}, \mu_A(x)\right), \min\left(\eta^{\frac{1}{\delta}}, \nu_A(x)\right)\rangle : x \in X\right\}\right),\\
&= \left\{\langle x, \min\left(\alpha^{\frac{1}{\delta}}, \max(\gamma^{\frac{1}{\delta}}, \mu_A(x))\right), \max\left(\beta^{\frac{1}{\delta}}, \min(\eta^{\frac{1}{\delta}}, \nu_A(x))\right)\rangle : x \in X\right\},\\
&= \left\{\langle x, \max\left(\min(\alpha^{\frac{1}{\delta}}, \gamma^{\frac{1}{\delta}}), \min(\alpha^{\frac{1}{\delta}}, \mu_A(x))\right)\right.\\
&\quad \left., \min\left(\max(\beta^{\frac{1}{\delta}}, \eta^{\frac{1}{\delta}}), \max(\beta^{\frac{1}{\delta}}, \nu_A(x))\right)\rangle : x \in X\right\},\\
&= P_{\min(\alpha,\gamma),\max(\beta,\eta)}\left\{\langle x, \min(\alpha^{\frac{1}{\delta}}, \mu_A(x)), \max(\beta^{\frac{1}{\delta}}, \nu_A(x))\rangle : x \in X\right\},\\
&= P_{\min(\alpha,\gamma),\max(\beta,\eta)}(Q_{\alpha,\beta}(A)).
\end{aligned}$$

Since $\alpha + \beta \leq 1$, $\gamma + \eta \leq 1$ then $\min(\alpha, \gamma) + \max(\beta, \eta) \leq 1$.

(v)

$$\begin{aligned}
P_{\alpha,\beta}(Q_{\gamma,\eta}(A)) &= P_{\alpha,\beta}\left(\left\{\langle x, \min\left(\gamma^{\frac{1}{\delta}}, \mu_A(x)\right), \max\left(\eta^{\frac{1}{\delta}}, \nu_A(x)\right)\rangle : x \in X\right\}\right),\\
&= \left\{\langle x, \max\left(\alpha^{\frac{1}{\delta}}, \min(\gamma^{\frac{1}{\delta}}, \mu_A(x))\right), \min\left(\beta^{\frac{1}{\delta}}, \max(\eta^{\frac{1}{\delta}}, \nu_A(x))\right)\rangle : x \in X\right\},\\
&= \left\{\langle x, \min\left(\max(\alpha^{\frac{1}{\delta}}, \gamma^{\frac{1}{\delta}}), \max(\alpha^{\frac{1}{\delta}}, \mu_A(x))\right)\right.\\
&\quad \left., \max\left(\min(\beta^{\frac{1}{\delta}}, \eta^{\frac{1}{\delta}}), \min(\beta^{\frac{1}{\delta}}, \nu_A(x))\right)\rangle : x \in X\right\},\\
&= Q_{\max(\alpha,\gamma),\min(\beta,\eta)}\left\{\langle x, \max(\alpha^{\frac{1}{\delta}}, \mu_A(x)), \min(\beta^{\frac{1}{\delta}}, \nu_A(x))\rangle : x \in X\right\},\\
&= Q_{\max(\alpha,\gamma),\min(\beta,\eta)}(P_{\alpha,\beta}(A)).
\end{aligned}$$

Since $\alpha + \beta \leq 1$, $\gamma + \eta \leq 1$ then $\max(\alpha, \gamma) + \min(\beta, \eta) \leq 1$. The proof is completed. The Proofs of (vi), (vii) and (viii) are similar to that of (iv). The Proof of (ix) is obvious.

□

Theorem 3.3. *For every $A \in GIFS_B$ and $\alpha, \beta \in [0, 1]$, where $\alpha + \beta \leq 1$ we have*

i. $P_{\alpha,\beta}(A \cap B) = P_{\alpha,\beta}(A) \cap P_{\alpha,\beta}(B)$,

ii. $P_{\alpha,\beta}(A \cup B) = P_{\alpha,\beta}(A) \cup P_{\alpha,\beta}(B)$,

iii. $Q_{\alpha,\beta}(A \cap B) = Q_{\alpha,\beta}(A) \cap Q_{\alpha,\beta}(B)$,

iv. $Q_{\alpha,\beta}(A \cup B) = Q_{\alpha,\beta}(A) \cup Q_{\alpha,\beta}(B)$.

Proof .

