Int. J. Nonlinear Anal. Appl. 8 (2017) No. 1, 301-314
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.1460.1370

Periodic boundary value problems for controlled nonlinear impulsive evolution equations on Banach spaces

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(Communicated by S. Abbasbandy)

Abstract

This paper deals with the Periodic boundary value problems for Controlled nonlinear impulsive evolution equations. By using the theory of semigroup and fixed point methods, some conditions ensuring the existence and uniqueness. Finally, two examples are provided to demonstrate the effectiveness of the proposed results.

Keywords: impulsive evolution equations; Periodic boundary value problems; Control; Mild solutions.
2010 MSC: Primary 37L05; Secondary 65L10.

1. Introduction

The theory of impulsive differential equations has become an important area of investigation in recent years, stimulated by their numerous applications to problems from mechanics, electrical engineering, medicine, biology, ecology, etc. Ordinary differential equations of first-and second-order with impulses have been treated in several works and we refer the reader to ([1, 11]) and the references therein related to this matter. First-order partial differential equations with impulses are studied in Bainov et al. [2] and Liu [5] among others. The Global solutions for impulsive abstract partial differential equations is studied in Hernandez [3].

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Received: June 2016 *Revised:* October 2016

Hernandez and O?Regan [4] and Pierri et al [9] studied, with more details, the existence of problem

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in (s_i, t_{i+1}], \quad i = 0, 1, 2, \dots, m, \\ x(t) = g_i(t, x(t)), & t \in (t_i, s_i], \quad i = 1, 2, \dots, m, \\ x(0) = x(a) \in X, \end{cases}$$

There are many papers discussing the impulsive differential equations and impulsive optimal controls with the classic initial condition: $x(0) = x_0$ (see [6, 7, 13, 14, 10]). In [16] Lanping Zhu and Qianglian Huang studied the controlled nonlocal impulsive equation :

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)) + B(t)c(t), & t \in [0, T], \quad t \neq t_i, \quad c \in \mathcal{U}_{ad}. \\ u(0) + g(u) = u_0, \\ \Delta u(t_i) = I_i(u(t_i)), & i = 1, \dots, p, \quad t_1 < t_2 < \dots < t_p < T, \end{cases}$$

where $c \in \mathcal{U}_{ad}$ is a control set which we will introduce later and $A : D(A) \subseteq X \longrightarrow X$ is the infinitesimal generator of strongly continuous semigroup $\{T(t), t \geq 0\}$ in a real Banach space X, f is a nonlinear perturbation, $I_i, \quad i = 1, \dots, p$ is a nonlinear map and $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, g is a given X-valued function.

In [15] Xiulan Yu, JinRong Wang studied the impulsive equation :

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), & t \in (s_i, t_{i+1}], \quad i = 0, 1, 2, \dots, m, \\ u(t) = x_i + T(t_i) \int_{t_i}^t g_i(s, u(s)) ds, & t \in (t_i, s_i], \quad i = 1, 2, \dots, m, \quad x_i \in X \\ u(0) = u(a) \in X, , \end{cases}$$

In this paper, we consider the following problems for nonlinear impulsive evolution equations with Periodic boundary value:

$$(\mathbf{IEE}) \begin{cases} u'(t) = Au(t) + f(t, u(t), u(\rho(t))) + B(t)c(t), & t \in (s_i, t_{i+1}], \\ \qquad i = 0, 1, 2, \dots, m, \quad c \in \mathcal{U}_{ad} \\ u(t) = x_i + T(t_i) \int_{t_i}^t g_i(s, u(s)) ds, & t \in (t_i, s_i], \quad i = 1, 2, \dots, m, \quad x_i \in X \\ u(0) = u(a) \in X, \end{cases}$$

Provided, the operator $A : D(A) : \quad X \longrightarrow X$ is the generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ on a Banach space X with a norm $\|.\|$, and the fixed points s_i and t_i satisfy

$$0 = s_0 < t_1 \leq s_1 \leq t_2 < \dots < t_m \leq s_m \leq t_{m+1} = a$$

are pre-fixed numbers, $f : [0, a] \times X \times X \longrightarrow X$ is continuous, $\rho : [0, a] \longrightarrow [0, a]$ is continuous and $g_i : [t_i, s_i] \times X \longrightarrow X$ is continuous for all $i = 1, 2, \dots, m$.

2. Preliminaries

Next, we review some basic concepts, notations and technical results that are necessary in our study. Throughout this paper, $I = [0, a]$, $\mathcal{C}(I, X)$ be the Banach space of all continuous functions from I into X with the norm
$\|u\|_{\mathcal{C}} = \sup_{t \in I} \{\|u(t)\| : t \in I\}$ for $u \in \mathcal{C}(I, X)$, and we consider the space

$$\mathcal{PC}(I, X) = \{u : I \longrightarrow X : u \in \mathcal{C}((t_i, t_{i+1}], X), i = 0, 1, \dots, m \text{ and there exist } u(t_i^-) \text{ and } u(t_i^+), i = 1, \dots, m \text{ with } u(t_i^-) = u(t_i)\},$$

endowed with the Chebyshev PC-norm $\|u\|_{\mathcal{PC}} = \sup_{t\in I}\{\|u(t)\| : t \in I\}$ for $u \in \mathcal{PC}(I, X)$. Denote $M = \sup_{t\in I} \|T(t)\|$.

Let Y be another separable reflexive Banach space where controls c take values. Denoted $P_f(Y)$ by a class of nonempty closed and convex subsets of Y. We suppose that the multivalued map $w : [0, T] \longrightarrow P_f(Y)$ is measurable,
$w(.) \subset E$, where E is a bounded set of Y, and the admissible control set

$$\mathcal{U}_{ad} = \{c \in L^p(E) : \quad c(t) \in w(t), \quad a.e\}, \quad p > 1.$$

Then $\mathcal{U}_{ad} \neq \emptyset$ which can be found in [12].

