Int. J. Nonlinear Anal. Appl. 8 (2017) No. 1, 337-351
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.1355.1334

Some extensions of Darbo's theorem and solutions of integral equations of Hammerstein type

Reza Allahyari[a,*], Asadollah Aghajani[b]

[a]*Department of Mathematics, Mashhad Branch, Islamic Azad University,mashhad, Iran*
[b]*School of Mathematics, Iran University of Science and Technology, Narmark, Tehran 16846 13114, Iran*

(Communicated by M. Eshaghi)

Abstract

In this brief note, using the technique of measures of noncompactness, we give some extensions of Darbo fixed point theorem. Also we prove an existence result for a quadratic integral equation of Hammerstein type on an unbounded interval in two variables which includes several classes of nonlinear integral equations of Hammerstein type. Furthermore, an example is presented to show the efficiency of our result.

Keywords: Measure of noncompactness; Quadratic integral equation; Darbo fixed point theorem.
2010 MSC: Primary 47H09; Secondary 47H10.

---

1. Introduction

Applications of measures of noncompactness to nonlinear differential and integral equations were considered by many investigators and some basic results have been obtained [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29]. Banaś, O'Regan and Sadarangani [10] studied the existence and behavior of solutions of a quadratic Hammerstein integral equation on an unbounded interval having the form

$$x(t) = p(t) + f(t, x(t)) \int_0^\infty g(t, \tau)h(\tau, x(\tau))d\tau, \ t \geq 0. \tag{1.1}$$

Eq.(1.1) is a generalization of the following classical Hammerstein integral equation on an unbounded interval

$$x(t) = p(t) + \int_0^\infty g(t, \tau)h(\tau, x(\tau))d\tau, \ t \geq 0.$$

---

[*] Corresponding author

Email addresses: `rezaallahyari@mshdiau.ac.ir` (Reza Allahyari), `aghajani@iust.ac.ir` (Asadollah Aghajani)

Received: April 2016 *Revised:* January 2017

In this paper, we study existence of solutions of the following nonlinear quadratic Hammerstein integral equation

$$x(t,s)=f\bigg(t,s,x(t,s),\int_0^\infty\int_0^\infty g(t,s,v,w)h(v,w,x(v,w))dvdw\bigg),\quad t,s\geq 0 \tag{1.2}$$

which will be considered on a Banach space of all bounded and continuous real functions on $\mathbb{R}_+\times\mathbb{R}_+$. This equation is a general form of the nonlinear quadratic Hammerstein integral equation on an unbounded interval in two variables. The principal tools employed in this paper are the method of measure of noncompactness and some extensions of Darbo fixed point theorem that we will prove. Please note that the gist of my paper is to use some new extensions of Darbo fixed point theorem because the solvability of the functional integral equation on the space $BC(\Omega)$ $(\Omega\subseteq\mathbb{R}^n)$ has been investigated (see [6, 7]). Also we provide an example in order to illustrate the efficiency of our main results.

2. Notation and auxiliary facts

In this section, we assume that E is an infinite dimensional Banach Space. If X is a subset of E then the symbols $\overline{X}$, $ConvX$ denote the closure and closed convex hull of X, respectively. Moreover, we indicate by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ the subfamily consisting of all relatively compact subsets of E.

Definition 2.1. [13] A mapping $\mu:\mathfrak{M}_E\longrightarrow\mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:
(A_1) The family $Ker\mu=\{X\in\mathfrak{M}_E:\mu(X)=0\}$ is nonempty and $Ker\mu\subseteq\mathfrak{N}_E$.
(A_2) $X\subset Y\Longrightarrow\mu(X)\leq\mu(Y)$.
(A_3) $\mu(\overline{X})=\mu(X)$.
(A_4) $\mu(ConvX)=\mu(X)$.
(A_5) $\mu(\lambda X+(1-\lambda)Y)\leq\mu(X)+(1-\lambda)\mu(Y)\ for\ \lambda\in[0,1]$.
(A_6) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1}\subseteq X_n$, $(n\geq1)$ and if $\lim\limits_{n\longrightarrow\infty}\mu(X_n)=0$ then the intersection set $X_\infty=\bigcap_{n=1}^\infty X_n$ is nonempty. The family $Ker\mu$ described in (A_1) is said to be the kernel of the measure of noncompactness μ. Observe that the intersection set X_∞ from (A_4) is a member of the family $Ker\mu$. In fact, Since $\mu(X_\infty)\leq\mu(X_n)$ for any n, we infer that $\mu(X_\infty)=0$. This yields that $X_\infty\in Ker\mu$.

Let $BC(\mathbb{R}_+\times\mathbb{R}_+)$ be the Banach space of all bounded and continuous functions on $\mathbb{R}_+\times\mathbb{R}_+$ equipped with the standard norm

$$\|x\|=\sup\{|x(t,s)|:t,s\geq0\}.$$

For any nonempty bounded subset X of $BC(\mathbb{R}_+\times\mathbb{R}_+)$, $x\in X$, $T>0$ and $\varepsilon>0$, let

$$\begin{aligned}&\omega^T(x,\varepsilon)=\sup\{|x(t,s)-x(u,v)|:t,s,u,v\in[0,T],\ |t-u|\leq\varepsilon,\ |s-v|\leq\varepsilon\}\\&\omega^T(X,\varepsilon)=\sup\left\{\omega^T(x,\varepsilon):x\in X\right\},\\&\omega_0^T(X)=\lim\nolimits_{\varepsilon\to0}\omega^T(X,\varepsilon),\\&\omega_0(X)=\lim\nolimits_{T\to\infty}\omega_0^T(X),\\&X(t,s)=\{x(t,s):x\in X\}\end{aligned}$$

and

$$\mu(X)=\omega_0(X)+\Gamma(X), \tag{2.1}$$

where

$$\Gamma(X)=\lim_{T\to\infty}\{\sup_{x\in X}\{\sup\{|x(t,s)|:t,s\geq T\}\}\}.$$

Similar to [11] (cf. also [13]), it can be shown that the function μ is a measure of noncompactness in the space $BC(\mathbb{R}_+\times\mathbb{R}_+)$ (in the sense of Definition 2.1).

