Int. J. Nonlinear Anal. Appl. 8 (2017) No. 1, 245-260
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.2504

Strong and Δ-convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces

G.S. Saluja[a], H.K. Nashine[b,*], Y.R. Singh[c]

[a]Department of Mathematics, Govt. Nagarjuna P.G. College of Science, Raipur - 492010 (C.G.), India
[b]Department of Mathematics, Texas A & M University - Kingsville - 78363-8202, Texas, USA
[c]National Institute of Technology Manipur, Takyelpat, Imphal-795001, Manipur, India

(Communicated by M. Eshaghi)

Abstract

In this work we use the Noor iteration process for total asymptotically nonexpansive mapping to establish the strong and Δ-convergence theorems in the framework of CAT(0) spaces. By doing this, some of the results existing in the current literature generalize, unify and extend.

Keywords: total asymptotically nonexpansive mapping; Δ-convergence; strong convergence; Noor iteration process; CAT(0) space.
2010 MSC: Primary 54H25; Secondary 54E40.

1. Introduction

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space.

Kirk [18, 19] was the first who studied fixed point theory in CAT(0) space and showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. It is worth mentioning that the results in CAT(0) spaces can

*Corresponding author
Email addresses: saluja1963@gmail.com (G.S. Saluja), drhknashine@gmail.com (H.K. Nashine), ymnehor2008@yahoo.com (Y.R. Singh)

Received: July 2015 *Revised:* December 2015

be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(m) space for every $m \geq k$ (see [2]).

In [22], authors proved the demiclosedness principle for asymptotically nonexpansive mappings and gave the Δ-convergence theorem of the modified Mann iteration process for above mentioned mappings in a CAT(0) space. In 2010, Niwongsa and Panyanak [23] studied the Noor iteration scheme in CAT(0) spaces and they proved some Δ and strong convergence theorems for asymptotically nonexpansive mappings which extend and improve some recent results from the literature. In 2012, Saluja [27] studied Noor iteration scheme for generalized asymptotically quasi nonexpansive mappings and established some strong convergence theorems in the framework of CAT(0) spaces. Recently, in the paper [4], Chang et al. introducedthe concept of total asymptotically nonexpansive mappings and proved the demiclosed principle for said mapping in a CAT(0) space. In addition, the Δ-convergence theorem of the modified Mann iteration process for total asymptotically nonexpansive mappings in a CAT(0) space has also been studied Chang et al. .

Very recently, Başarir and Şahin [26] studied the modified S-iteration process, modified two-step iteration process and established strong and Δ-convergence theorems for total asymptotically nonexpansive mappings in the framework of CAT(0) spaces (see, also [5], [9]).

Algorithm 1. The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$\begin{aligned} z_n &= \gamma_n T^n x_n \oplus (1-\gamma_n)x_n, \\ y_n &= \beta_n T^n z_n \oplus (1-\beta_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n \oplus (1-\alpha_n)x_n, \quad n \geq 1, \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ are appropriate sequences in [0,1] is called modified Noor iterative sequence (see [30]).

If $\gamma_n = 0$ for all $n \geq 1$, then Algorithm 1 reduces to the following.

Algorithm 2. The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$\begin{aligned} y_n &= \beta_n T^n x_n \oplus (1-\beta_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n \oplus (1-\alpha_n)x_n, \quad n \geq 1, \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are appropriate sequences in [0,1] is called an Ishikawa iterative sequence (see [16]).

If $\beta_n = 0$ for all $n \geq 1$, then Algorithm 2 reduces to the following.

Algorithm 3. The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$x_{n+1} = \alpha_n T^n x_n \oplus (1-\alpha_n)x_n, \quad n \geq 1, \tag{1.3}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in (0,1) is called a Mann iterative sequence (see [21]).

Iteration procedures in fixed point theory are lead by the considerations in summability theory. For example, if a given sequence converges, then we don't look for the convergence of the sequence of its arithmetic means. Similarly, if the sequence of Picard iterates of any mapping T converges, then we don't look for the convergence of other iteration procedures.

The three-step iterative approximation problems were studied extensively by Noor [24, 25], Glowinsky and Le Tallec [12], and Haubruge et al [14]. It has been shown [12] that three-step iterative scheme gives better numerical results than the two step and one step approximate iterations. Thus we conclude that three step scheme plays an important and significant role in solving various problems, which arise in pure and applied sciences.

Motivated by Chang et al. [4], Başarir and Şahin [26] and some others, in this paper, we establish strong and Δ-convergence theorems of Noor iteration process for total asymptotically nonexpansive mappings in the framework of CAT(0) spaces. Our results extend and improve many known results from the current existing literature (see, e.g., [4, 22, 23, 26, 30] and many others).

2. Preliminaries and lemmas

Let (X, d) be a metric space and K be its nonempty subset. Let $T\colon K \to K$ be a mapping. A point $x \in K$ is called a fixed point of T if $Tx = x$ and we denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$.

Definition 2.1. Let (X, d) be a metric space and K be its nonempty subset. Then $T\colon K \to K$ said to be

(1) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$;

(2) asymptotically nonexpansive [10] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $d(T^n x, T^n y) \leq k_n\, d(x, y)$ for all $x, y \in K$ and $n \geq 1$;

(3) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leq L\, d(x, y)$ for all $x, y \in K$ and $n \geq 1$;

(4) semi-compact if for a sequence $\{x_n\}$ in K with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$;

(5) a sequence $\{x_n\}$ in K is called approximate fixed point sequence for T (AFPS in short) if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Chang et al. [4] defined the concept of total asymptotically nonexpansive mapping as follows.