Some of our results are proved using the next well-known results.

Theorem 2.1. (Krasnoselskii's fixed point theorem). Assume that K is a closed bounded convex subset of a Banach space X. Furthermore assume that Γ_1 and Γ_2 are mappings from K into X such that

1. $\Gamma_1(u) + \Gamma_2(v) \in K$ for all $u, v \in K$,
2. Γ_1 is a contraction,
3. Γ_2 is continuous and compact.

Then $\Gamma_1 + \Gamma_2$ has a fixed point in K.

To begin our discussion, we need to introduce the concept of a mild solution for (**IEE**) .
Assume that $u : [0, a] \longrightarrow X$ is a solution of

$$u'(t) = Au(t) + f(t, u(t), u(\rho(t))) + B(t)c(t), \qquad 0 \leq t \leq a,$$

From a strongly continuous semigroups theory, we get

$$\begin{aligned}
u(t) &= T(t)u(0) + \int_0^t T(t-s)(f(s, u(s), u(\rho(s))) + B(s)c(s))ds \\
&= T(t)u(a) + \int_0^t T(t-s)(f(s, u(s), u(\rho(s))) + B(s)c(s))ds \\
&= T(t)\left[T(a-s_m)u(s_m) + \int_{s_m}^a T(a-s)(f(s, u(s), u(\rho(s))) + B(s)c(s))ds\right] \\
&+ \int_0^t T(t-s)(f(s, u(s), u(\rho(s))) + B(s)c(s))ds \\
&= T(t)\Big[T(a-s_m)\left(x_m + T(t_m)\int_{t_m}^{s_m} g_m(s, u(s))ds\right) \\
&+ \int_{s_m}^a T(a-s)(f(s, u(s), u(\rho(s))) + B(s)c(s))ds\Big] \\
&+ \int_0^t T(t-s)(f(s, u(s), u(\rho(s))) + B(s)c(s))ds \\
&= T(t)\Big[T(a-s_m)x_m + T(a-s_m+t_m)\int_{t_m}^{s_m} g_m(s, u(s))ds \\
&+ \int_{s_m}^a T(a-s)(f(s, u(s), u(\rho(s))) + B(s)c(s))ds\Big] \\
&+ \int_0^t T(t-s)(f(s, u(s), u(\rho(s))) + B(s)c(s))ds \quad \text{for all} \quad t \in [0, t_1],
\end{aligned}$$

and

$$\begin{aligned}
u(t) &= T(t-s_i)u(s_i) + \int_{s_i}^{t} T(t-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds \\
&= T(t-s_i)\left(x_i + T(t_i)\int_{t_i}^{s_i} g_i(s,u(s))ds\right) \\
&+ \int_{s_i}^{t} T(t-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds \\
&= T(t-s_i)x_i + T(t-s_i+t_i)\int_{t_i}^{s_i} g_i(s,u(s))ds \\
&+ \int_{s_i}^{t} T(t-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds,
\end{aligned}$$

for all $t \in (s_i, t_{i+1}],\quad i = 1,2,\dots,m$. This expression motivates the following definition.

Definition 2.2. We say that a function $u \in \mathcal{PC}(I,X)$ is called a mild solution of the problem (**IEE**), if u satisfies

$$\begin{cases}
u(t) = T(t)\Big[T(a-s_m)x_m + T(a-s_m+t_m)\int_{t_m}^{s_m} g_m(s,u(s))ds \\
\qquad + \int_{s_m}^{a} T(a-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds\Big] \\
\qquad + \int_0^t T(t-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds, \quad t \in [0,t_1]; \\
u(t) = x_i + T(t_i)\int_{t_i}^{t} g_i(s,u(s))ds, \qquad t \in (t_i, s_i], \quad i = 1,2,\dots,m; \\
u(t) = T(t-s_i)x_i + T(t-s_i+t_i)\int_{t_i}^{s_i} g_i(s,u(s))ds \\
\qquad + \int_{s_i}^{t} T(t-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds, \quad t \in (s_i, t_{i+1}], \quad i = 1,2,\dots,m.
\end{cases}$$

3. Existence and Uniqueness of mild solutions

To establish our results, we introduce the following assumptions :

- $\mathbf{H}_0$.
  1 . $A : D(A) \subseteq X \longrightarrow X$ is the generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ on X with a norm $\|.\|$.
  2 . $B : [0,a] \longrightarrow \mathcal{L}(Y,X)$ is essentially bounded, i.e., $B \in L^\infty([0,a], \mathcal{L}(Y,X))$.
- $\mathbf{H}_1$. The functions $f \in \mathcal{C}(I \times X \times X, X)$, $g_i \in \mathcal{C}([t_i,s_i] \times X, X)$, $i = 1,2,\dots,m$ and $\rho : I \longrightarrow I$ is continuous.
- $\mathbf{H}_2$. There is a constant $C_f, L_f > 0$ such that

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \leq C_f\|u_1 - u_2\| + L_f\|v_1 - v_2\|,$$

  for each $t \in [s_i, t_{i+1}]$, $u_1,u_2,v_1,v_2 \in X$ and $i = 0,1,\dots,m$.

- $\mathbf{H}_3$. There is a constant $L > 0$ such that

  $$\|f(t,u,v)\| \leq L(1 + \|u\|^{\mu} + \|v\|^{\nu}),$$

  for all $t \in [s_i, t_{i+1}]$ and all $u, v \in X$, $i = 0, 1, \dots, m$, $\mu, \nu \in [0,1]$.