On the other hand, we recall two important theorems playing a key role in fixed point theory (cf. [1, 9]).

Theorem 2.2. (Schauder [1]) Let C be a closed, convex subset of a Banach space E. Then every compact, continuous map $F:C\longrightarrow C$ has at least one fixed point in the set C.

Theorem 2.3. (Darbo [13]) Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T:C\longrightarrow C$ be a continuous mapping. Assume that there exists a constant $k\in[0,1)$ such that

$$\mu(T(X))\leq k\mu(X)$$

for any subset X of C, then T has a fixed point in the set C.

3. The main results

This section is devoted to prove some extensions of Darbo's theorem using control functions.

Definition 3.1. [22] Let $\Re$ denote the class of those functions $\beta:\mathbb{R}_+\longrightarrow[0,1)$ which satisfy the condition $\beta(t_n)\longrightarrow 1$ implies $t_n\longrightarrow 0$.

Let Ψ denote the class of functions $\psi:\mathbb{R}_+=[0,\infty)\longrightarrow\mathbb{R}_+$ satisfying the following conditions:
(a) ψ is nondecreasing.
(b) ψ is continuous.
(c) $\psi(t)=0\Longrightarrow t=0$.
Using this class, we prove the following main theorem.

Theorem 3.2. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and $T:C\longrightarrow C$ be a continuous function satisfying*

$$\psi(\mu(T(X)))\leq\beta(\mu(X))\psi(\mu(X)) \tag{3.1}$$

for any subset X of C, where μ is an arbitrary measure of noncompactness, $\beta\in\Re$ and $\psi\in\Psi$. Then T has at least one fixed point in C.

Proof . By induction, we define a sequence $\{C_n\}$ by letting $C_0=C$ and $C_n=Conv(TC_{n-1})$, $n\geq 1$. Then we have

$$TC_0=TC\subseteq C=C_0, C_1=Conv(TC_0)\subseteq C=C_0$$

and by continuing this process we obtain

$$C_0\supseteq C_1\supseteq C_2\supseteq\cdots.$$

If there exists an integer $N \geq 0$ such that $\mu(C_N) = 0$, then C_N is relatively compact and since $TC_N \subseteq Conv(TC_N) = C_{N+1} \subseteq C_N$, Theorem 2.2 implies that T has a fixed point. So we assume that $\mu(C_n) \neq 0$ for $n \geq 0$. From (3.1) we have

$$\begin{aligned}\psi(\mu(C_{n+1})) &= \psi(\mu(Conv(TC_n))) \\ &= \psi(\mu(TC_n)) \\ &\leq \beta(\mu(C_n))\psi(\mu(C_n)) \\ &< \psi(\mu(C_n)). \end{aligned} \tag{3.2}$$

Since ψ is nondecreasing, so $\mu(C_n)$ is a positive decreasing sequence of real numbers, thus, there is an $r \geq 0$ such that $\mu(C_n) \longrightarrow r$ as $n \longrightarrow \infty$. We show that $r = 0$. Suppose, to the contrary, that $r \neq 0$. Then from (3.2) we obtain

$$\frac{\psi(\mu(C_{n+1}))}{\psi(\mu(C_n))} \leq \beta(\mu(C_n)) < 1,$$

for every $n \geq 0$. From the continuity of ψ we have $\lim_{n\to\infty} \frac{\psi(\mu(C_{n+1}))}{\psi(\mu(C_n))} = \frac{\psi(r)}{\psi(r)} = 1$, thus the above inequalities imply that

$$\beta(\mu(C_n)) \longrightarrow 1 \ \ as \ \ n \longrightarrow \infty.$$

Since $\beta \in \Re$ we obtain $\mu(C_n) \longrightarrow 0$, as $n \longrightarrow \infty$, a contradiction. Thus $r = 0$. On the other hand, since $C_{n+1} \subseteq C_n$ and $TC_n \subseteq C_n$ for all $n \geq 1$, then from condition (A_6) of Definition 2.1, $C_\infty = \bigcap_{n=1}^{\infty} C_n$ is a nonempty convex closed set, invariant under T and belongs to $Ker\mu$. Now Theorem 2.2 completes the proof. □

Corollary 3.3. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and $T : C \longrightarrow C$ be a continuous function satisfying*

$$\psi(\mu(T(X))) \leq \varphi(\mu(X))$$

for any subset X of C, where μ is an arbitrary measure of noncompactness and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a nondecreasing and upper semicontinuous function such that $\varphi(t) < \psi(t)$ whenever $t > 0$, $\psi \in \Psi$ and $\lim\limits_{t\longrightarrow\infty} \psi(t) = \infty$. Then T has at least one fixed point in set C.

