Int. J. Nonlinear Anal. Appl. 8 (2017) No. 1, 261-267
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.1231.1289

A note on the Young type inequalities

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(Communicated by M. Eshaghi)

Abstract

In this paper, we present some refinements of the famous Young type inequality. As application of our result, we obtain some matrix inequalities for the Hilbert-Schmidt norm and the trace norm. The results obtained in this paper can be viewed as refinement of the derived results by H. Kai [Young type inequalities for matrices, J. East China Norm. Univ. 4 (2012) 12–17].

Keywords: Young inequality; Hilbert-Schmidt norm; Positive semidefinite matrices; Refinements.
2010 MSC: Primary 15A45; Secondary 15A60.

1. Introduction and preliminaries

As is known to all, the well-known Young inequality for scalars says that if $a, b \geq 0$ and $0 \leq \nu \leq 1$, then

$$a^{\nu} b^{1-\nu} \leq \nu a + (1-\nu) b, \tag{1.1}$$

with equality if and only if $a = b$. Recently, F. Kittaneh and Y. Manasrah [8] obtained a refinement of inequality (1.1) which can be stated as follows:

$$a^{\nu} b^{1-\nu} + r_0(\sqrt{a} - \sqrt{b})^2 \leq \nu a + (1-\nu) b \tag{1.2}$$

where $r_0 = min\{\nu, 1-\nu\}$.

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Received: February 2016 *Revised:* January 2017

After a short time, H. Kai [7] gave the the following Young type inequalities

$$[(\nu a)^{\nu} b^{1-\nu}]^2 + \nu^2 (a-b)^2 \leq \nu^2 a^2 + (1-\nu)^2 b^2, \qquad 0 \leq \nu \leq \frac{1}{2} \tag{1.3}$$

$$[a^{\nu}[(1-\nu)b]^{1-\nu}]^2 + (1-\nu)^2 (a-b)^2 \leq \nu^2 a^2 + (1-\nu)^2 b^2, \qquad 0 \leq \nu \leq \frac{1}{2}. \tag{1.4}$$

Let $M_n(\mathbb{C})$ be the space of $n \times n$ complex matrices. The unitarily invariance of the $\|.\|$ means that $\|UAV\| = \|A\|$ for all $A \in M_n(\mathbb{C})$ and for all unitary matrices $U, V \in M_n(\mathbb{C})$. For $A = [a_{ij}] \in M_n(\mathbb{C})$, the Hilbert-Schmidt (or Frobenius) norm and the trace norm of A are defined by

$$\|A\|_2 = \sqrt{\sum_{j=1}^{n} s_j^2(A)}, \quad \|A\|_1 = \sum_{j=1}^{n} s_j(A),$$

respectively, where $s_1(A) \geq s_2(A) \geq \ldots \geq s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive matrix $|A| = (A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. It is well known that $\|.\|_2$ is unitarily invariant.

Based on the refined Young inequalities (1.3) and (1.4), H. Kai [7] have showed that if $A, B, X \in M_n(\mathbb{C})$ with A and B positive semidefinite, then

$$\begin{aligned} \nu^{2\nu} \left\| A^{\nu} X B^{1-\nu} \right\|_2^2 + 2\nu(1-\nu) \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right\|_2^2 + \nu^2 \|AX - XB\|_2^2 \\ \leq \|\nu AX + (1-\nu) XB\|_2^2, \qquad 0 \leq \nu \leq \frac{1}{2}, \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} (1-\nu)^{2-2\nu} \left\| A^{\nu} X B^{1-\nu} \right\|_2^2 + 2\nu(1-\nu) \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right\|_2^2 + (1-\nu)^2 \|AX - XB\|_2^2 \\ \leq \|\nu AX + (1-\nu) XB\|_2^2, \qquad \frac{1}{2} \leq \nu \leq 1. \end{aligned} \tag{1.6}$$

For a detailed study of the classical Young inequality (1.1) and associated matrix inequality along with their history of origin, refinements and applications, one may refer to [1, 2, 4, 6] and [9, 14].

2. Main results

We shall categorize our main results into two categories, where refinements of the classical Young inequality (1.1) take first place. Then, the corresponding matrices inequalities for the Hilbert-Schmidt norm and the trace norm are studied.

2.1. Inequalities for scalars

We start this section with the following theorem for scalars.