(i)

$$P_{\alpha,\beta}(A \cap B) = \left\{ \langle x, \max(\alpha^{\frac{1}{\delta}}, \min(\mu_A(x), \mu_B(x))), \min(\beta^{\frac{1}{\delta}}, \max(\nu_A(x), \nu_B(x))) \rangle : x \in X \right\},$$

$$= \left\{ \langle x, \min\left(\max(\alpha^{\frac{1}{\delta}}, \mu_A(x)), \max(\alpha^{\frac{1}{\delta}}, \mu_B(x)) \right), \max\left(\min(\beta^{\frac{1}{\delta}}, \nu_A(x)), \min(\beta^{\frac{1}{\delta}}, \nu_B(x)) \right) \rangle : \right.$$

$$\left. x \in X \right\} = \left\{ \langle x, \max(\alpha^{\frac{1}{\delta}}, \mu_A(x)), \min(\beta^{\frac{1}{\delta}}, \nu_A(x)) \rangle : x \in X \right\} \cap$$

$$\left\{ \langle x, \max(\alpha^{\frac{1}{\delta}}, \mu_B(x)), \min(\beta^{\frac{1}{\delta}}, \nu_B(x)) \rangle : x \in X \right\} = P_{\alpha,\beta}(A) \cap P_{\alpha,\beta}(B).$$

(ii)

$$P_{\alpha,\beta}(A \cup B) = \left\{ \langle x, \max(\alpha^{\frac{1}{\delta}}, \max(\mu_A(x), \mu_B(x))), \min(\beta^{\frac{1}{\delta}}, \min(\nu_A(x), \nu_B(x))) \rangle : x \in X \right\},$$

$$= \left\{ \langle x, \max\left(\max(\alpha^{\frac{1}{\delta}}, \mu_A(x)), \max(\alpha^{\frac{1}{\delta}}, \mu_B(x)) \right), \min\left(\min(\beta^{\frac{1}{\delta}}, \nu_A(x)), \min(\beta^{\frac{1}{\delta}}, \nu_B(x)) \right) \rangle : \right.$$

$$\left. x \in X \right\} = \left\{ \langle x, \max(\alpha^{\frac{1}{\delta}}, \mu_A(x)), \min(\beta^{\frac{1}{\delta}}, \nu_A(x)) \rangle : x \in X \right\} \cup$$

$$\left\{ \langle x, \max(\alpha^{\frac{1}{\delta}}, \mu_B(x)), \min(\beta^{\frac{1}{\delta}}, \nu_B(x)) \rangle : x \in X \right\} = P_{\alpha,\beta}(A) \cup P_{\alpha,\beta}(B)$$

The proofs of (iii) and (iv) are similar to that of (i). □

Definition 3.4. Let $\alpha \in [0,1]$ and $A \in GIFS_B$, we define modal-like operators as follows

i. $\boxplus_\alpha A = \{\langle x, \alpha^{\frac{1}{\delta}}\mu_A(x), (\alpha\nu_A(x)^\delta + 1 - \alpha)^{\frac{1}{\delta}} \rangle : x \in X\}$,

ii. $\boxtimes_\alpha A = \{\langle x, (\alpha\mu_A(x)^\delta + 1 - \alpha)^{\frac{1}{\delta}}, \alpha^{\frac{1}{\delta}}\nu_A(x) \rangle : x \in X\}$.

Definition 3.5. Let $\alpha, \beta \in [0,1]$ and $A \in GIFS_B$, we define the first extension modal-like operators as follows

i. $\boxplus_{\alpha,\beta} A = \{\langle x, \alpha^{\frac{1}{\delta}}\mu_A(x), (\alpha\nu_A(x)^\delta + \beta)^{\frac{1}{\delta}} \rangle : x \in X\}$, where $\alpha + \beta \leq 1$,

ii. $\boxtimes_{\alpha,\beta} A = \{\langle x, (\alpha\mu_A(x)^\delta + \beta)^{\frac{1}{\delta}}, \alpha^{\frac{1}{\delta}}\nu_A(x) \rangle : x \in X\}$, where $\alpha + \beta \leq 1$.

Definition 3.6. Let $\alpha, \beta \in [0,1]$ and $A \in GIFS_B$, we define the operator $E_{\alpha,\beta}A$ as follows

$$E_{\alpha,\beta}A = \{\langle x, (\beta(\alpha\mu_A(x)^\delta + 1 - \alpha))^{\frac{1}{\delta}}, (\alpha(\beta\nu_A(x)^\delta + 1 - \beta))^{\frac{1}{\delta}} \rangle : x \in X\}.$$

Definition 3.7. Let $\alpha, \beta, \gamma, \eta \in [0,1]$ and $A \in GIFS_B$, where $\alpha\beta + \beta\gamma + \alpha\gamma \leq 1$ we define the operator $E_{\alpha,\beta,\gamma,\eta}A$ as follows

$$E_{\alpha,\beta,\gamma,\eta}A = \{\langle x, (\beta(\alpha\mu_A(x)^\delta + \gamma))^{\frac{1}{\delta}}, (\alpha(\beta\nu_A(x)^\delta + \eta))^{\frac{1}{\delta}} \rangle : x \in X\}.$$

Theorem 3.8. *For every $A \in GIFS_B$ and $\alpha, \beta, \gamma, \eta \in [0,1]$, where $\alpha\beta + \beta\gamma + \alpha\eta \leq 1$, we have* *i.* $E_{\alpha,\beta,\gamma,\eta}A \in GIFS_B$,

ii. $E_{\alpha,1,\gamma,0}A = \boxtimes_{\alpha,\gamma}A$,

iii. $E_{1,\beta,0,\eta}A = \boxplus_{\beta,\eta}A$,

iv. $E_{\alpha,1,1-\alpha,0}A = \boxtimes_{\alpha}A$,

v. $E_{1,\beta,0,1-\beta}A = \boxplus_{\beta}A$.

