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New integral inequalities for s-preinvex functions

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Abstract

In this note, we give some estimate of the generalized quadrature formula of Gauss-Jacobi

$$\int_{a}^{a+\eta(b,a)} (x-a)^{p} (a+\eta(b,a)-x)^{q} f(x)\, dx$$

in the cases where f and $|f|^{\lambda}$ for $\lambda > 1$, are s-preinvex functions in the second sense.

Keywords: integral inequality; s-preinvex function; Hölder inequality; power mean inequality.
2010 MSC: Primary 26D15; Secondary 26D20, 26A51.

1. Introduction

It is well known that the convexity plays an important and very central role in many areas, such as economics, finances, optimization, and game theory. Due to its diverse applications this concept has been extended and generalized in several directions.

Now we recall some definitions: A nonempty closed set I in $\mathbb{R}$ is said to be invex at x with respect to η, if $x + t\eta(y,x) \in I$ holds for all $x, y \in I$ and $t \in [0,1]$, where η is a continuous bi-function $\eta : I \times I \to \mathbb{R}$, if the set I is invex at each $x \in I$, then it is said that I is an invex set, see [17]. We said that a nonnegative function $f : I \subset [0,\infty) \to [0,\infty)$ is said to be s-peinvex in the second sense with respect to η, for some fixed $s \in (0,1]$, if the following inequality $f(x + t\eta(y,x)) \leq (1-t)^{s} f(x) + t^{s} f(y)$ holds for all $x, y \in I$ and $t \in [0,1]$, see [16]. The special cases which can be derived from s-preinvexity are as follows: the preinvex functions see [17] are the functions which satisfy the above inequality for $s = 1$. And the case where $s = 0$, recapture the

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concept of P-preinvex functions [11]. We also recall that the prequasiinvex functions are those that satisfy the following inequality $f(x+t\eta(y,x)) \le \max\{f(x), f(y)\}$ see [14].

The generalized quadrature formula of Gauss-Jacobi type has the following form

$$\int_a^b (x-a)^p (b-x)^q f(x)\,dx = \sum_{k=0}^{m} B_{m,k} f(\gamma_k) + \Re_m[f], \tag{1.1}$$

where $B_{m,k}$ are the Christoffel coefficients, γ_k are the roots of the Jacobi polynomial of degree m, and $\Re_m[f]$ is the remainder term, see[15].

Özdemir et al. [13] gave the estimate of the left hand sides of equality (1.1) when the function f is quasi-convex on $[a,b] \subset \mathbb{R}^+$ with $0 \le a < b < \infty$, as follows

$$\int_a^b (x-a)^p (b-x)^q f(x)\,dx \le (b-a)^{p+q+1} \beta(p+1,q+1) \max\{f(a), f(b)\}.$$

Ahmad [1] gave the estimates of the left hand side of the equality (1.1).

The first result concern the case where $|f|$ is P-preinvex function, is

$$\int_a^{a+\eta(b,a)} (x-a)^p (a+\eta(b,a)-x)^q f(x)\,dx \le (\eta(b,a))^{p+q+1} \beta(p+1,q+1) (|f(a)| + |f(b)|).$$

And obtained the following results when a certain power of modulus of f be P-preinvex function

$$\begin{aligned}\int_a^{a+\eta(b,a)} (x-a)^p (a+\eta(b,a)-x)^q f(x)\,dx &\le (\eta(b,a))^{p+q+1} [\beta(kp+1,kq+1)]^{\frac{1}{k}} \\ &\quad \times \left(|f(a)|^{\frac{k}{k-1}} + |f(b)|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}, k>1.\end{aligned}$$

And

$$\begin{aligned}\int_a^{a+\eta(b,a)} (x-a)^p (a+\eta(b,a)-x)^q f(x)\,dx &\le (\eta(b,a))^{p+q+1} \beta(p+1,q+1) \\ &\quad \times \left(|f(a)|^{l} + |f(b)|^{l}\right), l>1.\end{aligned}$$