Proof . Define

$$\Lambda(t) = \begin{cases} \varphi(t) & \text{for } 0 \leq t \leq \mu(C), \\ \\ \mu(C) & \text{for } t > \mu(C), \end{cases}$$

and $\beta(t) = \frac{\Lambda(t)}{\psi(t)}$ *for* $t > 0$ and $\beta(0) = \frac{1}{2}$. To see that $\beta(t)$ is in the class $\Re$, suppose $\beta(t_n) \longrightarrow 1$. Then $\{t_n\}$ must be bounded (otherwise, since $\lim\limits_{t_n\longrightarrow\infty} \psi(t_n) = \infty$, consequently $\beta(t_n) \longrightarrow 0$) and has a convergent subsequence say t_{n_k}. Now, we may assume that $t_{n_k} \longrightarrow t_0$. But since φ is upper semicontinuous, therefore

$$\psi(t_0) = \limsup_{k\longrightarrow\infty} \psi(t_{n_k}) = \limsup_{k\longrightarrow\infty} \varphi(t_{n_k}) \leq \varphi(t_0).$$

Now since $\varphi(t) < \psi(t)$ for $t > 0$, this implies that $t_0 = 0$, i.e., $t_{n_k} \longrightarrow 0$. So any convergent subsequence of the original sequence $\{t_n\}$ must converge to 0. It follows that $t_n \longrightarrow 0$, proving that β is in the class $\Re$. On the other hand, since

$$\psi(\mu(T(X))) \leq \varphi(\mu(X)) = \beta(\mu(X))\psi(\mu(X)),$$

by using Theorem 3.2 the proof is complete. □

Corollary 3.4. (Darbo [13]) *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : C \longrightarrow C$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that*

$$\mu(T(X)) \leq k\mu(X)$$

for any subset X of C, then T has a fixed point in set C.

Proof . In Theorem 3.2, by taking $\psi(t) = t$ and $\beta(t) = k$ where $0 \leq k < 1$, we get Corollary 3.4. □

4. Application

In this section, we study the nonlinear quadratic Hammerstein integral equation (1.2) with the following assumptions:

(i) $f : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. Moreover there exists a nondecreasing continuous function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\varphi(t) < t$ for all $t > 0$, $\varphi(0) = 0$, $\varphi(t) + \varphi(s) \leq \varphi(t + s)$ and a continuous function $m(t, s) : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$|f(t, s, x, y) - f(t, s, u, z)| \leq \varphi(|x - u|) + m(t, s)|y - z|$$

for all $x, y, u, z \in \mathbb{R}$ and for any $t, s \in \mathbb{R}_+$.

(ii) $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function.

(iii) $h : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exist a continuous function $a : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and a continuous and nondecreasing function $b : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$|h(t, s, x)| \leq a(t, s)b(|x|)$$

for $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Also the function $(v, w) \longrightarrow a(v, w)|g(t, s, v, w)|$ is integrable over $\mathbb{R}_+ \times \mathbb{R}_+$ for any fixed $t, s \in \mathbb{R}_+$.

(iv) The function $D : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ defined by the formula

$$D(t, s) = \int_0^\infty \int_0^\infty a(v, w)|g(t, s, v, w)|dvdw$$

is bounded on $\mathbb{R}_+ \times \mathbb{R}_+$, and

$$\overline{D} = \sup\{D(t, s) : t, s \in \mathbb{R}_+\} < \infty.$$

Moreover, $\lim_{t,s\longrightarrow\infty} f(t, s, 0, 0) = 0$ and

$$K = \sup\{f(t, s, 0, 0), t, s \in \mathbb{R}_+\} < \infty.$$

(v) The function $M : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ defined by the formula

$$M(t, s) = m(t, s) \int_0^\infty \int_0^\infty a(v, w)|g(t, s, v, w)|dvdw$$

is bounded on $\mathbb{R}_+ \times \mathbb{R}_+$, $\lim_{t,s\longrightarrow\infty} M(t, s) = 0$ and

$$\overline{M} = \sup\{M(t, s) : t, s \in \mathbb{R}_+\} < \infty.$$

(vi) The following equalities are hold:

$$\lim_{T\longrightarrow\infty}\left\{\sup\{m(t,s)\int_T^\infty\int_0^T a(v,w)|g(t,s,v,w)|dvdw : t,s\in\mathbb{R}_+\}\right\}=0,$$

$$\lim_{T\longrightarrow\infty}\left\{\sup\{m(t,s)\int_T^\infty\int_T^\infty a(v,w)|g(t,s,v,w)|dvdw : t,s\in\mathbb{R}_+\}\right\}=0,$$

$$\lim_{T\longrightarrow\infty}\left\{\sup\{m(t,s)\int_0^T\int_T^\infty a(v,w)|g(t,s,v,w)|dvdw : t,s\in\mathbb{R}_+\}\right\}=0.$$

(vii) There exists a positive solution r_0 of the inequality

$$\varphi(r)+b(r)\overline{M}+K\le r.$$

Theorem 4.1. *Under the assumptions* (i) − (vii), *Eq.* (1.2) *has at least one solution in the space* $BC(\mathbb{R}_+\times\mathbb{R}_+)$.

Proof . Consider the operator H defined on the space $BC(\mathbb{R}_+\times\mathbb{R}_+)$ by the formula

$$H(x)(t,s)=f\left(t,s,x(t,s),\int_0^\infty\int_0^\infty g(t,s,v,w)h(v,w,x(v,w))dvdw\right),\quad t,s\ge 0.$$