Definition 2.2. ([4] Definition 2.1) Let (X, d) be a metric space, K be its nonempty subset and let $T\colon K \to K$ be a mapping. T is said to be a total asymptotically nonexpansive mapping if there exist non-negative real sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\zeta\colon [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x, y) + \nu_n \zeta(d(x, y)) + \mu_n$$

for all $x, y \in K$ and $n \geq 1$.

Remark 2.3. From the above definitions, it is clear that each nonexpansive mapping is an asymptotically nonexpansive mapping with the constant sequence $\{k_n\} = \{1\}$, $\forall\, n \geq 1$, each asymptotically nonexpansive mapping is a total asymptotically nonexpansive mapping with $\mu_n = 0$, $\nu_n = k_n - 1$ for all $n \geq 1$, $\zeta(t) = t$, $t \geq 0$ and each asymptotically nonexpansive mapping is a uniformly L-Lipschitzian mapping with $L = \sup_{n\geq 1}\{k_n\}$.

We now give the definition and some basic properties of CAT(0) space.

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry, and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say that X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denote by $[x, y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison* *triangle* for the geodesic triangle $\triangle(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1,x_2,x_3) := \triangle(\overline{x_1},\overline{x_2},\overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i},\overline{x_j}) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$. Such a triangle always exists (see [2]).

CAT(0) space. A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x,y \in \triangle$ and all comparison points $\overline{x},\overline{y} \in \overline{\triangle}$,

$$d(x,y) \;\leq\; d_{\mathbb{R}^2}(\overline{x},\overline{y}). \tag{2.1}$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [17]). If x,y_1,y_2 are points of a $CAT(0)$ space and y_0 is the midpoint of the segment $[y_1,y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \;\leq\; \frac{1}{2}\,d^2(x,y_1) + \frac{1}{2}\,d^2(x,y_2) - \frac{1}{4}\,d^2(y_1,y_2). \tag{2.2}$$

Inequality (2.2) is the (CN) inequality of Bruhat and Tits [3]. The above inequality was extended in [8] as

$$\begin{aligned} d^2(z,\alpha x \oplus (1-\alpha)y) \;\leq\; & \alpha d^2(z,x) + (1-\alpha)d^2(z,y) \\ & -\alpha(1-\alpha)d^2(x,y), \end{aligned} \tag{2.3}$$

for any $\alpha \in [0,1]$ and $x,y,z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [2, p.163]). Moreover, if X is a $CAT(0)$ metric space and $x,y \in X$, then for any $\alpha \in [0,1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x,y]$ such that

$$d(z,\alpha x \oplus (1-\alpha)y) \;\leq\; \alpha d(z,x) + (1-\alpha)d(z,y), \tag{2.4}$$

for any $z \in X$ and $[x,y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0,1]\}$.

A subset K of a $CAT(0)$ space X is convex if for any $x,y \in K$, we have $[x,y] \subset K$.

In the sequel, we need the following definitions and useful lemmas to prove our main results.

Lemma 2.4. (See [23]) Let X be a CAT(0) space.

(i) For $x,\, y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,\, y]$ such that

$$d(x,\, z) = t\, d(x,\, y) \quad \text{and} \quad d(y,\, z) = (1-t)\, d(x,\, y). \quad (A)$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (A).

(ii) For $x,\, y \in X$ and $t \in [0,1]$, we have

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

Let $\{x_n\}$ be a bounded sequence in a closed convex subset K of a CAT(0) space X. For $x \in X$, set

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \big\{x \in X : r(\{x_n\}) = r(x,\{x_n\})\big\}.$$

It is known that, in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point [[6], Proposition 7].

We now recall the definition of Δ-convergence and weak convergence ($\rightharpoonup$) in CAT(0) space.

Definition 2.5. ([20]) A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{x_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

In this case we write $\Delta - \lim_n x_n = x$ and call x is the Δ-limit of $\{x_n\}$.

Recall that a bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In the Banach space it is known that, every bounded sequence has a regular subsequence [[11], Lemma 15.2].

Since in a CAT(0) space every regular sequence Δ-converges, we see that every bounded sequence in X has a Δ-convergent subsequence, also it is noticed that [[20], p.3690].

Lemma 2.6. (See [1]) Given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$, then

$$\limsup_n d(x_n, x) < \limsup_n d(x_n, y).$$

In a Banach space the above condition is known as the Opial property.

Now, recall the definition of weak convergence in a CAT(0) space.

Definition 2.7. (See [15]) Let K be a closed convex subset of a CAT(0) space X. A bounded sequence $\{x_n\}$ in K is said to converge weakly to $q \in K$ if and only if $\Phi(q) = \inf_{x \in K} \Phi(x)$, where $\Phi(x) = \limsup_{n\to\infty} d(x_n, x)$.

Note that $\{x_n\} \rightharpoonup q$ if and only if $A_K\{x_n\} = \{q\}$.

Nanjaras and Panyanak [22] established the following relation between Δ-convergence and weak convergence in a CAT(0) space.

Lemma 2.8. (See [22], Proposition 3.12) Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X and let K be a closed convex subset of X which contains $\{x_n\}$. Then

(i) $\Delta\text{-}\lim_{x_n} = x$ implies $x_n \rightharpoonup x$.

(ii) The converse of (i) is true if $\{x_n\}$ is regular.

Lemma 2.9. (See [8], Lemma 2.8) If $\{x_n\}$ is a bounded sequence in a CAT(0) space X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.

Lemma 2.10. (See [7], Proposition 2.1) If K is a closed convex subset of a CAT(0) space X and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.

Lemma 2.11. (See [4], Theorem 3.8) Let K be closed convex subset of a complete CAT(0) space X and let $T\colon K \to K$ be a total asymptotically nonexpansive and uniformly L-Lipschitzian mapping. Let $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = p$. Then $Tp = p$.