He also obtained the following inequalities for prequasiinvex functions

$$\int_a^{a+\eta(b,a)} (x-a)^p (a+\eta(b,a)-x)^q f(x)\,dx \le (\eta(b,a))^{p+q+1} \beta(p+1,q+1) \times \max\{|f(a)|, |f(b)|\}.$$

And when certain power of $|f|$ is a prequasiinvex function

$$\begin{aligned}\int_a^{a+\eta(b,a)} (x-a)^p (a+\eta(b,a)-x)^q f(x)\,dx &\le (\eta(b,a))^{p+q+1} [\beta(kp+1,kq+1)]^{\frac{1}{k}} \\ &\quad \max\left\{|f(a)|^{\frac{k}{k-1}}, |f(b)|^{\frac{k}{k-1}}\right\}^{\frac{k-1}{k}}, k>1.\end{aligned}$$

And

$$\int_a^{a+\eta(b,a)} (x-a)^p (a+\eta(b,a)-x)^q f(x)\,dx \leq (\eta(b,a))^{p+q+1} \beta(p+1,q+1) \times \max\left\{|f(a)|^l, |f(b)|^l\right\}^{\frac{1}{l}}, l>1.$$

In [6] Liu discussed the cases where certain power of the modulus of the function f is quasi-convex, and (α, m)-convex. Also he established in [7] the estimate of the left hand side of (1.1) in the case where $|f|$, and $|f|^l$ are P-convex functions. Iscan et al. [3] treated the equality (1.1) in the cases where f and $|f|^\lambda$ are harmonically convex functions. In [9] Muddassar et al. discussed the equality (1.1) in the cases where $|f|, |f|^{\frac{k}{k-1}}$ and $|f|^l$ are s-(α, m)-convex functions. In [10] Noor et al. established the estimate of left hand side of (1.1) for strongly generalized harmonic convex function with modulus $c>0$. About some recent papers related to this subject, one can see [2, 4, 5, 8, 12].

Motivated by the above results, in the present note we extend the result obtained by Ahmad [1] for s-preinvex functions in the second sense.

2. Main results

In order to prove the results we need the following Lemma

Lemma 2.1. [1] Let $f : S = [a, a+\eta(b,a)] \to \mathbb{R}$ be continuous function on the interval of real numbers S°(interior of S) with $\eta(b,a) > 0$ $[a, a+\eta(b,a)]$ such that $f \in L([a, a+\eta(b,a)])$. Then the equality

$$\int_a^{a+\eta(b,a)} (x-a)^p (a+\eta(b,a)-x)^q f(x)\,dx = (\eta(b,a))^{p+q+1} \int_0^1 (1-t)^q t^p f(a+t\eta(b,a))\,dt$$

holds for some fixed $p, q > 0$.

Theorem 2.2. *Let $f : [a, a+\eta(b,a)] \subset [0,\infty) \to [0,\infty)$ be integrable function on $[a, a+\eta(b,a)]$ with $\eta(b,a) > 0$. If f is s-preinvex in the second sense for some fixed $s \in (0,1]$ and $p, q > 0$, then we have*

$$\int_a^{a+\eta(b,a)} (x-a)^p (a+\eta(b,a)-x)^q f(x)\,dx$$
$$\leq (\eta(b,a))^{p+q+1} \left[f(a)\beta(p+1, q+s+1) + f(b)\beta(p+s+1, q+1)\right]. \tag{2.1}$$