- $\mathbf{H}_4$. There is a constant $C_{g_i} > 0$, $i = 1, 2, \dots, m$ such that

  $$\|g_i(t,u) - g_i(t,v)\| \leq C_{g_i}\|u - v\|,$$

  for each $t \in [t_i, s_i]$, and all $u, v \in E^n$, $i = 1, 2, \dots, m$.

- $\mathbf{H}_5$. There is a function $t \longmapsto \psi_i(t)$, $i = 1, 2, \dots, m$ such that

  $$\|g_i(t,u)\| \leq \psi_i(t),$$

  for each $t \in [t_i, s_i]$ and all $u \in X$.

We put $C = \max\limits_{1 \leq i \leq m} C_{g_i}$ and $N_i = \sup\limits_{t \in [t_i, s_i]} \psi_i(t) < +\infty$.

Remark 3.1. From the assumption $\mathbf{H}_0 - 2$ and the definition of $\mathcal{U}_{ad}$, it is also easy to verify that $Bc \in L^p([0,a]; X)$ with $p > 1$ for all $c \in \mathcal{U}_{ad}$. Therefore, $Bc \in L^1([0,a]; X)$ and $\|Bc\|_{L^1} < \infty$.

Now, we can establish our first existence result.

Theorem 3.2. *Let assumptions $\boldsymbol{H}_0, \boldsymbol{H}_1, \boldsymbol{H}_2$ and $\boldsymbol{H}_4$ be satisfied. Suppose, in addition, that the following properties is verified*

$$\begin{aligned}\lambda := M \max\Big\{ & M C_{g_m}(s_m - t_m) + (C_f + L_f)(M(a - s_m) + t_1) \\ & , \max_{1 \leq i \leq m} \{C(s_i - t_i) + (C_f + L_f)(t_{i+1} - s_i)\} \Big\} \\ < 1. &\end{aligned}$$

Then, the problem (**IEE**) *has a unique mild solution.*

Proof . Define a mapping $\Gamma : \mathcal{PC}(I, X) \longrightarrow \mathcal{PC}(I, X)$ by

$$(\Gamma u)(t) = \begin{cases} T(t)\Big[T(a - s_m)x_m + T(a - s_m + t_m)\int_{t_m}^{s_m} g_m(s, u(s))ds \\ \qquad + \int_{s_m}^{a} T(a - s)(f(s, u(s), u(\rho(s))) + B(s)c(s))ds\Big] \\ \quad + \int_0^t T(t - s)(f(s, u(s), u(\rho(s))) + B(s)c(s))ds, \quad t \in [0, t_1]; \\ \\ x_i + T(t_i)\int_{t_i}^{t} g_i(s, u(s))ds, \qquad t \in (t_i, s_i], \quad i = 1, 2, \dots, m; \\ \\ T(t - s_i)x_i + T(t - s_i + t_i)\int_{t_i}^{s_i} g_i(s, u(s))ds \\ \quad + \int_{s_i}^{t} T(t - s)(f(s, u(s), u(\rho(s))) + B(s)c(s))ds, \quad t \in (s_i, t_{i+1}], \quad i = 1, 2, \dots, m. \end{cases}$$

Let $h > 0$ very small and $u \in \mathcal{PC}(I, X)$, we have

Case 1: For $t \in [0, t_1]$, we have

$$\begin{aligned}
\|(\Gamma u)(t+h)-(\Gamma u)(t)\| = & \Bigg\| T(t+h)\left[T(a-s_m)x_m + T(a-s_m+t_m)\int_{t_m}^{s_m} g_m(s,u(s))ds\right.\\
& \left. + \int_{s_m}^{a} T(a-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds\right]\\
& + \int_0^{t+h} T(t+h-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds\\
& - T(t)\left[T(a-s_m)x_m + T(a-s_m+t_m)\int_{t_m}^{s_m} g_m(s,u(s))ds\right.\\
& \left. + \int_{s_m}^{a} T(a-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds\right]\\
& - \int_0^{t} T(t-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds\Bigg\|\\
& \leq M\Bigg\| T(h)\left[T(a-s_m)x_m + T(a-s_m+t_m)\int_{t_m}^{s_m} g_m(s,u(s))ds\right.\\
& \left. + \int_{s_m}^{a} T(a-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds\right]\\
& - \left[T(a-s_m)x_m + T(a-s_m+t_m)\int_{t_m}^{s_m} g_m(s,u(s))ds\right.\\
& \left. + \int_{s_m}^{a} T(a-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds\right]\Bigg\|\\
& + M\int_0^{h} (\|f(s,u(s),u(\rho(s))) + B(s)c(s)\|)ds\\
& + M\int_0^{t} \|B(s+h)c(s+h) - B(s)c(s)\|ds\\
& + M\int_0^{t} \|f(s+h,u(s+h),u(\rho(s+h))) - f(s,u(s),u(\rho(s)))\|ds \to 0 \text{ as } h \to 0.
\end{aligned}$$

Case 2: For $t \in (t_i, s_i]$, $i = 1, \ldots, m$, we have

$$\begin{aligned}
\|(\Gamma u)(t+h)-(\Gamma u)(t)\| &= \left\| x_i + T(t_i)\int_{t_i}^{t+h} g_i(s,u(s))ds - x_i - T(t_i)\int_{t_i}^{t} g_i(s,u(s))ds\right\|\\
&\leq M\int_t^{t+h} \|g_i(s,u(s))\|\, ds \to 0 \text{ as } h \to 0.
\end{aligned}$$