Lemma 2.12. (See [29]) Suppose that $\{a_n\}$, $\{b_n\}$ and $\{r_n\}$ are sequences of nonnegative numbers such that $a_{n+1} \leq (1 + b_n)a_n + r_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

3. Main Results

Now, we prove the following lemma using modified Noor iteration scheme (1.1) needed in the sequel.

Lemma 3.1. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon K \to K$ be a total asymptotically nonexpansive and uniformly L-Lipschitzian mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (1.1). If the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;

(ii) there exists a constant $M_1 > 0$ such that $\zeta(r) \leq M_1 r$, $r \geq 0$.

Then $\lim_{n\to\infty} d(x_n, p)$ and $\lim_{n\to\infty} d(x_n, F(T))$ exist for all $p \in F(T)$.

Proof . Let $p \in F(T)$. From (1.1) and Lemma 2.4(ii), we have

$$\begin{aligned} d(z_n, p) &= d(\gamma_n T^n x_n \oplus (1-\gamma_n)x_n, p) \\ &\leq \gamma_n\, d(T^n x_n, p) + (1-\gamma_n)d(x_n, p) \\ &\leq \gamma_n[d(x_n, p) + \nu_n(d(x_n, p)) + \mu_n] + (1-\gamma_n)d(x_n, p) \\ &\leq \gamma_n[d(x_n, p) + \nu_n M_1 d(x_n, p) + \mu_n] + (1-\gamma_n)d(x_n, p) \\ &= \gamma_n(1+M_1\nu_n)d(x_n, p) + \gamma_n\mu_n + (1-\gamma_n)d(x_n, p) \\ &\leq \gamma_n(1+M_1\nu_n)d(x_n, p) + \gamma_n\mu_n \\ &\quad +(1-\gamma_n)(1+M_1\nu_n)d(x_n, p) \\ &\leq (1+M_1\nu_n)d(x_n, p) + \mu_n. \end{aligned} \tag{3.1}$$

Again using (1.1), (3.1) and Lemma 2.4(ii), we have

$$\begin{aligned} d(y_n, p) &= d(\beta_n T^n z_n \oplus (1-\beta_n)x_n, p) \\ &\leq \beta_n\, d(T^n z_n, p) + (1-\beta_n)d(x_n, p) \\ &\leq \beta_n[d(z_n, p) + \nu_n(d(z_n, p)) + \mu_n] + (1-\beta_n)d(x_n, p) \\ &\leq \beta_n[d(z_n, p) + \nu_n M_1 d(z_n, p) + \mu_n] + (1-\beta_n)d(x_n, p) \\ &= \beta_n(1+M_1\nu_n)d(z_n, p) + \beta_n\mu_n + (1-\beta_n)d(x_n, p) \\ &\leq \beta_n(1+M_1\nu_n)[(1+M_1\nu_n)d(x_n, p) + \mu_n] + \beta_n\mu_n \\ &\quad +(1-\beta_n)d(x_n, p) \\ &= \beta_n(1+M_1\nu_n)^2 d(x_n, p) + \beta_n(1+M_1\nu_n)\mu_n + \beta_n\mu_n \\ &\quad +(1-\beta_n)d(x_n, p) \\ &\leq \beta_n(1+M_1\nu_n)^2 d(x_n, p) + \beta_n(1+M_1\nu_n)\mu_n + \beta_n\mu_n \\ &\quad +(1-\beta_n)(1+M_1\nu_n)^2 d(x_n, p) \\ &\leq (1+M_1\nu_n)^2 d(x_n, p) + (2+M_1\nu_n)\mu_n. \end{aligned} \tag{3.2}$$

Now using (1.1), (3.2) and Lemma 2.4(ii), we get

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n T^n y_n \oplus (1-\alpha_n)x_n, p) \\ &\leq \alpha_n\, d(T^n y_n, p) + (1-\alpha_n)d(x_n, p) \\ &\leq \alpha_n[d(y_n, p) + \nu_n(d(y_n, p)) + \mu_n] + (1-\alpha_n)d(x_n, p) \end{aligned}$$

and so

$$\begin{aligned} d(x_{n+1},p) &\leq \alpha_n[d(y_n,p)+\nu_n M_1 d(y_n,p)+\mu_n]+(1-\alpha_n)d(x_n,p) \\ &= \alpha_n(1+M_1\nu_n)d(y_n,p)+\alpha_n\mu_n+(1-\alpha_n)d(x_n,p) \\ &\leq \alpha_n(1+M_1\nu_n)[(1+M_1\nu_n)^2 d(x_n,p)+(2+M_1\nu_n)\mu_n] \\ &\quad +\alpha_n\mu_n+(1-\alpha_n)d(x_n,p) \\ &\leq \alpha_n(1+M_1\nu_n)^3 d(x_n,p)+\alpha_n(1+M_1\nu_n)(2+M_1\nu_n)\mu_n+\alpha_n\mu_n \\ &\quad +(1-\alpha_n)(1+M_1\nu_n)^3 d(x_n,p) \\ &\leq (1+M_1\nu_n)^3 d(x_n,p)+(3+M_1\nu_n)\mu_n \\ &= (1+h_n)d(x_n,p)+t_n, \end{aligned} \tag{3.3}$$

where

$$h_n = 3M_1\nu_n + 3M_1^2\nu_n^2 + M_1^3\nu_n^3$$

and $t_n = (3+M_1\nu_n)\mu_n$. Since by assumption of the theorem $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\sum_{n=1}^{\infty}\nu_n < \infty$, it follows that $\sum_{n=1}^{\infty} h_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. Equation (3.3) implies that

$$d(x_{n+1},F(T)) \leq (1+h_n)d(x_n,F(T))+t_n. \tag{3.4}$$

Hence from Lemma 2.12, (3.3) and (3.4), we get $\lim_{n\to\infty} d(x_n,p)$ and $\lim_{n\to\infty}$ $d(x_n,F(T))$ exist. □

Following the paper [13] and cited therein, in the following example, the considered mappings are total asymptotically nonexpansive and uniformly L-Lipschitzian which required for our Lemma 3.1.