Proof . From Lemma 2.1, and s-preinvexity in the second sense of f, we have

$$\begin{aligned}\int_a^{a+\eta(b,a)} (x-a)^p (a+\eta(b,a)-x)^q f(x)\,dx &= (\eta(b,a))^{p+q+1} \int_0^1 (1-t)^q t^p f(a+t\eta(b,a))\,dt \\ &\leq (\eta(b,a))^{p+q+1} \left(f(a)\int_0^1 (1-t)^{q+s} t^p dt + f(b)\int_0^1 (1-t)^q t^{p+s} dt\right) \\ &= (\eta(b,a))^{p+q+1} \left(f(a)\beta(p+1,q+s+1) + f(b)\beta(p+s+1,q+1)\right),\end{aligned}$$

which is the desired results. □

Theorem 2.3. *Let $f : [a, a+\eta(b,a)] \subset [0,\infty) \to [0,\infty)$ be integrable function on $[a, a+\eta(b,a)]$ with $\eta(b,a) > 0$ and let $\lambda > 1$. If $|f|^{\lambda}$ is s-preinvex in the second sense for some fixed $s \in (0,1]$ and $p, q > 0$, then we have*

$$\begin{aligned}
&\int_{a}^{a+\eta(b,a)} (x-a)^{p} (a+\eta(b,a)-x)^{q} f(x)\,dx \\
&\le (\eta(b,a))^{p+q+1} (\beta(p+1,q+1))^{1-\frac{1}{\lambda}} \times \Big(|f(a)|^{\lambda} \ \beta(p+1,q+s+1) \\
&+ |f(b)|^{\lambda} \beta(p+s+1,q+1) \Big)^{\frac{1}{\lambda}}.
\end{aligned} \tag{2.2}$$

Proof . From Lemma 2.1, properties of modulus, and power mean inequality, we have

$$\begin{aligned}
&\int_{a}^{a+\eta(b,a)} (x-a)^{p} (a+\eta(b,a)-x)^{q} f(x)\,dx \\
&\le (\eta(b,a))^{p+q+1} \left(\int_{0}^{1} (1-t)^{q} t^{p} dt \right)^{1-\frac{1}{\lambda}} \left(\int_{0}^{1} (1-t)^{q} t^{p} |f(a+t\eta(b,a))|^{\lambda} dt \right)^{\frac{1}{\lambda}} \\
&= (\eta(b,a))^{p+q+1} (\beta(p+1,q+1))^{1-\frac{1}{\lambda}} \left(\int_{0}^{1} (1-t)^{q} t^{p} |f(a+t\eta(b,a))|^{\lambda} dt \right)^{\frac{1}{\lambda}}.
\end{aligned} \tag{2.3}$$

Since $|f|^{\lambda}$ is s-preinvex in the second sense, we get

$$\begin{aligned}
&\int_{a}^{a+\eta(b,a)} (x-a)^{p} (a+\eta(b,a)-x)^{q} f(x)\,dx \\
&\le (\eta(b,a))^{p+q+1} (\beta(p+1,q+1))^{1-\frac{1}{\lambda}} \\
&\times \left(|f(a)|^{\lambda} \int_{0}^{1} (1-t)^{q+s} t^{p} dt + |f(b)|^{\lambda} \int_{0}^{1} (1-t)^{q} t^{p+s} dt \right)^{\frac{1}{\lambda}} \\
&= (\eta(b,a))^{p+q+1} (\beta(p+1,q+1))^{1-\frac{1}{\lambda}} \\
&\times \Big(|f(a)|^{\lambda} \ \beta(p+1,q+s+1) + |f(b)|^{\lambda} \beta(p+s+1,q+1) \Big)^{\frac{1}{\lambda}},
\end{aligned}$$

which is the desired result. $\square$

Theorem 2.4. *Suppose that all the assumptions of Theorem 2.3 are satisfied, then we have*

$$\begin{aligned}
&\int_{a}^{a+\eta(b,a)} (x-a)^{p} (a+\eta(b,a)-x)^{q} f(x)\,dx \\
&\le \frac{(\eta(b,a))^{p+q+1}}{(s+1)^{\frac{1}{\lambda}}} \times \Big(|f(a)|^{\lambda} \ + |f(b)|^{\lambda} \Big)^{\frac{1}{\lambda}} \left(\beta\left(\tfrac{p\lambda}{\lambda-1}+1, \tfrac{q\lambda}{\lambda-1}+1 \right) \right)^{1-\frac{1}{\lambda}}.
\end{aligned} \tag{2.4}$$