Theorem 2.1. *Let $a, b \geq 0$ and $\nu \in [0,1]$.*
If $0 \leq \nu \leq \frac{1}{2}$, then

$$\nu^2 a^2 + (1-\nu)^2 b^2 \geq (\nu a)^{2\nu} b^{2-2\nu} + \nu^2 (a-b)^2 + r_0 b(\sqrt{\nu a} - \sqrt{b})^2, \tag{2.1}$$

where, $r_0 = min\{2\nu, 1-2\nu\}$.
If $\frac{1}{2} \leq \nu \leq 1$, then

$$\nu^2 a^2 + (1-\nu)^2 b^2 \geq a^{2\nu}[(1-\nu)b]^{2-2\nu} + (1-\nu)^2 (a-b)^2 + r_0 a(\sqrt{a} - \sqrt{(1-\nu)b})^2 \tag{2.2}$$

where, $r_0 = min\{2\nu - 1, 2-2\nu\}$.

Proof . Firstly, we suppose $0 \leq \nu \leq \frac{1}{2}$. Observe that

$$\begin{aligned}
&\nu^2 a^2 + (1-\nu)^2 b^2 - (\nu a)^{2\nu} b^{2-2\nu} - \nu^2 (a-b)^2 - r_0 b(\sqrt{\nu a} - \sqrt{b})^2 \\
&= b[(1-2\nu)b + 2\nu(\nu a)] - (\nu a)^{2\nu} b^{2-2\nu} - r_0 b(\sqrt{\nu a} - \sqrt{b})^2 \\
&\geq b[(\nu a)^{2\nu} b^{1-2\nu} + r_0(\sqrt{\nu a} - \sqrt{b})^2] - (\nu a)^{2\nu} b^{2-2\nu} - r_0 b(\sqrt{\nu a} - \sqrt{b})^2 \qquad (by (1.2)) \\
&= 0.
\end{aligned}$$

That is

$$\nu^2 a^2 + (1-\nu)^2 b^2 \geq (\nu a)^{2\nu} b^{2-2\nu} + \nu^2 (a-b)^2 + r_0 b(\sqrt{\nu a} - \sqrt{b})^2.$$

So (2.1) holds.

Conversely, if $\frac{1}{2} \leq \nu \leq 1$, then we have

$$\begin{aligned}
&\nu^2 a^2 + (1-\nu)^2 b^2 - a^{2\nu}[(1-\nu)b]^{2-2\nu} - (1-\nu)^2 (a-b)^2 - r_0 a(\sqrt{a} - \sqrt{(1-\nu)b})^2 \\
&= a[(2\nu-1)a + 2(1-\nu)((1-\nu)b)] - a^{2\nu}[(1-\nu)b]^{2-2\nu} - r_0 a(\sqrt{a} - \sqrt{(1-\nu)b})^2 \\
&\geq a[a^{2\nu-1}((1-\nu)b)^{2-2\nu} + r_0(\sqrt{a} - \sqrt{(1-\nu)b})^2] - a^{2\nu}[(1-\nu)b]^{2-2\nu} \\
&- r_0 a(\sqrt{a} - \sqrt{(1-\nu)b})^2 \qquad (by (1.2)) \\
&= 0,
\end{aligned}$$

and so

$$\nu^2 a^2 + (1-\nu)^2 b^2 \geq a^{2\nu}[(1-\nu)b]^{2-2\nu} + (1-\nu)^2 (a-b)^2 + r_0 a(\sqrt{a} - \sqrt{(1-\nu)b})^2.$$

This completes the proof. □

Remark 2.2. It is obvious, that inequalities (2.1) and (2.2) are refinement of inequalities (1.3) and (1.4).

2.2. Inequalities for matrices

In this section, we yield some matrix inequalities for the Hilbert-Schmidt norm and trace norm by means of the derived inequalities (1.3) and (1.4). We begin our study with the following result.

Theorem 2.3. *Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite and $0 \leq \nu \leq 1$. If $0 \leq \nu \leq \frac{1}{2}$, then*

$$\begin{aligned}\|\nu AX + (1-\nu)XB\|_2^2 \geq \nu^{2\nu}\|A^\nu XB^{1-\nu}\|_2^2 + \nu^2\|AX + XB\|_2^2 \\ + r_0\left[\nu\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 + \|XB\|_2^2 - 2\sqrt{\nu}\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2\right] \\ + 2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2. \end{aligned} \tag{2.3}$$

where, $r_0 = min\{2\nu, 1-2\nu\}$.
If $\frac{1}{2} \leq \nu \leq 1$, then

$$\begin{aligned}\|\nu AX + (1-\nu)XB\|_2^2 \geq (1-\nu)^{2(1-\nu)}\|A^\nu XB^{1-\nu}\|_2^2 + (1-\nu)^2\|AX - XB\|_2^2 \\ + r_0\left[(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 + \|AX\|_2^2 - 2\sqrt{1-\nu}\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2\right] \\ + 2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2, \end{aligned} \tag{2.4}$$

where, $r_0 = min\{2\nu - 1, 2-2\nu\}$.