Case 3: For $t \in (s_i, t_{i+1}]$, $i = 1, \ldots, m$, we have

$$\begin{aligned}
\|(\Gamma u)(t+h)-(\Gamma u)(t)\| = & \Bigg\| T(t+h-s_i)x_i + T(t+h-s_i+t_i)\int_{t_i}^{s_i} g_i(s,u(s))ds\\
& + \int_{s_i}^{t+h} T(t+h-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds - T(t-s_i)x_i\\
& -T(t-s_i+t_i)\int_{t_i}^{s_i} g_i(s,u(s))ds + \int_{s_i}^{t} T(t-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds\Bigg\|
\end{aligned}$$

and so

$$\begin{aligned}
&\|(\Gamma u)(t+h)-(\Gamma u)(t)\| \\
&\leq M\,\|T(h)x_i-x_i\|+M\left\|T(h)\int_{t_i}^{s_i} g_i(s,u(s))ds-\int_{t_i}^{s_i} g_i(s,u(s))ds\right\| \\
&+M\int_{s_i}^{s_i+h}\|f(s,u(s),u(\rho(s)))+B(s)c(s)\|ds+M\int_{s_i}^{t}\|B(s+h)c(s+h)-B(s)c(s)\|ds \\
&+M\int_{s_i}^{t}\|f(s+h,u(s+h),u(\rho(s+h)))-f(s,u(s),u(\rho(s)))\|ds\to 0 \text{ as } h\to 0.
\end{aligned}$$

Then Γ is well defined and $\Gamma u\in\mathcal{PC}(I,X)$ for all $u\in\mathcal{PC}(I,X)$.
Now we only need to show that Γ is a contraction mapping:
Case 1: For $u,v\in\mathcal{PC}(I,X)$ and $t\in[0,t_1]$, we have

$$\begin{aligned}
\|(\Gamma u)(t)-(\Gamma v)(t)\| &= \Bigg\|T(t)\left[T(a-s_m)x_m+T(a-s_m+t_m)\int_{t_m}^{s_m} g_m(s,u(s))ds\right. \\
&\left.+\int_{s_m}^{a} T(a-s)(f(s,u(s),u(\rho(s)))+B(s)c(s))ds\right] \\
&+\int_0^t T(t-s)(f(s,u(s),u(\rho(s)))+B(s)c(s))ds \\
&-T(t)\left[T(a-s_m)x_m+T(a-s_m+t_m)\int_{t_m}^{s_m} g_m(s,v(s))ds\right. \\
&\left.+\int_{s_m}^{a} T(a-s)(f(s,v(s),v(\rho(s)))+B(s)c(s))ds\right] \\
&-\int_0^t T(t-s)(f(s,v(s),v(\rho(s)))+B(s)c(s))ds\Bigg\| \\
&\leq M\Big[MC_{g_m}(s_m-t_m)\|u(s)-v(s)\| \\
&+M\int_{s_m}^{a}\left(C_f\|u(s)-v(s)\|+L_f\|u(\rho(s))-v(\rho(s))\|\right)ds\Big] \\
&+M\int_0^t\left(C_f\|u(s)-v(s)\|+L_f\|u(\rho(s))-v(\rho(s))\|\right)ds \\
&\leq M\Big[MC_{g_m}(s_m-t_m)+(C_f+L_f)\,M(a-s_m)+(C_f+L_f)\,t_1\Big]\|u-v\|_{\mathcal{PC}} \\
&\leq \lambda\|u-v\|_{\mathcal{PC}}.
\end{aligned}$$

Case 2: For $u,v\in\mathcal{PC}(I,X)$ and $t\in(t_i,s_i]$, $i=1,\ldots,m$, we have

$$\begin{aligned}
\|(\Gamma u)(t)-(\Gamma v)(t)\| &= \left\|T(t_i)\int_{t_i}^{t} g_i(s,u(s))ds-T(t_i)\int_{t_i}^{t} g_i(s,v(s))ds\right\| \\
&\leq MC_{g_i}(s_i-t_i)\|u-v\|_{\mathcal{PC}} \\
&\leq MC(s_i-t_i)\|u-v\|_{\mathcal{PC}} \\
&\leq M\max_{1\leq i\leq m}\left\{C(s_i-t_i)+(C_f+L_f)(t_{i+1}-s_i)\right\}\|u-v\|_{\mathcal{PC}} \\
&\leq \lambda\|u-v\|_{\mathcal{PC}}.
\end{aligned}$$

Case 3: For $u, v \in \mathcal{PC}(I, X)$ and $t \in (s_i, t_{i+1}]$, $i = 1, \dots, m$, we have

$$\begin{aligned}
\|(\Gamma u)(t) - (\Gamma v)(t)\| &= \Big\| T(t - s_i + t_i) \int_{t_i}^{s_i} g_i(s, u(s))ds + \int_{s_i}^{t} T(t-s)(f(s, u(s), u(\rho(s))) + B(s)c(s))ds \\
&\quad - T(t - s_i + t_i) \int_{t_i}^{s_i} g_i(s, v(s))ds - \int_{s_i}^{t} T(t-s)(f(s, v(s), v(\rho(s))) + B(s)c(s))ds \Big\| \\
&\le M \left[C_{g_i}(s_i - t_i) + (C_f + L_f)(t_{i+1} - s_i) \right] \|u - v\|_{\mathcal{PC}} \\
&\le M \max_{1 \le i \le m} \{ C(s_i - t_i) + (C_f + L_f)(t_{i+1} - s_i) \} \|u - v\|_{\mathcal{PC}} \\
&\le \lambda \|u - v\|_{\mathcal{PC}}.
\end{aligned}$$

Therefore, we obtain

$$\|\Gamma u - \Gamma v\|_{\mathcal{PC}} \le \lambda \|u - v\|_{\mathcal{PC}}, \forall u, v \in \mathcal{PC}(I, X).$$

Finally, we find that Γ is a contraction mapping on $\mathcal{PC}(I, X)$, and there exists a unique $u \in \mathcal{PC}(I, X)$ such that $\Gamma u = u$.
So we conclude that u is the unique mild solution of (**IEE**). □

By using Krasnoselskii's fixed point theorem, we also obtain the existence of mild solution.