Proof . From Lemma 2.1, properties of modulus, and Hölder inequality, we have

$$\begin{aligned}
&\int_a^{a+\eta(b,a)} (x-a)^p \left(a+\eta(b,a)-x\right)^q f(x)\,dx \\
&\le (\eta(b,a))^{p+q+1} \left(\int_0^1 (1-t)^{\frac{q\lambda}{\lambda-1}} t^{\frac{p\lambda}{\lambda-1}} dt\right)^{1-\frac{1}{\lambda}} \left(\int_0^1 \left|f\left(a+t\eta(b,a)\right)\right|^\lambda dt\right)^{\frac{1}{\lambda}} \\
&= (\eta(b,a))^{p+q+1} \left(\beta\left(\tfrac{p\lambda}{\lambda-1}+1, \tfrac{q\lambda}{\lambda-1}+1\right)\right)^{1-\frac{1}{\lambda}} \left(\int_0^1 \left|f\left(a+t\eta(b,a)\right)\right|^\lambda dt\right)^{\frac{1}{\lambda}}.
\end{aligned} \tag{2.5}$$

Since $|f|^\lambda$is s-preinvex in the second sense, we get

$$\begin{aligned}
&\int_a^{a+\eta(b,a)} (x-a)^p \left(a+\eta(b,a)-x\right)^q f(x)\,dx \\
&\le (\eta(b,a))^{p+q+1} \left(\beta\left(\tfrac{p\lambda}{\lambda-1}+1, \tfrac{q\lambda}{\lambda-1}+1\right)\right)^{1-\frac{1}{\lambda}} \\
&\times \left(\int_0^1 \left((1-t)^s |f(a)|^\lambda + t^s |f(b)|^\lambda\right) dt\right)^{\frac{1}{\lambda}} \\
&= \frac{(\eta(b,a))^{p+q+1}}{(s+1)^{\frac{1}{\lambda}}} \left(|f(a)|^\lambda + |f(b)|^\lambda\right)^{\frac{1}{\lambda}} \left(\beta\left(\tfrac{p\lambda}{\lambda-1}+1, \tfrac{q\lambda}{\lambda-1}+1\right)\right)^{1-\frac{1}{\lambda}},
\end{aligned}$$

which is the desired result. □

Corollary 2.5. *Suppose that all the assumptions of Theorem* 2.4 *are satisfied, then we have*

$$\begin{aligned}
&\int_a^{a+\eta(b,a)} (x-a)^p \left(a+\eta(b,a)-x\right)^q f(x)\,dx \\
&\le \frac{(\eta(b,a))^{p+q+1}}{(s+1)^{\frac{1}{\lambda}}} \left(\beta\left(\tfrac{p\lambda}{\lambda-1}+1, \tfrac{q\lambda}{\lambda-1}+1\right)\right)^{1-\frac{1}{\lambda}} \left(|f(a)|+|f(b)|\right).
\end{aligned} \tag{2.6}$$

Proof . From Theorem 2.4 we have

$$\begin{aligned}
&\int_a^{a+\eta(b,a)} (x-a)^p \left(a+\eta(b,a)-x\right)^q f(x)\,dx \\
&\le \frac{(\eta(b,a))^{p+q+1}}{(s+1)^{\frac{1}{\lambda}}} \times \left(\beta\left(\tfrac{p\lambda}{\lambda-1}+1, \tfrac{q\lambda}{\lambda-1}+1\right)\right)^{1-\frac{1}{\lambda}} \left(|f(a)|^\lambda + |f(b)|^\lambda\right)^{\frac{1}{\lambda}}.
\end{aligned}$$

Using the following algebraic inequality

$$(a+b)^\alpha \le a^\alpha + b^\alpha$$

for all $a \ge 0$ and $b \ge 0$ and $0 \le \alpha \le 1$, we get the desired result. □

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