Proof . (i)

$$\begin{aligned}
&\mu_{E_{\alpha,\beta,\gamma,\eta}A}(x)^{\delta} + \nu_{E_{\alpha,\beta,\gamma,\eta}A}(x)^{\delta} = ((\beta(\alpha\mu_A(x)^{\delta}+\gamma))^{\frac{1}{\delta}})^{\delta} + ((\alpha(\beta\nu_A(x)^{\delta}+\eta))^{\frac{1}{\delta}})^{\delta},\\
&= (\alpha\beta\mu_A(x)^{\delta}+\beta\gamma) + (\alpha\beta\nu_A(x)^{\delta}+\alpha\eta) = \alpha\beta(\mu_A(x)^{\delta}+\nu_A(x)^{\delta}) + \beta\gamma + \alpha\eta,\\
&\leq \alpha\beta + \beta\gamma + \alpha\eta \leq 1.
\end{aligned}$$

The proof is completed. The Proofs of (ii), (iii), (iv) and (v) are obvious. □

Definition 3.9. Let $\alpha, \beta \in [0,1]$ and $A \in GIFS_B$, we define the second extension modal-like operators as follows

i. $\boxplus_{\alpha,\beta,\gamma}A = \left\{\langle x, \alpha^{\frac{1}{\delta}}\mu_A(x), (\beta\nu_A(x)^{\delta}+\gamma)^{\frac{1}{\delta}}\rangle : x \in X\right\}$, where $\max(\alpha,\beta)+\gamma \leq 1$.

ii. $\boxtimes_{\alpha,\beta,\gamma}A = \left\{\langle x, (\alpha\mu_A(x)^{\delta}+\gamma)^{\frac{1}{\delta}}, \beta^{\frac{1}{\delta}}\nu_A(x)\rangle : x \in X\right\}$, where $\max(\alpha,\beta)+\gamma \leq 1$.

Theorem 3.10. *For every $A \in GIFS_B$ and $\alpha,\beta,\gamma,\varepsilon,\eta,\lambda \in [0,1]$, where $\max(\alpha,\beta)+\gamma \leq 1$, and $\max(\varepsilon,\eta)+\lambda \leq 1$, we have*

i. $\boxplus_{\alpha,\beta,\gamma}A \in GIFS_B$,

ii. $\boxtimes_{\alpha,\beta,\gamma}A \in GIFS_B$,

iii. $\overline{\boxplus_{\alpha,\beta,\gamma}\bar{A}} = \boxtimes_{\beta,\alpha,\gamma}A$,

iv. $\boxplus_{\alpha,\beta,\gamma}\boxplus_{\varepsilon,\eta,\lambda}A = \boxplus_{\alpha\varepsilon,\beta\eta,\beta\lambda+\gamma}A$,

v. $\boxplus_{\alpha,\beta,\gamma}(A\cap B) = \boxplus_{\alpha,\beta,\gamma}A \cap \boxplus_{\alpha,\beta,\gamma}B$,

vi. $\boxplus_{\alpha,\beta,\gamma}(A\cup B) = \boxplus_{\alpha,\beta,\gamma}A \cup \boxplus_{\alpha,\beta,\gamma}B$,

vii. $\boxtimes_{\alpha,\beta,\gamma}(A\cap B) = \boxtimes_{\alpha,\beta,\gamma}A \cap \boxtimes_{\alpha,\beta,\gamma}B$,

viii. $\boxtimes_{\alpha,\beta,\gamma}(A\cup B) = \boxtimes_{\alpha,\beta,\gamma}A \cup \boxtimes_{\alpha,\beta,\gamma}B$,

ix. $\boxplus_{\alpha,\beta,\gamma}P_{\varepsilon,\eta}(A) = P_{\alpha\varepsilon,\beta\eta+\gamma}(\boxplus_{\alpha,\beta,\gamma}A)$, *if* $\alpha\varepsilon+\beta\eta+\gamma \leq 1$,

x. $\boxplus_{\alpha,\beta,\gamma}Q_{\varepsilon,\eta}(A) = Q_{\alpha\varepsilon,\beta\eta+\gamma}(\boxplus_{\alpha,\beta,\gamma}A)$, *if* $\alpha\varepsilon+\beta\eta+\gamma \leq 1$,