In view of the imposed assumptions we have that the function $H(x)$ is continuous on $\mathbb{R}_+\times\mathbb{R}_+$. Further, for arbitrarily fixed function $x\in BC(\mathbb{R}_+\times\mathbb{R}_+)$, using our assumptions, we obtain

$$\begin{aligned}\left|(Hx)(t,s)\right| &\le \Bigg|f\left(t,s,x(t,s),\int_0^\infty\int_0^\infty g(t,s,v,w)h(v,w,x(v,w))dvdw\right)\\ &\quad -f\left(t,s,0,0\right)\Bigg|+\left|f\left(t,s,0,0\right)\right|\\ &\le \varphi(|x(t,s)|)+m(t,s)\left|\int_0^\infty\int_0^\infty g(t,s,v,w)h(v,w,x(v,w))dvdw\right|+|f(t,s,0,0)|\\ &\le \varphi(|x(t,s)|)+m(t,s)\int_0^\infty\int_0^\infty |g(t,s,v,w)||h(v,w,x(v,w))|dvdw+|f(t,s,0,0)|\\ &\le \varphi(|x(t,s)|)+m(t,s)\int_0^\infty\int_0^\infty |g(t,s,v,w)|a(v,w)b|x(v,w)|dvdw+|f(t,s,0,0)|\\ &\le \varphi(|x(t,s)|)+b(\|x\|)m(t,s)\int_0^\infty\int_0^\infty |g(t,s,v,w)|a(v,w)dvdw+|f(t,s,0,0)|\\ &= \varphi(|x(t,s)|)+b(\|x\|)M(t,s)+|f(t,s,0,0)|.\end{aligned}\tag{4.1}$$

Hence by (iv), (v) we have

$$\|Hx\|\le\varphi(\|x\|)+b(\|x\|)\overline{M}+K.\tag{4.2}$$

Now, H is well defined and the estimate (4.2) yields H transforms the ball B_{r_0} into itself where r_0 is a constant appearing in assumption (vii). We also show that the map $H: B_{r_0}\longrightarrow B_{r_0}$ is continuous.

To this end fix an arbitrary number $\varepsilon > 0$. Then, for $x, y \in B_{r_0}$ such that $\|x - y\| \leq \varepsilon$, we obtain

$$\begin{aligned}
&\Big|(Hx)(t,s) - (Hy)(t,s)\Big| \leq \Big|f\Big(t, s, x(t,s), \int_0^\infty \int_0^\infty g(t,s,v,w)h(v,w,x(v,w))dvdw\Big)\\
&-f\Big(t, s, y(t,s), \int_0^\infty \int_0^\infty g(t,s,v,w)h(v,w,y(v,w))dvdw\Big)\Big|\\
\leq\ & \varphi(|x(t,s) - y(t,s)|)\\
&+m(t,s)|\int_0^\infty \int_0^\infty g(t,s,v,w)\{h(v,w,x(v,w) - h(v,w,y(v,w)\}dvdw|\\
\leq\ & \varphi(|x(t,s) - y(t,s)|)\\
&+m(t,s)\int_0^\infty \int_0^\infty |g(t,s,v,w)|[|h(v,w,x(v,w)| + |h(v,w,y(v,w)|]dvdw\\
\leq\ & \varphi(|x(t,s) - y(t,s)|) + m(t,s)\int_0^\infty \int_0^\infty |g(t,s,v,w)|(b(\|x\|) + b(\|y\|))a(v,w)dvdw\\
\leq\ & \varphi(|x(t,s) - y(t,s)|) + m(t,s)\int_0^\infty \int_0^\infty 2|g(t,s,v,w)|b(r_0)a(v,w)dvdw\\
\leq\ & \varphi(|x(t,s) - y(t,s)|) + 2b(r_0)m(t,s)\int_0^\infty \int_0^\infty |g(t,s,v,w)|a(v,w)dvdw\\
\leq\ & \varphi(|x(t,s) - y(t,s)|) + 2b(r_0)M(t,s).
\end{aligned}$$

Furthermore, considering assumption (v) there exists $T > 0$ such that for $t, s \geq T$ we have

$$\Big|(Hx)(t,s) - (Hy)(t,s)\Big| \leq \varphi(\varepsilon) + 2b(r_0)\frac{\varepsilon}{2b(r_0)} \leq \varepsilon + \varepsilon = 2\varepsilon.$$

Now, we assume that $t, s \in [0, T]$ and applying the assumptions, we have:

$$\begin{aligned}
&\Big|(Hx)(t,s) - (Hy)(t,s)\Big| \leq \Big|f\Big(t, s, x(t,s), \int_0^\infty \int_0^\infty g(t,s,v,w)h(v,w,x(v,w))dvdw\Big)\\
&-f\Big(t, s, y(t,s), \int_0^\infty \int_0^\infty g(t,s,v,w)h(v,w,y(v,w))dvdw\Big)\Big|\\
\leq\ & \varphi(|x(t,s) - y(t,s)|) + m(t,s)|\int_0^\infty \int_0^\infty g(t,s,v,w)\{h(v,w,x(v,w) - h(v,w,y(v,w)\}dvdw|\\
\leq\ & \varphi(\varepsilon) + m(t,s)\Bigg\{\int_0^\infty \Bigg\{\int_0^T |g(t,s,v,w)||h(v,w,x(v,w)) - h(v,w,y(v,w))|dv\\
&+\int_T^\infty |g(t,s,v,w)|[|h(v,w,x(v,w))| + h(v,w,y(v,w))|]dv\Bigg\}dw\Bigg\}\\
\leq\ & \varepsilon + m(t,s)\int_0^T \int_0^T |g(t,s,v,w)||h(v,w,x(v,w)) - h(v,w,y(v,w))|dvdw\\
&+m(t,s)\int_0^T \int_T^\infty |g(t,s,v,w)|[|h(v,w,x(v,w))| + |h(v,w,y(v,w))|]dvdw\\
&+m(t,s)\int_T^\infty \int_0^T |g(t,s,v,w)|[|h(v,w,x(v,w))| + |h(v,w,y(v,w))|]dvdw\\
&+m(t,s)\int_T^\infty \int_T^\infty |g(t,s,v,w)|[|h(v,w,x(v,w))| + |h(v,w,y(v,w))|]dvdw
\end{aligned}$$