Proof . Since $A, B \geq 0$, then by spectral theorem there are unitary matrices U, V and diagonal matrices $D_1 = diag(\lambda_i)$ and $D_2 = diag(\mu_i)$ for $i = 1, \ldots, n$, such that

$$A = UD_1U^*$$

and

$$B = VD_2V^*.$$

For our computations, let $Y = U^*XV = [y_{ij}]$. Then we have

$$\nu AX + (1-\nu)XB = U[(\nu\lambda_i + (1-\nu)\mu_j)y_{ij}]V^*,$$

$$A^{\frac{1}{2}}XB^{\frac{1}{2}} = U[\left(\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}}\right)y_{ij}]V^*$$

and

$$A^\nu XB^{1-\nu} = U[(\lambda_i^\nu\mu_j^{1-\nu})y_{ij}]V^*.$$

If $0 \leq \nu \leq \frac{1}{2}$, utilizing (2.1) and the unitarily invariant property of $\|.\|_2$, we have

$$\begin{aligned}
&\|\nu AX + (1-\nu)XB\|_2^2 \\
&= \sum_{i,j=1}^{n} (\nu\lambda_i + (1-\nu)\mu_j)^2 |y_{ij}|^2 \\
&\geq \nu^{2\nu} \sum_{i,j=1}^{n} \left(\lambda_i^{\nu}\mu_j^{1-\nu}\right)^2 |y_{ij}|^2 + \nu^2 \Sigma_{i,j=1}^{n} (\lambda_i - \mu_j)^2 |y_{ij}|^2 \\
&+ r_0 \left[\nu \sum_{i,j=1}^{n} \left(\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}}\right)^2 |y_{ij}|^2 + \sum_{i,j=1}^{n} \mu_j^2 |y_{ij}|^2 - 2\sqrt{\nu} \sum_{i,j=1}^{n} \left(\lambda_i^{\frac{1}{4}}\mu_j^{\frac{3}{4}}\right)^2 |y_{ij}|^2\right] \\
&+ 2\nu(1-\nu) \sum_{i,j=1}^{n} \left(\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}}\right)^2 |y_{ij}|^2 \\
&= \nu^{2\nu} \|A^{\nu}XB^{1-\nu}\|_2^2 + \nu^2 \|AX + XB\|_2^2 \\
&+ r_0 \left[\nu \left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 + \|XB\|_2^2 - 2\sqrt{\nu}\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2\right] \\
&+ 2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2.
\end{aligned}$$

So,

$$\begin{aligned}
\|\nu AX + (1-\nu)XB\|_2^2 \geq\ & \nu^{2\nu} \|A^{\nu}XB^{1-\nu}\|_2^2 + \nu^2 \|AX + XB\|_2^2 \\
&+ r_0 \left[\nu \left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 + \|XB\|_2^2 - 2\sqrt{\nu}\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2\right] \\
&+ 2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2.
\end{aligned}$$

The proof of inequality (2.4) is similar to that of inequality (2.3). Thus, we leave it to the reader. □

Remark 2.4. Clearly, inequalities (2.3) and (2.4) are refinement of inequalities (1.5) and (1.6).

In the end, we obtain refinements of the trace versions of Young type inequalities. To do this we need the following lemmas.

Lemma 2.5. (Cauchy-Schwarz inequality [3]) Let $a_i, b_i \geq 0 (1 \leq i \leq n)$. Then

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}.$$

Lemma 2.6. [3] Let $A, B \in M_n(\mathbb{C})$. Then

$$\sum_{j=1}^{n} s_j(AB) \leq \sum_{j=1}^{n} s_j(A)s_j(B).$$

We will prove the last result.