xi. $\boxtimes_{\alpha,\beta,\gamma}P_{\varepsilon,\eta}(A) = P_{\alpha\varepsilon,\beta\eta+\gamma}(\boxtimes_{\alpha,\beta,\gamma}A)$, *if* $\alpha\varepsilon+\beta\eta+\gamma \leq 1$,

xii. $\boxtimes_{\alpha,\beta,\gamma}Q_{\varepsilon,\eta}(A) = Q_{\alpha\varepsilon,\beta\eta+\gamma}(\boxtimes_{\alpha,\beta,\gamma}A)$, *if* $\alpha\varepsilon+\beta\eta+\gamma \leq 1$,

Proof . (i)

$$\begin{aligned}
\mu_{\boxplus_{\alpha,\beta,\gamma}A}(x)^{\delta}+\nu_{\boxplus_{\alpha,\beta,\gamma}A}(x)^{\delta} &= (\alpha^{\frac{1}{\delta}}\mu_A(x))^{\delta}+((\beta\nu_A(x)^{\delta}+\gamma)^{\frac{1}{\delta}})^{\delta},\\
=\alpha\mu_A(x)^{\delta}+\beta\nu_A(x)^{\delta}+\gamma &\leq \max(\alpha,\beta)\mu_A(x)^{\delta}+\max(\alpha,\beta)\nu_A(x)^{\delta}+\gamma,\\
&= \max(\alpha,\beta)(\mu_A(x)^{\delta}+\nu_A(x)^{\delta})+\gamma,\\
&\leq \max(\alpha,\beta)+\gamma\leq 1.
\end{aligned}$$

The proof is completed. The Proof of (ii) is similar to that of (i).

(iii) Since $\bar{A}=\{\langle x,\nu_A(x),\mu_A(x)\rangle : x\in X\}$, we have

$$\boxplus_{\alpha,\beta,\gamma}\bar{A}=\left\{\langle x,\alpha^{\frac{1}{\delta}}\nu_A(x),(\beta\mu_A(x)^{\delta}+\gamma)^{\frac{1}{\delta}}\rangle : x\in X\right\},$$

then

$$\overline{\boxplus_{\alpha,\beta,\gamma}\bar{A}}=\left\{\langle x,(\beta\mu_A(x)^{\delta}+\gamma)^{\frac{1}{\delta}},\alpha^{\frac{1}{\delta}}\nu_A(x)\rangle : x\in X\right\}=\boxtimes_{\beta,\alpha,\gamma}A.$$

The proof is completed.

(iv) Since

$$\boxplus_{\alpha,\beta,\gamma}A=\left\{\langle x,\alpha^{\frac{1}{\delta}}\mu_A(x),(\beta\nu_A(x)^{\delta}+\gamma)^{\frac{1}{\delta}}\rangle : x\in X\right\},$$

$$\boxplus_{\varepsilon,\eta,\lambda}A=\left\{\langle x,\varepsilon^{\frac{1}{\delta}}\mu_A(x),(\eta\nu_A(x)^{\delta}+\lambda)^{\frac{1}{\delta}}\rangle : x\in X\right\},$$

we have

$$\begin{aligned}
&\boxplus_{\alpha,\beta,\gamma}\boxplus_{\varepsilon,\eta,\lambda}A=\left\{\langle x,\alpha^{\frac{1}{\delta}}\varepsilon^{\frac{1}{\delta}}\mu_A(x),(\beta((\eta\nu_A(x)^{\delta}+\lambda)^{\frac{1}{\delta}})^{\delta}+\gamma)^{\frac{1}{\delta}}\rangle : x\in X\right\},\\
&=\left\{\langle x,\alpha^{\frac{1}{\delta}}\varepsilon^{\frac{1}{\delta}}\mu_A(x),(\beta\eta\nu_A(x)^{\delta}+\beta\lambda+\gamma)^{\frac{1}{\delta}}\rangle : x\in X\right\},\\
&=\boxplus_{\alpha\varepsilon,\beta\eta,\beta\lambda+\gamma}A.
\end{aligned}$$

Since $\max(\alpha,\beta)+\gamma\leq 1$, and $\max(\varepsilon,\eta)+\lambda\leq 1$, then $\max(\alpha\varepsilon,\beta\eta)+\beta\lambda+\gamma\leq 1$.