Example 3.2. Let $\mathbb{R}$ be the real line with the usual norm $\|.\|$ and $K = [-1,1]$. Define two mappings $T, S\colon K \to K$ by

$$T(x) = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0,1], \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1,0) \end{cases}$$

and

$$S(x) = \begin{cases} x, & \text{if } x \in [0,1], \\ -x, & \text{if } x \in [-1,0). \end{cases}$$

In addition, $F(T) = \{0\}$ and $F(S) = \{x \in K : 0 \leq x \leq 1\}$.

Theorem 3.3. *Let X, K, T, $\{x_n\}$ satisfy the hypothesis of Lemma 3.1. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if*

$$\liminf_{n\to\infty} d(x_n,F(T)) = 0,$$

where

$$d(x,F(T)) = \inf\{d(x,p) : p \in F(T)\}.$$

Proof . Necessity is obvious. Conversely, suppose that

$$\liminf_{n\to\infty} d(x_n,F(T)) = 0$$

As proved in Lemma 3.1, for all $p \in F(T)$,

$$\lim_{n\to\infty} d(x_n,F(T))$$

exists. Thus by hypothesis,

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in K. With the help of inequality $1 + x \leq e^x$, $x \geq 0$. For any integer $m \geq 1$, therefore from (3.2), we have

$$\begin{aligned} d(x_{n+m}, p) &\leq (1 + h_{n+m-1})d(x_{n+m-1}, p) + t_{n+m-1} \\ &\leq e^{h_{n+m-1}} d(x_{n+m-1}, p) + t_{n+m-1} \\ &\leq e^{h_{n+m-1}}[e^{h_{n+m-2}} d(x_{n+m-2}, p) + t_{n+m-2}] + t_{n+m-1} \\ &\leq e^{(h_{n+m-1}+h_{n+m-2})} d(x_{n+m-2}, p) + e^{h_{n+m-1}}[t_{n+m-2} + t_{n+m-1}] \\ &\leq \dots \\ &\leq \big(e^{\sum_{k=n}^{n+m-1} h_k}\big) d(x_n, p) + \big(e^{\sum_{k=n}^{n+m-1} h_k}\big) \sum_{k=n}^{n+m-1} t_k \\ &\leq \big(e^{\sum_{n=1}^{\infty} h_n}\big) d(x_n, p) + \big(e^{\sum_{n=1}^{\infty} h_n}\big) \sum_{k=n}^{n+m-1} t_k \\ &= R\, d(x_n, p) + R \sum_{k=n}^{n+m-1} t_k, \end{aligned} \tag{3.5}$$

where $R = e^{\sum_{n=1}^{\infty} h_n}$.

Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, therefore for any $\varepsilon > 0$, there exists a natural number n_0 such that $d(x_n, F(T)) < \varepsilon/8R$ and $\sum_{k=n}^{n+m-1} t_k < \varepsilon/2R$ So, we can find $p^* \in F(T)$ such that $d(x_{n_0}, p^*) < \varepsilon/4R$. Hence, for all $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(x_n, p^*) \\ &\leq R\, d(x_{n_0}, p^*) + R \sum_{k=n}^{n+m-1} t_k \\ &\quad + R\, d(x_{n_0}, p^*) \\ &= 2R\, d(x_{n_0}, p^*) + R \sum_{k=n}^{n+m-1} t_k \\ &< 2R.\frac{\varepsilon}{4R} + R.\frac{\varepsilon}{2R} = \varepsilon. \end{aligned} \tag{3.6}$$

This proves that $\{x_n\}$ is a Cauchy sequence in K. Thus, the completeness of X implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = q$. Since K is closed, therefore $q \in K$. Next, we show that $q \in F(T)$. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ we get $d(q, F(T)) = 0$, closedness of $F(T)$ gives that $q \in F(T)$. Thus $\{x_n\}$ converges strongly to a point in $F(T)$. This completes the proof. □

Lemma 3.4. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $T\colon K \to K$ be a total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (1.1). *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$. *If the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;

(ii) there exists a constant $M_1 > 0$ such that $\zeta(r) \leq M_1 r$, $r \geq 0$.

Then $\lim_{n\to\infty} d(T^n y_n, x_n) = 0$ *and* $\lim_{n\to\infty} d(T^n z_n, x_n) = 0$.

Proof . Let $p \in F(T)$. Then by Lemma 3.1, we have $\lim_{n\to\infty} d(x_n, p)$ exists, so we can assume that $\lim_{n\to\infty} d(x_n, p) = a$, where $a > 0$. We claim that

$$\lim_{n\to\infty} d(T^n y_n, x_n) = 0, \quad \lim_{n\to\infty} d(T^n z_n, x_n) = 0.$$