Theorem 3.3. *Let assumptions* $\boldsymbol{H}_0, \boldsymbol{H}_1, \boldsymbol{H}_3$ *and* $\boldsymbol{H}_5$ *be satisfied. Suppose, in addition, that the semigroup* $\{T(t), t \ge 0\}$ *is compact and*

$$\alpha := LM \max \{ (M(a - s_m) + t_1), (t_{i+1} - s_i) \} < \frac{1}{2}, \quad i = 1, \dots, m$$

$$\beta := M \max \{ MC_{g_m}(s_m - t_m), C_{g_i}(s_i - t_i) \} < 1.$$

Then the problem (**IEE**) *has at least one mild solution.*

Proof . Let $N = \max(N_1, N_2, \dots, N_m)$ and $B_r = \{u \in \mathcal{PC}(I, X) : \|u\|_{\mathcal{PC}} < r\}$ the ball with radius $r > 0$,
where

$$r \ge M \max \{\lambda_1, \lambda_2\}$$

with

$$\lambda_1 = \frac{M\|x_m\| + MN_m(s_m - t_m) + (M+1)\|Bc\|_{L^1} + L(M(a - s_m) + t_1)}{1 - 2\alpha},$$

and

$$\lambda_2 = \frac{1}{1 - 2\alpha} \max_{1 \le i \le m} \{ \|x_i\| + N(s_i - t_i) + \|Bc\|_{L^1} + L(t_{i+1} - s_i) \}.$$

We introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$, where

$$(\Gamma_1 u)(t) = \begin{cases} T(t) \left[T(a - s_m)x_m + T(a - s_m + t_m) \int_{t_m}^{s_m} g_m(s, u(s))ds \right], \\ \qquad t \in [0, t_1]; \\ x_i + T(t_i) \int_{t_i}^{t} g_i(s, u(s))ds, \qquad t \in (t_i, s_i], \quad i = 1, 2, \dots, m; \\ T(t - s_i)x_i + T(t - s_i + t_i) \int_{t_i}^{s_i} g_i(s, u(s))ds, \quad t \in (s_i, t_{i+1}], \quad i = 1, 2, \dots, m. \end{cases}$$

and

$$(\Gamma_2 u)(t)=\begin{cases} T(t)\int_{s_m}^{a} T(a-s)(f(s,u(s),u(\rho(s)))+B(s)c(s))ds \\ \quad +\int_0^t T(t-s)(f(s,u(s),u(\rho(s)))+B(s)c(s))ds \quad t\in[0,t_1]. \\ 0, \qquad t\in(t_i,s_i], \quad i=1,2,\ldots,m. \\ \int_{s_i}^t T(t-s)(f(s,u(s),u(\rho(s)))+B(s)c(s))ds, \quad t\in(s_i,t_{i+1}], \quad i=1,2,\ldots,m. \end{cases}$$

We divide the proof into several steps:
Step 1. We prove that $\Gamma u=\Gamma_1 u+\Gamma_2 u\in B_r$ for all $u\in B_r$. Indeed:
Case 1. For $t\in[0,t_1]$, we have

$$\begin{aligned}
\|(\Gamma_1 u+\Gamma_2 u)(t)\| &\le \|T(t)\|\left\|T(a-s_m)x_m+T(a-s_m+t_m)\int_{t_m}^{s_m} g_m(s,u(s))ds\right\| \\
&+\|T(t)\|\int_{s_m}^{a}\|T(a-s)\|(\|f(s,u(s),u(\rho(s)))\|+\|B(s)c(s)\|)ds \\
&+\int_0^t\|T(t-s)\|(\|f(s,u(s),u(\rho(s)))\|+\|B(s)c(s)\|)ds \\
&\le M\left[M\|x_m\|+MN_m(s_m-t_m)\right]+LM^2\int_{s_m}^{a}(1+\|u(s)\|^{\mu}+\|u(\rho(s))\|^{\nu}))ds+M^2\|Bc\|_{L^1} \\
&+LM\int_0^t(1+\|u(s)\|^{\mu}+\|u(\rho(s))\|^{\nu})ds+M\|Bc\|_{L^1} \\
&\le M^2\|x_m\|+M^2N_m(s_m-t_m)+LM^2(1+2r)(a-s_m)+M(M+1)\|Bc\|_{L^1}+LM(1+2r)t_1 \\
&= M^2\|x_m\|+M^2N_m(s_m-t_m)+M(M+1)\|Bc\|_{L^1}+LM(M(a-s_m)+t_1) \\
&+2rLM(M(a-s_m)+t_1) \\
&\le r(1-2\alpha)+2r\alpha=r.
\end{aligned}$$

Case 2. For $t\in(t_i,s_i]$, $i=1,\ldots,m$, we have

$$\begin{aligned}
\|(\Gamma_1 u+\Gamma_2 u)(t)\| &\le \|x_i\|+\|T(t_i)\|\int_{t_i}^t\|g_i(s,u(s))\|ds \\
&\le \|x_i\|+MN_i(s_i-t_i) \\
&\le \|x_i\|+MN(s_i-t_i) \\
&\le M\|x_i\|+MN(s_i-t_i)\le r.
\end{aligned}$$