(v) Since

$$A\cap B=\{\langle x,\min(\mu_A(x),\mu_B(x)),\max(\nu_A(x),\nu_B(x))\rangle : x\in X\},$$

we have

$$\begin{aligned}
&\boxplus_{\alpha,\beta,\gamma}(A\cap B)=\left\{\langle x,\alpha^{\frac{1}{\delta}}\min(\mu_A(x),\mu_B(x)),(\beta(\max(\nu_A(x),\nu_B(x)))^{\delta}+\gamma)^{\frac{1}{\delta}}\rangle : x\in X\right\},\\
&=\left\{\langle x,\min(\alpha^{\frac{1}{\delta}}\mu_A(x),\alpha^{\frac{1}{\delta}}\mu_B(x)),(\max((\beta\nu_A(x)^{\delta}+\gamma)^{\frac{1}{\delta}},(\beta\nu_B(x)^{\delta}+\gamma)^{\frac{1}{\delta}}\rangle : x\in X\right\},\\
&=\boxplus_{\alpha,\beta,\gamma}A\cap\boxplus_{\alpha,\beta,\gamma}B.
\end{aligned}$$

The proof is completed. The Proofs of (vi), (vii) and (viii) are similar to that of (v)

(ix) Since

$$P_{\varepsilon,\eta}(A)=\left\{\langle x,\max(\varepsilon^{\frac{1}{\delta}},\mu_A(x)),\min(\eta^{\frac{1}{\delta}},\nu_A(x))\rangle : x\in X\right\},$$

we have

$$\begin{aligned}&\boxplus_{\alpha,\beta,\gamma} P_{\varepsilon,\eta}(A) = \left\{\langle x, \alpha^{\frac{1}{\delta}} \max\left(\varepsilon^{\frac{1}{\delta}}, \mu_A(x)\right), (\beta(\min(\eta^{\frac{1}{\delta}}, \nu_A(x)))^{\delta} + \gamma)^{\frac{1}{\delta}}\rangle : x \in X\right\},\\ &= \left\{\langle x, \max\left(\alpha^{\frac{1}{\delta}}\varepsilon^{\frac{1}{\delta}}, \alpha^{\frac{1}{\delta}}\mu_A(x)\right), (\min(\beta\eta+\gamma, \beta\nu_A(x)^{\delta}+\gamma))^{\frac{1}{\delta}}\rangle : x \in X\right\},\\ &= P_{\alpha\varepsilon,\beta\eta+\gamma}(\boxplus_{\alpha,\beta,\gamma}A).\end{aligned}$$

The proof is completed. Proofs (x), (xi) and (xii) are similar to that of (ix). □

If in operators $\boxplus_{\alpha,\beta,\gamma}A$ and $\boxtimes_{\alpha,\beta,\gamma}A$, we choose $\beta = \alpha$ and $\gamma = \beta$, then they are converted to the operators $\boxplus_{\alpha,\beta}A$ and $\boxtimes_{\alpha,\beta}A$, respectively. Therefore operators $\boxplus_{\alpha,\beta,\gamma}A$ and $\boxtimes_{\alpha,\beta,\gamma}A$ are the extension of the $\boxplus_{\alpha,\beta}A$ and $\boxtimes_{\alpha,\beta}A$, respectively. If in operators $\boxplus_{\alpha,\beta}A$ and $\boxtimes_{\alpha,\beta}A$. we choose$\beta = \alpha$ and $\beta = 1-\alpha$, then they are converted to the operators $\boxplus_{\alpha}A$ and $\boxtimes_{\alpha}A$, respectively. Therefore operators $\boxplus_{\alpha,\beta}A$ and $\boxtimes_{\alpha,\beta}A$ are the extension of the $\boxplus_{\alpha}A$ and $\boxtimes_{\alpha}A$, respectively. That is

i. $\boxplus_{\alpha,\alpha,\beta}A = \boxplus_{\alpha,\beta}A$, $\boxplus_{\alpha,\alpha,1-\alpha}A = \boxplus_{\alpha,1-\alpha}A = \boxplus_{\alpha}A$,

ii. $\boxtimes_{\alpha,\alpha,\beta}A = \boxtimes_{\alpha,\beta}A$, $\boxtimes_{\alpha,\alpha,1-\alpha}A = \boxtimes_{\alpha,1-\alpha}A = \boxtimes_{\alpha}A$.

Therefore using Theorem 3.10, we have the following proposition:

Proposition 3.11. For every $A \in GIFS_B$ and $\alpha, \beta, \varepsilon, \eta \in [0, 1]$, where $\alpha + \beta \leq 1$, and $\varepsilon + \eta \leq 1$, we have

i. $\boxplus_{\alpha,\beta}A \in GIFS_B$,

ii. $\boxtimes_{\alpha,\beta}A \in GIFS_B$,

iii. $\overline{\boxplus_{\alpha,\beta}\bar{A}} = \boxtimes_{\beta,\alpha}A$,

iv. $\boxplus_{\alpha,\beta} \boxplus_{\varepsilon,\eta} A = \boxplus_{\alpha\varepsilon,\alpha\eta+\beta}A$,

v. $\boxplus_{\alpha,\beta}(A \cap B) = \boxplus_{\alpha,\beta}A \cap \boxplus_{\alpha,\beta}B$,