From (1.1) and (2.3), we have

$$\begin{aligned}
d^2(z_n, p) &= d^2(\gamma_n T^n x_n \oplus (1-\gamma_n)x_n, p) \\
&\leq \gamma_n d^2(T^n x_n, p) + (1-\gamma_n) d^2(x_n, p) \\
&\quad -\gamma_n(1-\gamma_n) d(T^n x_n, x_n) \\
&\leq \gamma_n [d(x_n, p) + \nu_n(d(x_n, p)) + \mu_n]^2 + (1-\gamma_n) d^2(x_n, p) \\
&\quad -\gamma_n(1-\gamma_n) d(T^n x_n, x_n) \\
&\leq \gamma_n [d(x_n, p) + \nu_n M_1 d(x_n, p) + \mu_n]^2 + (1-\gamma_n) d^2(x_n, p) \\
&\quad -\gamma_n(1-\gamma_n) d(T^n x_n, x_n) \\
&= \gamma_n [(1+\nu_n M_1) d(x_n, p) + \mu_n]^2 + (1-\gamma_n) d^2(x_n, p) \\
&\quad -\gamma_n(1-\gamma_n) d(T^n x_n, x_n) \\
&\leq (1+\nu_n M_1)^2 \gamma_n d^2(x_n, p) + (1+\nu_n M_1)^2 (1-\gamma_n) d^2(x_n, p) \\
&\quad +\gamma_n [2(1+\nu_n M_1)\mu_n d(x_n, p) + \mu_n^2] - \gamma_n(1-\gamma_n) d(T^n x_n, x_n) \\
&\leq (1+\nu_n M_1)^2 d^2(x_n, p) + \gamma_n [2(1+\nu_n M_1)\mu_n d(x_n, p) + \mu_n^2].
\end{aligned}$$

This implies that

$$d^2(z_n, p) \leq d^2(x_n, p) + P\nu_n + Q\mu_n, \tag{3.7}$$

for some P, $Q > 0$.

Again from (1.1) and (2.3), we have

$$\begin{aligned}
d^2(y_n, p) &= d^2(\beta_n T^n z_n \oplus (1-\beta_n)x_n, p) \\
&\leq \beta_n d^2(T^n z_n, p) + (1-\beta_n) d^2(x_n, p) \\
&\quad -\beta_n(1-\beta_n) d^2(T^n z_n, x_n) \\
&\leq \beta_n [d(z_n, p) + \nu_n(d(z_n, p)) + \mu_n]^2 + (1-\beta_n) d^2(x_n, p) \\
&\quad -\beta_n(1-\beta_n) d^2(T^n z_n, x_n) \\
&\leq \beta_n [d(z_n, p) + \nu_n M_1 d(z_n, p) + \mu_n]^2 + (1-\beta_n) d^2(x_n, p) \\
&\quad -\beta_n(1-\beta_n) d^2(T^n z_n, x_n) \\
&= \beta_n [(1+\nu_n M_1) d(z_n, p) + \mu_n]^2 + (1-\beta_n) d^2(x_n, p) \\
&\quad -\beta_n(1-\beta_n) d^2(T^n z_n, x_n) \\
&\leq \beta_n (1+\nu_n M_1)^2 d^2(z_n, p) + (1-\beta_n)(1+\nu_n M_1)^2 d^2(x_n, p) \\
&\quad +\beta_n [2\mu_n(1+\nu_n M_1) d(z_n, p) + \mu_n^2] - \beta_n(1-\beta_n) d^2(T^n z_n, x_n).
\end{aligned} \tag{3.8}$$

Substituting (3.7) into (3.8), we have

$$\begin{aligned}
d^2(y_n, p) &\leq \beta_n (1+\nu_n M_1)^2 [d^2(x_n, p) + P\nu_n + Q\mu_n] + (1-\beta_n)(1+\nu_n M_1)^2 d^2(x_n, p) \\
&\quad +\beta_n [2\mu_n(1+\nu_n M_1) d(z_n, p) + \mu_n^2] - \beta_n(1-\beta_n) d^2(T^n z_n, x_n) \\
&\leq d^2(x_n, p) + L\nu_n + M\mu_n - \beta_n(1-\beta_n) d^2(T^n z_n, x_n).
\end{aligned} \tag{3.9}$$

for some L, $M > 0$. This implies that

$$d^2(y_n, p) \leq d^2(x_n, p) + L\nu_n + M\mu_n. \tag{3.10}$$

Finally, from (1.1) and (2.3), we have

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(\alpha_n T^n y_n \oplus (1-\alpha_n)x_n, p) \\ &\leq \alpha_n d^2(T^n y_n, p) + (1-\alpha_n)d^2(x_n, p) \\ &\quad -\alpha_n(1-\alpha_n)d^2(T^n y_n, x_n) \\ &\leq \alpha_n[d(y_n, p) + \nu_n(d(y_n, p)) + \mu_n]^2 + (1-\alpha_n)d^2(x_n, p) \\ &\quad -\alpha_n(1-\alpha_n)d^2(T^n y_n, x_n) \\ &\leq \alpha_n[d(y_n, p) + \nu_n M_1 d(y_n, p) + \mu_n]^2 + (1-\alpha_n)d^2(x_n, p) \\ &\quad -\alpha_n(1-\alpha_n)d^2(T^n y_n, x_n) \\ &= \alpha_n[(1+\nu_n M_1)d(y_n, p) + \mu_n]^2 + (1-\alpha_n)d^2(x_n, p) \\ &\quad -\alpha_n(1-\alpha_n)d^2(T^n y_n, x_n) \\ &\leq \alpha_n(1+\nu_n M_1)^2 d^2(y_n, p) + (1+\nu_n M_1)^2(1-\alpha_n)d^2(x_n, p) \\ &\quad +\alpha_n[2\mu_n(1+\nu_n M_1)d(y_n, p) + \mu_n^2] \\ &\quad -\alpha_n(1-\alpha_n)d^2(T^n y_n, x_n). \end{aligned} \tag{3.11}$$

Substituting (3.10) into (3.11), we have

$$\begin{aligned} d^2(x_{n+1}, p) &\leq \alpha_n(1+\nu_n M_1)^2[d^2(x_n, p) + L\nu_n + M\mu_n] \\ &\quad +(1+\nu_n M_1)^2(1-\alpha_n)d^2(x_n, p) \\ &\quad +\alpha_n[2\mu_n(1+\nu_n M_1)d(y_n, p) + \mu_n^2] \\ &\quad -\alpha_n(1-\alpha_n)d^2(T^n y_n, x_n) \\ &\leq d^2(x_n, p) + R\nu_n + T\mu_n \\ &\quad -\alpha_n(1-\alpha_n)d^2(T^n y_n, x_n), \end{aligned} \tag{3.12}$$

for some R, $T > 0$.