and so

$$\begin{aligned}&\Big|(Hx)(t,s)-(Hy)(t,s)\Big| \le \varepsilon + m_T g_T {w_{r_0}}^T(h,\varepsilon)T^2\\ &+2b(r_0)m(t,s)\int_0^T\int_T^\infty a(v,w)|g(t,s,v,w)|dvdw\\ &+2b(r_0)m(t,s)\int_T^\infty\int_0^T a(v,w)|g(t,s,v,w)|dvdw\\ &+2b(r_0)m(t,s)\int_T^\infty\int_T^\infty a(v,w)|g(t,s,v,w)|dvdw,\end{aligned} \tag{4.3}$$

where we denoted

$$m_T=\sup\{m(t,s):t,s\in[0,T]\},$$

$$g_T=\max\{|g(t,s,v,w)|:t,s,v,w\in[0,T]\},$$

$${w_{r_0}}^T(h,\varepsilon)=\sup\{|h(v,w,x)-h(v,w,y)|:v,w,t,s\in[0,T],x,y\in[-r_0,r_0],|x-y|\le\varepsilon\}.$$

Observe that ${w_{r_0}}^T(h,\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$ which is a simple consequence of the uniform continuity of the function $h(v,w,x)$ on the compact set $[0,T]\times[0,T]\times[-r_0,r_0]$. Moreover, in view of assumption (vi) we can choose T in such a way that three last terms of the estimate (4.3) are sufficiently small. Thus H is continuous on B_{r_0}.

Further, let us take a nonempty X of the ball B_{r_0}. Next, fix arbitrarily $T>0$ and $\varepsilon>0$. Choose a function $x\in X$ and take $t_1,t_2,s_1,s_2\in[0,T]$ such that $|t_2-t_1|\le\varepsilon$, $|s_2-s_1|\le\varepsilon$. Then, by the assumptions we have:

$$\begin{aligned}&\Big|(Hx)(t_2,s_2)-(Hx)(t_1,s_1)\Big|\\ \le\ &\Big|f\Big(t_2,s_2,x(t_2,s_2),\int_0^\infty\int_0^\infty g(t_2,s_2,v,w)h(v,w,x(v,w))dvdw\Big)\\ &-f\Big(t_1,s_1,x(t_1,s_1),\int_0^\infty\int_0^\infty g(t_1,s_1,v,w)h(v,w,x(v,w))dvdw\Big)\Big|\\ \le\ &\Big|f\Big(t_2,s_2,x(t_2,s_2),\int_0^\infty\int_0^\infty g(t_2,s_2,v,w)h(v,w,x(v,w))dvdw\Big)\\ &-f\Big(t_2,s_2,x(t_1,s_1),\int_0^\infty\int_0^\infty g(t_2,s_2,v,w)h(v,w,x(v,w))dvdw\Big)\Big|\\ &+\Big|f\Big(t_2,s_2,x(t_1,s_1),\int_0^\infty\int_0^\infty g(t_2,s_2,v,w)h(v,w,x(v,w))dvdw\Big)\\ &-f\Big(t_1,s_1,x(t_1,s_1),\int_0^\infty\int_0^\infty g(t_2,s_2,v,w)h(v,w,x(v,w))dvdw\Big)\Big|\\ &+\Big|f\Big(t_1,s_1,x(t_1,s_1),\int_0^\infty\int_0^\infty g(t_2,s_2,v,w)h(v,w,x(v,w))dvdw\Big)\\ &-f\Big(t_1,s_1,x(t_1,s_1),\int_0^\infty\int_0^\infty g(t_1,s_1,v,w)h(v,w,x(v,w))dvdw\Big)\Big|\\ \le\ &\varphi(|x(t_2,s_2)-x(t_1,s_1)|)+w^T{}_{r,D_1}(f,\varepsilon)\\ &+m(t_1,s_1)|\int_0^\infty\int_0^\infty[g(t_2,s_2,v,w)-g(t_1,s_1,v,w)]h(v,w,x(v,w))dvdw|\end{aligned}$$

and so

$$\begin{aligned}
&\left|(Hx)(t_2,s_2)-(Hx)(t_1,s_1)\right| \le \varphi(|x(t_2,s_2)-x(t_1,s_1)|)+{w^T}_{r,D_1}(f,\varepsilon)\\
&+m(t_1,s_1)\Bigg\{\int_0^\infty\Bigg\{\int_0^T |g(t_2,s_2,v,w)-g(t_1,s_1,v,w)||h(v,w,x(v,w))|dv\\
&+\int_T^\infty |g(t_2,s_2,v,w)-g(t_1,s_1,v,w)||h(v,w,x(v,w))|dv\Bigg\}dw\Bigg\}\\
\le\ & \varphi(|x(t_2,s_2)-x(t_1,s_1)|)+{w^T}_{r,D_1}(f,\varepsilon)\\
&+m(t_1,s_1)\int_0^T\int_0^T |g(t_2,s_2,v,w)-g(t_1,s_1,v,w)||h(v,w,x(v,w))|dvdw\\
&+m(t_1,s_1)\int_0^T\int_T^\infty |g(t_2,s_2,v,w)-g(t_1,s_1,v,w)||h(v,w,x(v,w))|dvdw\\
&+m(t_1,s_1)\int_T^\infty\int_0^T |g(t_2,s_2,v,w)-g(t_1,s_1,v,w)||h(v,w,x(v,w))|dvdw\\
&+m(t_1,s_1)\int_T^\infty\int_T^\infty |g(t_2,s_2,v,w)-g(t_1,s_1,v,w)||h(v,w,x(v,w))|dvdw\\
\le\ & \varphi(|x(t_2,s_2)-x(t_1,s_1)|)+{w^T}_{r,D_1}(f,\varepsilon)+m_T{w_1}^T(g,\varepsilon)a_{v,w}b(r_0)T^2\\
&+m(t_1,s_1)\int_0^T\int_T^\infty [|g(t_2,s_2,v,w)|+|g(t_1,s_1,v,w)|]|h(v,w,x(v,w))|dvdw\\
&+m(t_1,s_1)\int_T^\infty\int_0^T [|g(t_2,s_2,v,w)|+|g(t_1,s_1,v,w)|]|h(v,w,x(v,w))|dvdw\\
&+m(t_1,s_1)\int_T^\infty\int_T^\infty [|g(t_2,s_2,v,w)|+|g(t_1,s_1,v,w)|]|h(v,w,x(v,w))|dvdw
\end{aligned} \tag{4.4}$$