vi. $\boxplus_{\alpha,\beta}(A \cup B) = \boxplus_{\alpha,\beta}A \cup \boxplus_{\alpha,\beta}B$,

vii. $\boxtimes_{\alpha,\beta}(A \cap B) = \boxtimes_{\alpha,\beta}A \cap \boxtimes_{\alpha,\beta}B$,

viii. $\boxtimes_{\alpha,\beta}(A \cup B) = \boxtimes_{\alpha,\beta}A \cup \boxtimes_{\alpha,\beta}B$,

ix. $\boxplus_{\alpha,\beta}P_{\varepsilon,\eta}(A) = P_{\alpha\varepsilon,\alpha\eta+\beta}(\boxplus_{\alpha,\beta}A)$, if $\alpha\varepsilon + \alpha\eta + \beta \leq 1$,

x. $\boxplus_{\alpha,\beta}Q_{\varepsilon,\eta}(A) = Q_{\alpha\varepsilon,\alpha\eta+\beta}(\boxplus_{\alpha,\beta}A)$, if $\alpha\varepsilon + \alpha\eta + \beta \leq 1$,

xi. $\boxtimes_{\alpha,\beta}P_{\varepsilon,\eta}(A) = P_{\alpha\varepsilon,\alpha\eta+\beta}(\boxtimes_{\alpha,\beta}A)$, if $\alpha\varepsilon + \alpha\eta + \beta \leq 1$,

xii. $\boxtimes_{\alpha,\beta}Q_{\varepsilon,\eta}(A) = Q_{\alpha\varepsilon,\alpha\eta+\beta}(\boxtimes_{\alpha,\beta}A)$, if $\alpha\varepsilon + \alpha\eta + \beta \leq 1$.

Proposition 3.12. For every $A \in GIFS_B$ and $\alpha, \beta, \varepsilon, \eta \in [0, 1]$, where $\varepsilon + \eta \leq 1$, we have

i. $\boxplus_{\alpha}A \in GIFS_B$,

ii. $\boxtimes_{\alpha}A \in GIFS_B$,

iii. $\overline{\boxplus_\alpha \overline{A}} = \boxtimes_{1-\alpha} A$,

iv. $\boxplus_\alpha \boxplus_\varepsilon A = \boxplus_{1-\alpha\varepsilon} A$,

v. $\boxplus_\alpha (A \cap B) = \boxplus_\alpha A \cap \boxplus_\alpha B$,

vi. $\boxplus_\alpha (A \cup B) = \boxplus_\alpha A \cup \boxplus_\alpha B$,

vii. $\boxtimes_\alpha (A \cap B) = \boxtimes_\alpha A \cap \boxtimes_\alpha B$,

viii. $\boxtimes_\alpha (A \cup B) = \boxtimes_\alpha A \cup \boxtimes_\alpha B$,

ix. $\boxplus_\alpha P_{\varepsilon,\eta}(A) = P_{\alpha\varepsilon,\alpha\eta+1-\alpha}(\boxplus_\alpha A)$,

x. $\boxplus_\alpha Q_{\varepsilon,\eta}(A) = Q_{\alpha\varepsilon,\alpha\eta+1-\alpha}(\boxplus_\alpha A)$,

xi. $\boxtimes_\alpha P_{\varepsilon,\eta}(A) = P_{\alpha\varepsilon,\alpha\eta+1-\alpha}(\boxtimes_\alpha A)$,

xii. $\boxtimes_\alpha Q_{\varepsilon,\eta}(A) = Q_{\alpha\varepsilon,\alpha\eta+1-\alpha}(\boxtimes_\alpha A)$.

Definition 3.13. Let $\alpha, \beta, \gamma, \eta \in [0, 1]$ and $A \in GIFS_B$, we define the third extension modal-like operator as follows

$$\boxdot_{\alpha,\beta,\gamma,\eta} A = \left\{ \langle x, (\alpha\mu_A(x)^\delta + \gamma)^{\frac{1}{\delta}}, (\beta\nu_A(x)^\delta + \eta)^{\frac{1}{\delta}} \rangle : x \in X \right\}, \max(\alpha, \beta) + \gamma + \eta \leq 1.$$