Equation (3.12) yielding

$$\alpha_n(1-\alpha_n)d^2(T^n y_n, x_n) \leq d^2(x_n, p) - d^2(x_{n+1}, p) + R\nu_n + T\mu_n.$$

Since $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, we have

$$\alpha_n(1-\alpha_n)d^2(T^n y_n, x_n) < \infty.$$

This implies by $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ that

$$\lim_{n\to\infty} d(T^n y_n, x_n) = 0.$$

Now, we have

$$\begin{aligned} d(x_n, p) &\leq d(x_n, T^n y_n) + d(T^n y_n, p) \\ &\leq d(x_n, T^n y_n) + d(y_n, p) + \nu_n(d(y_n, p)) + \mu_n \\ &\leq d(x_n, T^n y_n) + d(y_n, p) + \nu_n M_1 d(y_n, p) + \mu_n \\ &= d(x_n, T^n y_n) + (1+\nu_n M_1)d(y_n, p) + \mu_n \end{aligned}$$

from which we deduce that

$$a \leq \liminf_{n\to\infty} d(y_n, p).$$

On the other hand, from (3.2), we have

$$\limsup_{n\to\infty} d(y_n, p) \leq a.$$

Therefore $\lim_{n\to\infty} d(y_n, p) = a$.

Now, consider (3.9), we have

$$d^2(y_n, p) \leq d^2(x_n, p) + L\nu_n + M\mu_n - \beta_n(1-\beta_n)d^2(T^n z_n, x_n). \tag{3.13}$$

Equation (3.13) yielding

$$\beta_n(1-\beta_n)d^2(T^n z_n, x_n) \leq d^2(x_n, p) - d^2(y_n, p) + L\nu_n + M\mu_n.$$

Since $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, we have

$$\beta_n(1-\beta_n)d^2(T^n z_n, x_n) < \infty.$$

This implies by $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ that

$$\lim_{n\to\infty} d(T^n z_n, x_n) = 0.$$

This completes the proof. □

Lemma 3.5. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $T\colon K \to K$ be a uniformly continuous and total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (1.1)*. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in* $(0,1)$ *such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. If the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;

(ii) there exists a constant $M_1 > 0$ such that $\zeta(r) \leq M_1 r$, $r \geq 0$.

Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof . From Lemma 3.4, we have $\lim_{n\to\infty} d(T^n y_n, x_n) = 0$ and $\lim_{n\to\infty} d(T^n z_n, x_n) = 0$.

Now, note that

$$d(x_n, y_n) \leq \beta_n d(x_n, T^n z_n) \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

By the uniform continuity of T, we have

$$\lim_{n\to\infty} d(T^n x_n, T^n y_n) = 0. \tag{3.15}$$

Thus

$$d(T^n x_n, x_n) \leq d(T^n x_n, T^n y_n) + d(T^n y_n, x_n). \tag{3.16}$$

Using (3.15) and $\lim_{n\to\infty} d(T^n y_n, x_n) = 0$ in (3.16), we obtain

$$\lim_{n\to\infty} d(T^n x_n, x_n) = 0. \tag{3.17}$$

Now by the definitions of x_{n+1} and y_n, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, T^n y_n) \\ &\leq d(x_n, T^n x_n) + d(T^n x_n, T^n y_n) \\ &\leq d(x_n, T^n x_n) + d(x_n, y_n) + \nu_n(d(x_n, y_n)) + \mu_n \\ &\leq d(x_n, T^n x_n) + d(x_n, y_n) + \nu_n M_1 d(x_n, y_n) + \mu_n \\ &\leq d(x_n, T^n x_n) + (1 + \nu_n M_1) d(x_n, y_n) + \mu_n \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.18}$$

By (3.17), (3.18) and uniform continuity of T, we have

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) \\ &\quad + d(T^{n+1} x_{n+1}, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) \\ &\quad + d(x_{n+1}, x_n) + +\nu_{n+1}(d(x_{n+1}, x_n)) + \mu_{n+1} \\ &\quad + d(T^{n+1} x_n, Tx_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) \\ &\quad + d(x_{n+1}, x_n) + \nu_{n+1} M_1 d(x_{n+1}, x_n) + \mu_{n+1} \\ &\quad + d(T^{n+1} x_n, Tx_n) \\ &= (2 + \nu_{n+1} M_1) d(x_{n+1}, x_n) + d(x_{n+1}, T^{n+1} x_{n+1}) \\ &\quad + d(T^{n+1} x_n, Tx_n) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.19}$$

This completes the proof. □

Now, we are in a position to prove the Δ-convergence and strong convergence results.

Theorem 3.6. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon K \to K$ be a uniformly continuous and total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (1.1). *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in* $(0,1)$ *such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. *If the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;

(ii) there exists a constant $M_1 > 0$ such that $\zeta(r) \leq M_1 r$, $r \geq 0$.

Then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

Proof . Let $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We can complete the proof by showing that $\omega_w(x_n) \subseteq F(T)$ and $\omega_w(x_n)$ consists of exactly one point. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.10, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in K$. Hence $v \in F(T)$ by Lemma 2.11. Since by Lemma 3.1, $\lim_{n\to\infty} d(x_n, v)$ exists, so by Lemma 2.9, $v = u$, i.e., $\omega_w(x_n) \subseteq F(T)$.