where
$D_1 = b(r_0)\overline{D}$(see assumption (iv)),

${w^T}_{r,D_1}(f,\varepsilon) = \sup\{|f(t_2,s_2,x,y)-f(t_1,s_1,x,y)| : t_1,s_1,t_2,s_2 \in [0,T], |t_2-t_1| \le \varepsilon, |s_2-s_1| \le \varepsilon, x \in [-r_0,r_0], y \in [-D_1,D_1]\}$,

${w_1}^T(g,\varepsilon) = \sup\{|g(t_2,s_2,v,w)-g(t_1,s_1,v,w)| : t_1,s_1,t_2,s_2,v,w \in [0,T], |t_2-t_1| \le \varepsilon, |s_2-s_1| \le \varepsilon\}$,

$a_{v,w} = \sup\{a(v,w) : v,w \in [0,T]\}$.

On the other hand, we have the following estimate:

$$\begin{aligned}
&m(t_1,s_1)\int_0^T\int_T^\infty [|g(t_2,s_2,v,w)|+|g(t_1,s_1,v,w)|]|h(v,w,x(v,w))|dvdw\\
&= m(t_1,s_1)\int_0^T\int_T^\infty |g(t_1,s_1,v,w)||h(v,w,x(v,w))|dvdw\\
&+m(t_1,s_1)\int_0^T\int_T^\infty |g(t_2,s_2,v,w)||h(v,w,x(v,w))|dvdw
\end{aligned}$$

and so

$$\begin{aligned}
& m(t_1,s_1)\int_0^T\int_T^\infty [|g(t_2,s_2,v,w)|+|g(t_1,s_1,v,w)|]|h(v,w,x(v,w))|dvdw\\
& = m(t_1,s_1)\int_0^T\int_T^\infty |g(t_1,s_1,v,w)||h(v,w,x(v,w))|dvdw\\
& +[m(t_1,s_1)-m(t_2,s_2)+m(t_2,s_2)]\int_0^T\int_T^\infty |g(t_2,s_2,v,w)||h(v,w,x(v,w))|dvdw\\
& = m(t_1,s_1)\int_0^T\int_T^\infty |g(t_1,s_1,v,w)||h(v,w,x(v,w))|dvdw\\
& +m(t_2,s_2)\int_0^T\int_T^\infty |g(t_2,s_2,v,w)||h(v,w,x(v,w))|dvdw\\
& +[m(t_1,s_1)-m(t_2,s_2)]\int_0^T\int_T^\infty |g(t_2,s_2,v,w)||h(v,w,x(v,w))|dvdw\\
\leq\ & m(t_1,s_1)\int_0^T\int_T^\infty |g(t_1,s_1,v,w)||h(v,w,x(v,w))|dvdw\\
& +m(t_2,s_2)\int_0^T\int_T^\infty |g(t_2,s_2,v,w)||h(v,w,x(v,w))|dvdw\\
& +w^T(m,\varepsilon)\int_0^\infty\int_0^\infty |g(t_2,s_2,v,w)||h(v,w,x(v,w))|dvdw\\
\leq\ & m(t_1,s_1)\int_0^T\int_T^\infty a(v,w)b(|x(v,w)|)|g(t_1,s_1,v,w)|dvdw\\
& +m(t_2,s_2)\int_0^T\int_T^\infty a(v,w)b(|x(v,w)|)|g(t_2,s_2,v,w)|dvdw\\
& +w^T(m,\varepsilon)\int_0^\infty\int_0^\infty a(v,w)b(|x(v,w)|)|g(t_2,s_2,v,w)|dvdw\\
\leq\ & b(r_0)m(t_1,s_1)\int_0^T\int_T^\infty a(v,w)|g(t_1,s_1,v,w)|dvdw\\
& +b(r_0)m(t_2,s_2)\int_0^T\int_T^\infty a(v,w)|g(t_2,s_2,v,w)|dvdw\\
& +b(r_0)w^T(m,\varepsilon)\int_0^\infty\int_0^\infty a(v,w)|g(t_2,s_2,v,w)|dvdw\\
\leq\ & b(r_0)m(t_1,s_1)\int_0^T\int_T^\infty a(v,w)|g(t_1,s_1,v,w)|dvdw\\
& +b(r_0)m(t_2,s_2)\int_0^T\int_T^\infty a(v,w)|g(t_2,s_2,v,w)|dvdw+b(r_0)w^T(m,\varepsilon)\overline{D}. \qquad (4.5)
\end{aligned}$$