Theorem 3.14. *For every $A \in GIFS_B$ and $\alpha, \beta, \gamma, \eta, \varepsilon, \theta, \lambda, \tau \in [0, 1]$, $\max(\alpha, \beta) + \gamma + \eta \leq 1$, $\max(\varepsilon, \theta) + \lambda + \tau \leq 1$ and $\varepsilon + \theta \leq 1$, we have*

i. $\boxdot_{\alpha,\beta,\gamma,\eta} A \in GFIS_B$,

ii. $\boxdot_{\alpha,\beta,0,\eta} A = \boxplus_{\alpha,\beta,\gamma} A$,

iii. $\boxdot_{\alpha,\beta,\gamma,0} A = \boxtimes_{\alpha,\beta,\gamma} A$,

iv. $\overline{\boxdot_{\alpha,\beta,\gamma,\eta} \overline{A}} = \boxdot_{\beta,\alpha,\eta,\gamma} A$,

v. $\boxdot_{\alpha,\beta,\gamma,\eta} \boxdot_{\varepsilon,\theta,\lambda,\tau} A = \boxdot_{\alpha\varepsilon,\beta\theta,\alpha\lambda+\gamma,\beta\tau+\eta} A$,

vi. $\boxdot_{\alpha,\beta,\gamma,\eta}(A \cap B) = \boxdot_{\alpha,\beta,\gamma,\eta} A \cap \boxdot_{\alpha,\beta,\gamma,\eta} B$,

vii. $\boxdot_{\alpha,\beta,\gamma,\eta}(A \cup B) = \boxdot_{\alpha,\beta,\gamma,\eta} A \cup \boxdot_{\alpha,\beta,\gamma,\eta} B$,

viii. $\boxdot_{\alpha,\beta,\gamma,\eta} P_{\varepsilon,\theta}(A) = P_{\alpha\varepsilon+\gamma,\beta\theta+\eta}(\boxdot_{\alpha,\beta,\gamma,\eta} A)$,

ix. $\boxdot_{\alpha,\beta,\gamma,\eta} Q_{\varepsilon,\theta}(A) = Q_{\alpha\varepsilon+\gamma,\beta\theta+\eta}(\boxdot_{\alpha,\beta,\gamma,\eta} A)$,

Proof . (i)

$$\begin{aligned}
&\mu_{\boxdot_{\alpha,\beta,\gamma,\eta} A}(x)^\delta + \nu_{\boxdot_{\alpha,\beta,\gamma,\eta} A}(x)^\delta = ((\alpha\mu_A(x)^\delta + \gamma)^{\frac{1}{\delta}})^\delta + ((\beta\nu_A(x)^\delta + \eta)^{\frac{1}{\delta}})^\delta, \\
&= \alpha\mu_A(x)^\delta + \gamma + \beta\nu_A(x)^\delta + \eta \leq \max(\alpha, \beta)\mu_A(x)^\delta + \max(\alpha, \beta)\nu_A(x)^\delta + \gamma + \eta, \\
&= \max(\alpha, \beta)(\mu_A(x)^\delta + \nu_A(x)^\delta) + \gamma + \eta \leq \max(\alpha, \beta) + \gamma + \eta \leq 1.
\end{aligned}$$

The proof is completed. The Proofs of (ii) and (iii) are obvious.

(iv) Since $\bar{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$ we have

$$\boxdot_{\alpha,\beta,\gamma,\eta}\bar{A} = \left\{\langle x, (\alpha\nu_A(x)^\delta + \gamma)^{\frac{1}{\delta}}, (\beta\mu_A(x)^\delta + \eta)^{\frac{1}{\delta}}\rangle : x \in X\right\},$$

then

$$\overline{\boxdot_{\alpha,\beta,\gamma,\eta}\bar{A}} = \left\{\langle x, (\beta\mu_A(x)^\delta + \eta)^{\frac{1}{\delta}}, (\alpha\nu_A(x)^\delta + \gamma)^{\frac{1}{\delta}}, \rangle : x \in X\right\} = \boxdot_{\beta,\alpha,\eta,\gamma}A.$$

The proof is completed.

(v) Since

$$\boxdot_{\alpha,\beta,\gamma,\eta} A = \left\{\langle x, (\alpha\mu_A(x)^\delta + \gamma)^{\frac{1}{\delta}}, (\beta\nu_A(x)^\delta + \eta)^{\frac{1}{\delta}}\rangle : x \in X\right\},$$
$$\boxdot_{\varepsilon,\theta,\lambda,\tau} A = \left\{\langle x, (\varepsilon\mu_A(x)^\delta + \lambda)^{\frac{1}{\delta}}, (\theta\nu_A(x)^\delta + \tau)^{\frac{1}{\delta}}\rangle : x \in X\right\},$$

hence

$$\begin{aligned}\boxdot_{\alpha,\beta,\gamma,\eta} \boxdot_{\alpha\varepsilon,\theta,\lambda,\tau} A &= \left\{\langle x, (\varepsilon\mu_A(x)^\delta + \alpha\lambda + \gamma)^{\frac{1}{\delta}}, (\beta\theta\nu_A(x)^\delta + \beta\tau + \eta)^{\frac{1}{\delta}}\rangle : x \in X\right\},\\ &= \boxdot_{\alpha\varepsilon,\beta\theta,\alpha\lambda+\gamma,\beta\tau+\eta}A\end{aligned}$$

The proof is completed.