To show that $\{x_n\}$ Δ-converges to a fixed point of T, it is sufficient to show that $\omega_w(x_n)$ consists of exactly one point.

Let $\{w_n\}$ be a subsequence of $\{x_n\}$ with $A(\{w_n\}) = \{w\}$ and let $A(\{x_n\}) = \{x\}$. Since $w \in \omega_w(x_n) \subseteq F(T)$ and by Lemma 3.1, $\lim_{n\to\infty} d(x_n, w)$ exists. Again by Lemma 2.9, we have $x = w \in$

$F(T)$. Thus $\omega_w(x_n) = \{x\}$. This shows that $\{x_n\}$ Δ-converges to a fixed point of T. This completes the proof. □

As a consequence of Theorem 3.6, we obtain the following.

Corollary 3.7. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $T\colon K \to K$ be a uniformly continuous and total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (1.2). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n\beta_n(1-\beta_n) > 0$. If the following conditions are satisfied:*
(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;
(ii) there exists a constant $M_1 > 0$ such that $\zeta(r) \leq M_1 r$, $r \geq 0$.
Then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

Proof . The proof of corollary 3.7 immediately follows from Theorem 3.6 by taking $\gamma_n = 0$ for all $n \geq 1$. This completes the proof. □

Theorem 3.8. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $T\colon K \to K$ be a uniformly continuous and total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (1.1). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. If the following conditions are satisfied:*
(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;
(ii) there exists a constant $M_1 > 0$ such that $\zeta(r) \leq M_1 r$, $r \geq 0$.
Suppose that T^m is semi-compact for some $m \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof . By Lemma 3.5, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since T is uniformly continuous, we have

$$d(x_n, T^m x_n) \leq d(x_n, Tx_n) + d(Tx_n, T^2 x_n) + \cdots + d(T^{m-1}x_n, T^m x_n) \to 0$$

as $n \to \infty$. That is, $\{x_n\}$ is an AFPS for T^m. By the semi-compactness of T^m, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $p \in K$ such that $\lim_{j\to\infty} x_{n_j} = p$. Again, by the uniform continuity of T, we have

$$d(Tp, p) \leq d(Tp, Tx_{n_j}) + d(Tx_{n_j}, x_{n_j}) + d(x_{n_j}, p) \to 0 \quad \text{as } j \to \infty.$$

That is $p \in F(T)$. By Lemma 3.1, $d(x_n, p)$ exists, thus p is the strong limit of the sequence $\{x_n\}$ itself. This shows that the sequence $\{x_n\}$ converges strongly to a fixed point of T. This completes the proof. □

Senter and Dotson [28] introduced the concept of Condition (A) as follows.

Definition 3.9. (See [28]) A mapping $T\colon K \to K$ is said to satisfy Condition (A) if there exists a non-decreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x, Tx) \geq f(d(x, F(T)))$, for all $x \in K$.

As an application of Theorem 3.3, we establish another strong convergence result employing Condition (A).

Theorem 3.10. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon K \to K$ be a uniformly continuous and total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (1.1). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. If the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;

(ii) there exists a constant $M_1 > 0$ such that $\zeta(r) \leq M_1 r$, $r \geq 0$.

If T satisfies Condition (A), then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof . As in the proof of Lemma 3.1, we have that $\lim_{n\to\infty} d(x_n, F(T))$ exists. Again by Lemma 3.5, we know that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. So Condition (A) guarantees that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since f is a non-decreasing function and $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, Theorem 3.3 implies that $\{x_n\}$ converges strongly to a fixed point of T. This completes the proof. □

Remark 3.11. Theorem 3.6 contains Theorem 5.7 of Nanjaras and Panyanak [22] and Theorem 3.5 of Niwongsa and Panyanak [23] since each asymptotically nonexpansive mapping is a total asymptotically nonexpansive mapping. Also, Theorem 3.6 contains Theorem 3.5 of Chang et al. [4] since the modified Noor iteration reduces to the modified Mann iteration.

Remark 3.12. Theorem 3.6 also extends Theorem 4 of Başarir and Şahin [26] to the case of modified Noor iteration scheme considered in this paper.

Example 3.13. Let $\mathbb{R}$ be the real line with the usual norm $|.|$ and $C = [0, \infty)$. Define a mapping $T\colon C \to C$ by $T(x) = sin\, x$ for all $x \in C$. Let ψ be the strictly increasing continuous function such that $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$. Let $\{\mu_n\}_{n\geq 1}$ and $\{\nu_n\}_{n\geq 1}$ be two nonnegative real sequences defined by $\mu_n = \frac{1}{n^2}$ and $\nu_n = \frac{1}{n^3}$ for all $n \geq 1$ with $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$. Since $T(x) = sin\, x$ for $x \in C$, we have

$$|T^n x - T^n y| \leq |x - y|.$$

For all $x, y \in C$, we obtain

$$|T^n x - T^n y| - |x - y| - \nu_n \psi(|x - y|) - \mu_n$$

$$\leq |x - y| - |x - y| - \nu_n \psi(|x - y|) - \mu_n \leq 0.$$

for all $n = 1, 2, \ldots$, $\{\nu_n\}_{n\geq 1}$ and $\{\mu_n\}_{n\geq 1}$ with $\nu_n, \mu_n \to 0$ as $n \to \infty$ and so T is a total asymptotically nonexpansive mapping. Also, T is uniformly L-Lipschitzian with $L = 1$. Clearly $F(T) = \{0\}$. Let $\alpha_n = \frac{n}{2n+1}$, $\beta_n = \frac{n}{3n+1}$ and $\gamma_n = \frac{n}{4n+1}$ for all $n \geq 1$. Therefore, the conditions of Theorem 3.1 and 3.2 are fulfilled.