Similarly, we get:

$$\begin{aligned}
& m(t_1,s_1)\int_T^\infty\int_0^T [|g(t_2,s_2,v,w)|+|g(t_1,s_1,v,w)|]|h(v,w,x(v,w))|dvdw\\
\leq\ & b(r_0)m(t_1,s_1)\int_T^\infty\int_0^T a(v,w)|g(t_1,s_1,v,w)|dvdw\\
& +b(r_0)m(t_2,s_2)\int_T^\infty\int_0^T a(v,w)|g(t_2,s_2,v,w)|dvdw+b(r_0)w^T(m,\varepsilon)\overline{D}, \qquad (4.6)
\end{aligned}$$

and also

$$\begin{aligned}
& m(t_1,s_1)\int_T^\infty\int_T^\infty [|g(t_2,s_2,v,w)|+|g(t_1,s_1,v,w)|]|h(v,w,x(v,w))|dvdw \\
\leq\ & b(r_0)m(t_1,s_1)\int_T^\infty\int_T^\infty a(v,w)|g(t_1,s_1,v,w)|dvdw \\
& +b(r_0)m(t_2,s_2)\int_T^\infty\int_T^\infty a(v,w)|g(t_2,s_2,v,w)|dvdw + b(r_0)w^T(m,\varepsilon)\overline{D}.
\end{aligned} \tag{4.7}$$

In the sequel, from linking (4.4), (4.5), (4.6) and (4.7) we obtain:

$$\begin{aligned}
& \Big|(Hx)(t_2,s_2)-(Hx)(t_1,s_1)\Big| \\
\leq\ & \varphi(|x(t_2,s_2)-x(t_1,s_1)|)+w^T{}_{r,D_1}(f,\varepsilon)+m_T w_1{}^T(g,\varepsilon)a_{v,w}b(r_0)T^2 \\
& +b(r_0)m(t_1,s_1)\int_0^T\int_T^\infty a(v,w)|g(t_1,s_1,v,w)|dvdw \\
& +b(r_0)m(t_2,s_2)\int_0^T\int_T^\infty a(v,w)|g(t_2,s_2,v,w)|dvdw + b(r_0)w^T(m,\varepsilon)\overline{D} \\
& +b(r_0)m(t_1,s_1)\int_T^\infty\int_0^T a(v,w)|g(t_1,s_1,v,w)|dvdw \\
& +b(r_0)m(t_2,s_2)\int_T^\infty\int_0^T a(v,w)|g(t_2,s_2,v,w)|dvdw + b(r_0)w^T(m,\varepsilon)\overline{D} \\
& +b(r_0)m(t_1,s_1)\int_T^\infty\int_T^\infty a(v,w)|g(t_1,s_1,v,w)|dvdw \\
& +b(r_0)m(t_2,s_2)\int_T^\infty\int_T^\infty a(v,w)|g(t_2,s_2,v,w)|dvdw + b(r_0)w^T(m,\varepsilon)\overline{D}.
\end{aligned}$$

By using the above estimate we have

$$\begin{aligned}
w^T(H(X),\varepsilon) \leq\ & \varphi(w^T(X,\varepsilon))+w^T{}_{r,D_1}(f,\varepsilon)+m_T w_1{}^T(g,\varepsilon)a_{v,w}b(r_0)T^2+3b(r_0)w^T(m,\varepsilon)\overline{D} \\
& +b(r_0)m(t_1,s_1)\int_0^T\int_T^\infty a(v,w)|g(t_1,s_1,v,w)|dvdw \\
& +b(r_0)m(t_2,s_2)\int_0^T\int_T^\infty a(v,w)|g(t_2,s_2,v,w)|dvdw \\
& +b(r_0)m(t_1,s_1)\int_T^\infty\int_0^T a(v,w)|g(t_1,s_1,v,w)|dvdw \\
& +b(r_0)m(t_2,s_2)\int_T^\infty\int_0^T a(v,w)|g(t_2,s_2,v,w)|dvdw \\
& +b(r_0)m(t_1,s_1)\int_T^\infty\int_T^\infty a(v,w)|g(t_1,s_1,v,w)|dvdw \\
& +b(r_0)m(t_2,s_2)\int_T^\infty\int_T^\infty a(v,w)|g(t_2,s_2,v,w)|dvdw.
\end{aligned}$$

From the continuity of f and g on the compact sets $[0,T]\times[0,T]\times[-r_0,r_0]\times[-D_1,D_1]$ and $[0,T]\times[0,T]\times[0,T]\times[0,T]$, respectively, we find $w^T{}_{r,D_1}(f,\varepsilon)\longrightarrow 0$, $w_1{}^T(g,\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$.

Similarly we get $w^T(m,\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Then we obtain

$$\begin{aligned} {w_0}^T(H(X)) \;\; \leq \;\; & \varphi({w_0}^T(X)) \\ & + b(r_0)m(t_1,s_1)\int_0^T\int_T^\infty a(v,w)|g(t_1,s_1,v,w)|dvdw \\ & + b(r_0)m(t_2,s_2)\int_0^T\int_T^\infty a(v,w)|g(t_2,s_2,v,w)|dvdw \\ & + b(r_0)m(t_1,s_1)\int_T^\infty\int_0^T a(v,w)|g(t_1,s_1,v,w)|dvdw \\ & + b(r_0)m(t_2,s_2)\int_T^\infty\int_0^T a(v,w)|g(t_2,s_2,v,w)|dvdw \\ & + b(r_0)m(t_1,s_1)\int_T^\infty\int_T^\infty a(v,w)|g(t_1,s_1,v,w)|dvdw \\ & + b(r_0)m(t_2,s_2)\int_T^\infty\int_T^\infty a(v,w)|g(t_2,s_2,v,w)|dvdw. \end{aligned}$$

Now taking $T \longrightarrow \infty$ and by using assumption(vi) we get

$$w_0(H(X)) \leq \varphi(w_0(X)). \tag{4.8}$$

Also for an arbitrary function $x \in X$ and a number $T > 0$, from the estimate (4.1) we obtain

$$\begin{aligned} \sup\{|(Hx)(t,s)| : t,s \geq T\} \leq \varphi(\sup\{x(t,s) : t,s \geq T\}) + b(\|x\|)\sup\{M(t,s) : t,s \geq T\} \\ + \sup\{|f(t,s,0,0) : t,s \geq T\}. \end{aligned}$$