Theorem 2.7. *Let $A, B \in M_n(\mathbb{C})$ such that A and B are positive semidefinite and $0 \le \nu \le 1$. If $0 \le \nu \le \frac{1}{2}$, then*

$$tr(\nu^2 A^2 + (1-\nu)^2 B^2) \ge \nu^{2\nu}[\|A^\nu B^{1-\nu}\|_2^2] + \nu^2 \left[\|A\|_2^2 + \|B\|_2^2 - 2\|A\|_2\|B\|_2\right] + r_0\left[\nu\|AB\|_1 + \|B\|_2^2 - 2\sqrt{\nu}\left(\sqrt{\|A\|_1}\sqrt{\left\|B^{\frac{3}{2}}\right\|_1}\right)\right]. \tag{2.5}$$

If $\frac{1}{2} \le \nu \le 1$, then

$$tr(\nu^2 A^2 + (1-\nu)^2 B^2) \ge (1-\nu)^{2-2\nu}\left[\|A^\nu B^{1-\nu}\|_2^2\right] + (1-\nu)^2\left[\|A\|_2^2 + \|B\|_2^2 - 2\|A\|_2\|B\|_2\right] + r_0\left[(1-\nu)\|AB\|_1 + \|A\|_2^2 - 2\sqrt{1-\nu}\left(\sqrt{\left\|A^{\frac{3}{2}}\right\|_1}\sqrt{\left\|B^{\frac{1}{2}}\right\|_1}\right)\right]. \tag{2.6}$$

Proof . We shall prove the first inequality, and leave the second to the reader because the proof is very similar. If $0 \le \nu \le \frac{1}{2}$, then using Lemma2.5, Lemma2.6 and the inequality (2.1), we have

$$\begin{aligned}
& tr(\nu^2 A^2 + (1-\nu)^2 B^2) \\
& = \nu^2 trA^2 + (1-\nu)^2 trB^2 \\
& = \sum_{j=1}^n (\nu^2 s_j^2(A) + (1-\nu)^2 s_j^2(B)) \\
& \ge \nu^{2\nu}\sum_{j=1}^n [s_j(A^\nu)s_j(B^{1-\nu})]^2 \\
& + \nu^2\left[\sum_{j=1}^n s_j^2(A) + \sum_{j=1}^n s_j^2(B) - 2\sum_{j=1}^n s_j(A)s_j(B)\right] \\
& + r_0\left[\nu\sum_{j=1}^n s_j(A)s_j(B) + \sum_{j=1}^n s_j^2(B) - 2\sqrt{\nu}\left(\sum_{j=1}^n s_j^{\frac{1}{4}}(A)s_j^{\frac{3}{4}}(B)\right)^2\right] \\
& \ge \nu^{2\nu}\sum_{j=1}^n [s_j(A^\nu)s_j(B^{1-\nu})]^2 + \upsilon^2\left[\sum_{j=1}^n s_j^2(A) + \sum_{j=1}^n s_j^2(B) - 2\sqrt{\sum_{j=1}^n s_j^2(A)}\sqrt{\sum_{j=1}^n s_j^2(B)}\right] \\
& + r_0\left[\nu\sum_{j=1}^n s_j(AB) + \sum_{j=1}^n s_j^2(B) - 2\sqrt{\nu}\left(\sum_{j=1}^n s_j^{\frac{1}{2}}(A)\sum_{j=1}^n s_j^{\frac{3}{2}}(B)\right)\right] \\
& = \nu^{2\nu}\left[\|A^\nu B^{1-\nu}\|_2^2\right] + \nu^2\left[\|A\|_2^2 + \|B\|_2^2 - 2\|A\|_2\|B\|_2\right] \\
& + r_0\left[\nu\|AB\|_1 + \|B\|_2^2 - 2\sqrt{\nu}\left(\sqrt{\|A\|_1}\sqrt{\left\|B^{\frac{3}{2}}\right\|_1}\right)\right].
\end{aligned} \tag{2.7}$$

On the other hand, we have

$$tr(\nu^2 A^2 + (1-\nu)^2 B^2) = \nu^2 trA^2 + (1-\nu)^2 trB^2 = \nu^2\|A\|_2^2 + (1-\nu)^2\|B\|_2^2. \tag{2.8}$$

With the aid of inequalities (2.7) and (2.8), it follows that

$$\nu^2\|A\|_2^2+(1-\nu)^2\|B\|_2^2 \geq \nu^{2\nu}\left[\|A^\nu B^{1-\nu}\|_2^2\right]+\nu^2\left[\|A\|_2^2+\|B\|_2^2-2\|A\|_2\|B\|_2\right]$$
$$+r_0\left[\nu\|AB\|_1+\|B\|_2^2-2\sqrt{\nu}\left(\sqrt{\|A\|_1}\sqrt{\left\|B^{\frac{3}{2}}\right\|_1}\right)\right].$$

This completes the proof. □

Remark 2.8. Obviously, inequalities (2.5) and (2.6) are refinement of the well-known results in [7].

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