Case 3. For $t\in(s_i,t_{i+1}]$, $i=1,\ldots,m$, we have

$$\begin{aligned}
\|(\Gamma_1 u+\Gamma_2 u)(t)\| &\le \|T(t-s_i)\|\|x_i\|+\|T(t-s_i+t_i)\|\int_{t_i}^{s_i}\|g_i(s,u(s))\|ds \\
&+\int_{s_i}^t\|T(t-s)\|(\|f(s,u(s),u(\rho(s)))\|+\|B(s)c(s)\|)ds \\
&\le M\|x_i\|+MN_i(s_i-t_i)+LM(1+2r)(t_{i+1}-s_i)+M\|Bc\|_{L^1} \\
&= M\|x_i\|+M(N(s_i-t_i)+\|Bc\|_{L^1})+LM(t_{i+1}-s_i)+2rLM(t_{i+1}-s_i) \\
&\le r(1-2\alpha)+2r\alpha=r.
\end{aligned}$$

Then, we infer that $\Gamma_1 u+\Gamma_2 u\in B_r$.
Step 2. Γ_1 is contraction on B_r. Let $u,v\in B_r$,

Case 1. For $t \in [0, t_1]$, we have

$$
\begin{aligned}
&\|(\Gamma_1 u)(t) - (\Gamma_1 v)(t)\| \\
&\leq \|T(t)\| \left\| T(a - s_m + t_m) \int_{t_m}^{s_m} g_m(s, u(s)) ds - T(a - s_m + t_m) \int_{t_m}^{s_m} g_m(s, v(s)) ds \right\| \\
&\leq \|T(t)\| \|T(a - s_m + t_m)\| \int_{t_m}^{s_m} \|g_m(s, u(s)) - g_m(s, v(s))\| ds \\
&\leq M^2 C_{g_m} \int_{t_m}^{s_m} \|u(s) - v(s)\| ds \\
&\leq M^2 C_{g_m} (s_m - t_m) \|u - v\|_{\mathcal{PC}} \\
&\leq \beta \|u - v\|_{\mathcal{PC}}.
\end{aligned}
$$

Case 2. For $t \in (t_i, s_i]$, $i = 1, \dots, m$, we have

$$
\begin{aligned}
\|(\Gamma_1 u)(t) - (\Gamma_1 v)(t)\| &\leq \|T(t_i)\| \int_{t_i}^{t} \|g_i(s, u(s)) - g_i(s, v(s))\| \, ds \\
&\leq M C_{g_i} (s_i - t_i) \|u - v\|_{\mathcal{PC}} \\
&\leq \beta \|u - v\|_{\mathcal{PC}}.
\end{aligned}
$$

Case 3. For $t \in (s_i, t_{i+1}]$, $i = 1, \dots, m$, we have

$$
\begin{aligned}
\|(\Gamma_1 u)(t) - (\Gamma_1 v)(t)\| &\leq \|T(t - s_i + t_i)\| \int_{t_i}^{s_i} \|g_i(s, u(s)) - g_i(s, u(s))\| ds \\
&\leq M C_{g_i} (s_i - t_i) \|u - v\|_{\mathcal{PC}} \\
&\leq \beta \|u - v\|_{\mathcal{PC}}.
\end{aligned}
$$

Which implies that Γ_1 is a contraction.
Step 3. Γ_2 is continuous.
Let $(u_n)_{n \geq 0}$ be a sequence such that $\lim_{n \to +\infty} \|u_n - u\|_{\mathcal{PC}} = 0$.
Case 1. For $t \in [0, t_1]$, we have

$$
\begin{aligned}
\|(\Gamma_2 u_n)(t) - (\Gamma_2 u)(t)\| &\leq \|T(t)\| \int_{s_m}^{a} \|T(a - s)\| \|f(s, u_n(s), u_n(\rho(s))) - f(s, u(s), u(\rho(s)))\| ds \\
&+ \int_0^t \|T(t - s)\| \|f(s, u_n(s), u_n(\rho(s))) - f(s, u(s), u(\rho(s)))\| ds \\
&\leq M^2 (a - s_m) \|f(., u_n(.), u_n(\rho(.))) - f(., u(.), u(\rho(.)))\|_{\mathcal{PC}} \\
&+ M t_1 \|f(., u_n(.), u_n(\rho(.))) - f(., u(.), u(\rho(.)))\|_{\mathcal{PC}} \\
&= M \left[M(a - s_m) + t_1 \right] \|f(., u_n(.), u_n(\rho(.))) - f(., u(.), u(\rho(.)))\|_{\mathcal{PC}}
\end{aligned}
$$

Case 2. For $t \in (t_i, s_i]$, $i = 1, \dots, m$, we have

$$\|(\Gamma_2 u_n)(t) - (\Gamma_2 u)(t)\| = 0.$$

Case 3. For $t \in (s_i, t_{i+1}]$, $i = 1, \dots, m$, we have

$$
\begin{aligned}
\|(\Gamma_2 u_n)(t) - (\Gamma_2 u)(t)\| &\leq \int_{s_i}^{t} \|T(t - s)\| \|f(s, u_n(s), u_n(\rho(s))) - f(s, u(s), u(\rho(s)))\| ds \\
&= M(t_{i+1} - s_i) \|f(., u_n(.), u_n(\rho(.))) - f(., u(.), u(\rho(.)))\|_{\mathcal{PC}}.
\end{aligned}
$$

Which implies that $\lim_{n \to +\infty} \|\Gamma_2 u_n - \Gamma_2 u\|_{\mathcal{PC}} = 0$, then we infer thet Γ_2 is continuous.
Step 4. Γ_2 is compact.