(vi) Since $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_A(x))\rangle : x \in X\}$, we have

$$\begin{aligned}\boxdot_{\alpha,\beta,\gamma,\eta}(A \cap B) &= \left\{\langle x, (\alpha \min(\mu_A(x), \mu_B(x))^\delta + \gamma)^{\frac{1}{\delta}}, (\beta(\max(\nu_A(x), \nu_B(x)))^\delta + \eta)^{\frac{1}{\delta}}\rangle : x \in X\right\},\\ &= \Big\{\langle x, \min((\alpha\mu_A(x)^\delta + \gamma)^{\frac{1}{\delta}}, (\alpha\mu_B(x)^\delta + \gamma)^{\frac{1}{\delta}}), \max((\beta\nu_A(x)^\delta + \eta)^{\frac{1}{\delta}}, (\beta\nu_B(x)^\delta\\ &\quad + \gamma)^{\frac{1}{\delta}}\rangle : x \in X\Big\},\\ &= \boxdot_{\alpha,\beta,\gamma,\eta}A \cap \boxdot_{\alpha,\beta,\gamma,\eta}B,\end{aligned}$$

The proof is completed. The Proof of (vii) is similar to that of (vi)

(viii) Since

$$P_{\varepsilon,\theta}(A) = \left\{\langle x, \max(\varepsilon^{\frac{1}{\delta}}, \mu_A(x)), \min(\theta^{\frac{1}{\delta}}, \nu_A(x))\rangle : x \in X\right\},$$

we have

$$\begin{aligned}\boxdot_{\alpha,\beta,\gamma,\eta}P_{\varepsilon,\theta}(A) &= \left\{\langle x, (\alpha \max\left(\varepsilon^{\frac{1}{\delta}}, \mu_A(x)\right)^\delta + \gamma)^{\frac{1}{\delta}}, (\beta \min(\theta^{\frac{1}{\delta}}, \nu_A(x))^\delta + \eta)^{\frac{1}{\delta}}\rangle : x \in X\right\},\\ &= \left\{\langle x, \max(\alpha\varepsilon + \gamma, \alpha\mu_A(x)^\delta + \gamma)^{\frac{1}{\delta}}, (\min(\beta\theta + \eta, \beta\nu_A(x)^\delta + \eta)^{\frac{1}{\delta}}\rangle : x \in X\right\},\\ &= P_{\alpha\varepsilon+\gamma,\beta\theta+\eta} \boxdot_{\alpha,\beta,\gamma,\eta} A.\end{aligned}$$

The proof is completed. The Proof of (ix) is similar to that of (viii). □

4. Conclusions

We have introduced level operators and modal-like operators over $GIFS_B$ and proved their relationships. An open problem is as follows: definition of negation operator and other types over $GIFS_B$ and study their properties.

References

[1] K.T. Atanassov, *Intuitionistic fuzzy sets*, Springer Physica-Verlag, Heidelberg, 1999.

[2] K.T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets Syst. 20 (1986) 87–96.

[3] K.T. Atanassov, *Remark on two operations over intuitionistic fuzzy sets*, Int. J. of Uncertainty, Fuzziness Knowledge Syst. 9 (2001) 71–75.

[4] K.T. Atanassov, *The most general form of one type of intuitionistic fuzzy modal operators*, NIFS. 12 (2006) 36–38.

[5] K.T. Atanassov, *Some properties of the operators from one type of intuitionistic fuzzy modal operators*, Adv. Studies Contemporary Math. 15 (2007) 13–20.

[6] K.T. Atanassov, *On intuitionistic fuzzy negation and intuitionistic fuzzy level operators*, NIFS. 14 (2008) 15–19.

[7] E. Baloui Jamkhaneh and S. Nadarajah, *A new generalized intuitionistic fuzzy sets*, Hacettepe J. Math. Stat. 44 (2015) 1537–1551.

[8] G. Cuvalcoglu, *Some properties of $E_{\alpha,\beta}$ operator*, Adv. Studies Contemporary Math. 14 (2007) 305–310.

[9] G. Cuvalcoglu., S. Ylmaz and K.T. Atanassov, *Matrix representation of the second type of intuitionistic fuzzy modal operators*, Notes Intuitionistic Fuzzy Set. 20 (2014) 9–16.

[10] K. Dencheva, *Extension of intuitionistic fuzzy modal operators $\boxplus$ and $\boxtimes$*, In Intelligent Systems, 2004. Proceedings. 2004 2nd International IEEE Conference (Vol. 3, pp. 21–22). IEEE.

[11] F.G. Lupianez, *Intuitionistic fuzzy topology operators and topology*, IEEE Symp. Int. J. Pure Appl. Math. 17 (2004) 35–40.

[12] R. Parvathi and S.P. Geetha, *A note on properties of temporal intuitionistic fuzzy sets*, Notes Intuitionistic Fuzzy Set. 15 (2009) 42–48.