4. Conclusion

In this work we extensively used the notion of Noor iteration process for total asymptotically nonexpansive mapping to establish the strong and Δ-convergence theorems in the framework of CAT(0) spaces that differ from the iteration scheme of modified two-step used by Başarir and Şahin [26]. In this way our work not only extend and generalize the work of Başarir and Şahin [26] but also work done in the papers [22, 23, 26, 30] and many others related works.

Acknowledgements

The authors are highly indebted to the referees and managing editor for their careful observations of the manuscript and valuable suggestions for making different from the existing ones.

The second (corresponding) author is thankful to the United State-India Education Foundation, New Delhi, India and IIE/CIES, Washington, DC, USA for Fulbright-Nehru PDF Award (No. 2052/FNPDR/2015).

References

[1] M. Abbas, B.S. Thakur and D. Thakur, *Fixed points of asymptotically nonexpansive mappings in the intermediate sense in CAT(0) spaces*, Commun. Korean Math. Soc. 28 (2013) 107–121.

[2] M.R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Vol. 319 of Grundlehren der Mathematischen Wissenschaften, Springer, Berlin, Germany, 1999.

[3] F. Bruhat and J. Tits, *Groups reductifs sur un corps local*, Institut des Hautes Etudes Scientifiques, Publications Mathematiques 41 (1972) 5–251.

[4] S.S. Chang, L. Wang, H.W. Joesph Lee, C.K. Chan and L. Yang, *Demiclosed principle and $\triangle$-convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces*, Appl. Math. Comput. 219 (2012) 2611–2617.

[5] L. Cirić, A. Rafiq, S. Radenović, M. Rajović and J.S. Ume, *On Mann implicit iterations for strongly accreative and strongly pseudo-contractive mappings*, Appl. Math. Comput. 198 (2008) 128–137.

[6] S. Dhompongsa, W.A. Kirk and B. Sims, *Fixed points of uniformly Lipschitzian mappings*, Nonlinear Anal. 65 (2006) 762–772.

[7] S. Dhompongsa, W.A. Kirk and B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces*, J. Nonlinear Convex Anal. 8 (2007) 35–45.

[8] S. Dhompongsa and B. Panyanak, *On $\triangle$-convergence theorem in CAT(0) spaces*, Comput. Math. Appl. 56 (2008) 2572–2579.

[9] D. Dukić, L. Paunović and S. Radenović, *Convergence of iterates with errors of uniformly quasi-Lipschitzian mappings in cone metric spaces*, Kragujevac J. Math. 35 (2011) 399–410.

[10] K. Goebel and W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 35 (1972) 171–174.

[11] K. Goebel and W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.

[12] R. Glowinski and P. Le Tallec, *Augemented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics*, Siam, Philadelphia, 1989.

[13] W. P. Guo, Y. J. Cho and W. Guo, *Convergence theorems for mixed type asymptotically nonexpansive mappings*, Fixed Point Theory Appl. 2012 (2012):224.doi:10.1186/1687-1812-2012-224.

[14] S. Haubruge, V.H. Nguyen and J.J. Strodiot, *Convergence analysis and applications of the Glowinski Le Tallec splitting method for finding a zero of the sum of two maximal monotone operators*, J. Optim. Theory Appl. 97 (1998) 645–673.

[15] N. Hussain and M.A. Khamsi, *On asymptotic pointwise contractions in metric spaces*, Nonlinear Anal. 71 (2009) 4423–4429.

[16] S. Ishikawa, *Fixed point by a new iteration method*, Proc. Amer. Math. Soc. 44 (1974) 147–150.

[17] M.A. Khamsi and W. A. Kirk, *An introduction to metric spaces and fixed point theory*, Pure Appl. Math, Wiley-Interscience, New York, NY, USA, 2001.

[18] W.A. Kirk, *Geodesic geometry and fixed point theory*, in Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), Vol. 64 of Coleccion Abierta, 195-225, University of Seville Secretary of Publications, Seville, Spain, 2003.

[19] W.A. Kirk, Geodesic geometry and fixed point theory II, in International Conference on Fixed point Theory and Applications, 113-142, Yokohama Publishers, Yokohama, Japan, 2004.

[20] W.A. Kirk and B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Anal. 68 (2008) 3689–3696.

[21] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. 4 (1953) 506–510.

[22] B. Nanjaras and B. Panyanak, *Demiclosed principle for asymptotically nonexpansive mappings in CAT(0) spaces*, Fixed Point Theory Appl. 2010 (2010), Art. ID 268780.

[23] Y. Niwongsa and B. Panyanak, *Noor iterations for asymptotically nonexpansive mappings in CAT(0) spaces*, Int. J. Math. Anal. 4 (2010) 645–656.

[24] M.A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl. 251 (2000) 217–229.

[25] M.A. Noor, *Three-step iterative algorithms for multivalued quasi variational inclusions*, J. Math. Anal. Appl. 255 (2001) 589–604.

[26] M. Başarir and A. Şahin, *On the strong and $\triangle$-convergence for total asymptotically nonexpansive mappings on a CAT(0) space*, Carpathian Math. Pub. 5 (2013) 170–179.

[27] G.S. Saluja, *Strong convergence of Noor iteration for generalized asymptotically quasi-nonexpansive mappings in CAT(0) spaces*, Acta Math. Vietnamica 37 (2012) 341–355.

[28] H.F. Senter and W.G. Dotson,*Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc. 44 (1974) 375–380.

[29] K.K. Tan and H.K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. 178 (1993) 301–308.

[30] B.L. Xu and M.A. Noor, *Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. 267 (2002) 444–453.