1. We have $\Gamma_2 B_r \subseteq B_r$, then Γ_2 is uniformly bounded on B_r.
2. For $u \in B_r$, we have
Case 1. For $0 \le l_1 < l_2 \le t_1$, we have

$$\begin{aligned}
&\|(\Gamma_2 u)(l_2) - (\Gamma_2 u)(l_1)\| \\
&\le \|T(l_2) - T(l_1)\| \int_{s_m}^{a} \|T(a-s)\|(\|f(s,u(s),u(\rho(s)))\| + \|B(s)c(s)\|)ds \\
&+ \int_0^{l_1} \|T(l_2 - s) - T(l_1 - s)\|(\|f(s,u(s),u(\rho(s)))\| + \|B(s)c(s)\|)ds \\
&+ \int_{l_1}^{l_2} \|T(l_2 - s)\|(\|f(s,u(s),u(\rho(s)))\| + \|B(s)c(s)\|)ds \\
&\le M^2(L(1+2r)(a-s_m) + \|Bc\|_{L^1})\|T(l_2 - l_1) - I\| \\
&+ M(L(1+2r)t_1 + \|Bc\|_{L^1})\|T(l_2 - l_1) - I\| + LM(1+2r)(l_2 - l_1) \\
&+ M \int_{l_1}^{l_2} \|B(s)c(s)\|ds \\
&= (LM(1+2r)\left[M(a-s_m) + t_1\right] + (M^2 + M)\|Bc\|_{L^1}))\|T(l_2 - l_1) - I\| \\
&+ LM(1+2r)(l_2 - l_1) + M \int_{l_1}^{l_2} \|B(s)c(s)\|ds \to 0 \text{ as } l_2 \to l_1.
\end{aligned}$$

Since $\{T(t), t \ge 0\}$ is compact, then $\|T(l_2 - l_1) - I\| \to 0$ as $l_2 \to l_1$.
Case 2. For $t_i \le l_1 < l_2 \le s_i$, $i = 1, \ldots, m$, we have

$$\|(\Gamma_2 u)(l_2) - (\Gamma_2 u)(l_1)\| = 0.$$

Case 3. For $s_i \le l_1 < l_2 \le t_{i+1}$, $i = 1, \ldots, m$, we have

$$\begin{aligned}
&\|(\Gamma_2 u)(l_2) - (\Gamma_2 u)(l_1)\| \\
&= \left\| \int_{s_i}^{l_2} T(l_2 - s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds - \right. \\
&\left. \int_{s_i}^{l_1} T(l_1 - s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds \right\| \\
&\le \int_{l_1}^{l_2} \|T(l_2 - s)\|(\|f(s,u(s),u(\rho(s)))\| + \|B(s)c(s)\|)ds \\
&+ \int_{s_i}^{l_1} \|T(l_1 - s)\|\|T(l_2 - l_1) - I\|(\|f(s,u(s),u(\rho(s)))\| + \|B(s)c(s)\|)ds \\
&\le LM(1+2r)(l_2 - l_1) + M \int_{l_1}^{l_2} \|B(s)c(s)\|ds \\
&+ M(L(1+2r)t_{i+1} + \|Bc\|_{L^1})\|T(l_2 - l_1) - I\| \to 0 \text{ as } l_2 \to l_1.
\end{aligned}$$

This permit to conclude that Γ_2 is equicontinuous.
We have $\Gamma_2 B_r \subseteq B_r$, let $\Theta := \Gamma_2 B_r$, $\Theta(t) := \Gamma_2 B_r(t) = \{(\Gamma_2 u)(t) : u \in B_r\}$ for $t \in [0, a]$.
3. $\Theta(t)$ is relatively compact. Indeed:
We have $T(t)$ is compact, hence

$$\Theta(0) = \left\{ \int_{s_m}^{a} T(a-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds \right\},$$

is relatively compact. For $0 < \epsilon < t \le a$, define

$$\Theta_\epsilon(t) := \Gamma_2^\epsilon B_r(t) = \{T(\epsilon)(\Gamma_2 u)(t-\epsilon) : u \in B_r\}.$$

Clearly, $\Theta_\epsilon(t)$ is relatively compact for $t \in (\epsilon, a]$, since $T(t)$ is compact.

Case 1. For $t \in (0, t_1]$, we have

$$\begin{aligned}\Theta_\epsilon(t) := (\Gamma_2^\epsilon u)(t) = T(\epsilon)(\Gamma_2^\epsilon u)(t-\epsilon)\\ = \Big\{T(t)\left[\int_{s_m}^{a} T(t-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds\right]\\ + \int_0^{t-\epsilon} T(t-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds : \quad u \in B_r\Big\},\end{aligned}$$

and we get

$$\begin{aligned}\|(\Gamma_2 u)(t) - (\Gamma_2^\epsilon u)(t)\| &= \Big\|\int_0^t T(t-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds - \\ &\quad \int_0^{t-\epsilon} T(t-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds\Big\|\\ &\le \int_{t-\epsilon}^{t} \|T(t-s)\|(\|f(s,u(s),u(\rho(s)))\| + \|B(s)c(s)\|)ds\\ &\le LM(1+2r)\epsilon + \int_{t-\epsilon}^{t} \|B(s)c(s)\|ds \to 0 \quad \text{as } \epsilon \to 0.\end{aligned}$$

Case 2. For $t \in (t_i, s_i]$, $i = 1, \ldots, m$, we have

$$\Theta_\epsilon(t) := \{0, u \in B_r\},$$

in this case $\|(\Gamma_2 u)(t) - (\Gamma_2^\epsilon u)(t)\| = 0$.

Case 3. For $t \in (s_i, t_{i+1}]$, $i = 1, \ldots, m$, we have

$$\Theta_\epsilon(t) := (\Gamma_2^\epsilon u)(t) = \left\{\int_{s_i}^{t-\epsilon} T(t-s)f(s,u(s),u(\rho(s)))ds : \quad u \in B_r\right\},$$

and we get

$$\begin{aligned}\|(\Gamma_2 u)(t) - (\Gamma_2^\epsilon u)(t)\| &= \Big\|\textstyle\int_{s_i}^{t} T(t-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds - \\ &\quad \textstyle\int_{s_i}^{t-\epsilon} T(t-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))ds\Big\|\\ &\le \textstyle\int_{t-\epsilon}^{t} \|T(t-s)\|(\|f(s,u(s),u(\rho(s)))\| + \|B(s)c(s)\|)ds\\ &\le LM(1+2r)\epsilon + \textstyle\int_{t-\epsilon}^{t} \|B(s)c(s)\|ds \to 0 \quad \text{as } \epsilon \to 0.\end{aligned}$$

Now, from Arzela−Ascoli theorem we can conclude that $\Gamma_2 : B_r \longrightarrow B_r$ is completely continuous. The existence of a mild solution for (**IEE**) is now a consequence of Krasnoskii's fixed point theorem. □

4. Examples

In this section, we make examples to illustrate our abstract results in the previous section. Let $X = L^2(0,1)$, $I = [0,3]$, $0 = t_0 = s_0$, $t_1 = 1$, $s_1 = 2$ and $a = 3$. Define $Av = \dfrac{\partial^2}{\partial^2 x}v$ for

$$v \in D(A) = \left\{v \in X : \quad \frac{\partial v}{\partial x},\ \frac{\partial^2 v}{\partial^2 x} \in X,\ v(0) = v(1) = 0\right\}.$$

Then A is the infinitesimal generator of strongly continuous semigroup $\{T(t), t \geq 0\}$ on X. In addition $T(t)$ is compact and $\|T(t)\| \leq 1 = M$, for all $t \geq 0$ (see [8]).

Example 4.1. Consider

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial^2 x}u(t,x) + \frac{1}{12}\cos(u(t,x) + u(t^2,x)) + c(t,x), \quad x \in (0,1),\ t \in [0,1) \cup (2,3], \\ \frac{\partial}{\partial x}u(t,0) = \frac{\partial}{\partial x}u(t,1) = 0, \quad t \in [0,1) \cup (2,3], \\ u(0,x) = u(3,x), \quad x \in (0,1), \\ u(t,x) = e^{-x} + T(1)\int_1^t \frac{1}{4}\sin(u(s,x))ds, \quad x \in (0,1),\ t \in (1,2]. \end{cases}$$

Denote $v(t)(x) = u(t,x)$ and $B(t)c(t)(x) = c(t,x)$, this problem can be abstracted into

$$(1) \begin{cases} v'(t) = Av(t) + f(t, v(t), v(\rho(t))) + B(t)c(t), \quad t \in [s_0, t_1) \cup (s_1, a], \\ v(t) = y_1 + T(1)\int_1^t g_1(s, v(s))ds, \qquad t \in (t_1, s_1], y_1 \in X \\ v(0) = v(a) \in X, \end{cases}$$

Where, $\rho(t) = t^2$, $f(t, v(t), v(\rho(t))(x) = \frac{1}{12}\cos(v(t)(x) + v(t^2)(x))$ and $g_1(t, v(t))(x) = \frac{1}{4}\sin(v(t)(x))$.

In this case, we have, $C_f = L_f = \frac{1}{12}$, $C_{g_1} = \frac{1}{4}$ and

$$\lambda = M\left[MC_{g_1}(s_1 - t_1) + (C_f + L_f)(a - s_1) + (C_f + L_f)t_1\right] = \frac{7}{12} < 1.$$

This implies that all assuptions in theorem 3.1 are satisfied. Then, there exists an unique mild solution for this problem.

Example 4.2. Consider

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial^2 x}u(t,x) + \frac{1}{8}|u(t,x)|^{\frac{1}{2}} + \frac{1}{8}|u(t^2,x)|^{\frac{1}{2}}, \quad x \in (0,1),\ t \in [0,1) \cup (2,3], \\ \frac{\partial}{\partial x}u(t,0) = \frac{\partial}{\partial x}u(t,1) = 0, \quad t \in [0,1) \cup (2,3], \\ u(0,x) = u(3,x), \quad x \in (0,1), \\ u(t,x) = y_1 x + T(1)\int_0^1\int_1^t \frac{1}{2}\frac{|u(t,x)|}{1 + |u(t,x)|}dsdx, \quad x \in (0,1),\ t \in (1,2], y_1 \in X. \end{cases}$$

This problem can be abstracted into (1), with $\rho(t) = t^2$,

$$f(t, v(t), v(\rho(t))(x) = \frac{1}{8}|v(t)(x)|^{\frac{1}{2}} + \frac{1}{8}|v(t^2)(x)|^{\frac{1}{2}} \quad \text{and} \quad g_1(t, v(t))(x) = \frac{1}{2}\int_0^1 \frac{|v(t)(x)|}{1 + |v(t)(x)|}dx,$$

In this case, we have $L = \frac{1}{8}$, $N_1 = C_{g_1} = \frac{1}{2}$, $M = 1$, $\alpha = \frac{1}{4} < \frac{1}{2}$ and $\beta = \frac{1}{8} < 1$.

This implies that all assuptions in theorem 3.2 are satisfied. Then, this problem has at least one mild solution.

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