Hence, in view of assumptions (iv) and (v) we get

$$\Gamma(H(X)) \leq \varphi(\Gamma(X)). \tag{4.9}$$

Further, combining (4.8), (4.9), and the definition of the measure of noncompactness given by formula (2.1) with assumption (i) we get

$$w_0(H(X)) + \Gamma(H(X)) \leq \varphi(w_0(X) + \Gamma(X))$$

or, equivalently

$$\mu(H(X)) \leq \varphi(\mu(X)).$$

Now, by considering the function $\psi : [0,\infty) \longrightarrow [0,\infty)$ defined by

$$\psi(t) = t,$$

we get:

$$\psi(\mu(H(X))) \leq \varphi(\mu(X)). \tag{4.10}$$

Finally, from (4.10) and applying Corollary 3.3 we get the desired result. □

Example 4.2. Consider the following quadratic Hammerstein integral equation

$$x(t,s) = \frac{3tsx(t,s)}{2+8ts} + \frac{ts}{1+t^2s^2}\int_0^\infty\int_0^\infty \frac{t^2s^2}{(1+t^2)(1+s^2)}vwe^{-(v+w)}\sqrt[3]{|x(v,w)|}dvdw. \tag{4.11}$$

Observe that this equation is a special case of Eq. (1.2) with

$$f(t,s,x,y)=\frac{3tsx}{2+8ts}+\frac{yts}{1+t^2s^2},$$

$$g(t,s,v,w)=e^{-(v+w)}\frac{t^2s^2}{(1+t^2)(1+s^2)},$$

$$h(t,s,x)=ts\sqrt[3]{|x|}.$$

Taking $\varphi(t)=\frac{3}{8}t, m(t,s)=\frac{ts}{1+t^2s^2}, a(t,s)=ts, b(r)=\sqrt[3]{r}$, then by some simple calculations we show that assumptions (i)-(vii) of Theorem 4.1 hold. Suppose that $t,s\in\mathbb{R}_+$ and $x,y\in\mathbb{R}$ then we get

$$\begin{aligned}|f(t,s,x,y)-f(t,s,u,z)| &\leq \frac{3ts}{2+8ts}|x-u|+\frac{ts}{1+t^2s^2}|y-z|\\ &\leq \frac{3}{8}|x-u|+\frac{ts}{1+t^2s^2}|y-z|\\ &=\varphi(|x-u|)+m(t,s)|y-z|.\end{aligned}$$

Thus assumption (i) holds. Also, assumptions (ii), (iii) clearly are evident. Also we have:

$$\begin{aligned}\int_0^\infty\int_0^\infty a(v,w)|g(t,s,v,w)|dvdw &= \int_0^\infty\int_0^\infty \frac{t^2s^2}{(1+t^2)(1+s^2)}vwe^{-(v+w)}dvdw\\ &= \frac{t^2s^2}{(1+t^2)(1+s^2)}\int_0^\infty\int_0^\infty ve^{-v}we^{-w}dvdw\\ &= \frac{t^2s^2}{(1+t^2)(1+s^2)}.\end{aligned}$$

Thus, $D(t,s)=\frac{t^2s^2}{(1+t^2)(1+s^2)}$ and $\overline{D}=1$. Moreover, $f(t,s,0,0)=0$,

$$K=\sup\{f(t,s,0,0):t,s\in\mathbb{R}_+\}=0.$$

Consequently, assumption (iv) is satisfied. Now, let us check that assumption (v) is hold. In order to we get:

$$\begin{aligned}M(t,s) &= m(t,s)\int_0^\infty\int_0^\infty a(v,w)|g(t,s,v,w)|dvdw\\ &= \frac{ts}{1+t^2s^2}\int_0^\infty\int_0^\infty\frac{t^2s^2}{(1+t^2)(1+s^2)}vwe^{-(v+w)}dvdw\\ &= \frac{ts}{1+t^2s^2}\frac{t^2s^2}{(1+t^2)(1+s^2)}.\end{aligned}$$

Thus, $\overline{M}=\frac{1}{2}$ and $\lim_{t,s\longrightarrow\infty}M(t,s)=0$. This shows that assumption (v) holds. Further, for arbitrarily fixed $T>0$ we obtain:

$$\begin{aligned}m(t,s)\int_T^\infty\int_0^T a(v,w)|g(t,s,v,w)|dvdw &= \frac{ts}{1+t^2s^2}\int_T^\infty\int_0^T\frac{t^2s^2}{(1+t^2)(1+s^2)}vwe^{-(v+w)}dvdw\\ &= \frac{ts}{1+t^2s^2}\frac{t^2s^2}{(1+t^2)(1+s^2)}\int_T^\infty\int_0^T ve^{-v}we^{-w}dvdw\\ &\leq [-Te^{-T}-e^{-T}+1][Te^{-T}+e^{-T}].\end{aligned}$$

Similarly, we get:

$$\begin{aligned}
m(t,s)\int_T^\infty\int_T^\infty a(v,w)|g(t,s,v,w)|dvdw &= \frac{ts}{1+t^2s^2}\int_T^\infty\int_T^\infty \frac{t^2s^2}{(1+t^2)(1+s^2)}vwe^{-(v+w)}dvdw\\
&= \frac{ts}{1+t^2s^2}\frac{t^2s^2}{(1+t^2)(1+s^2)}\int_T^\infty\int_T^\infty ve^{-v}we^{-w}dvdw\\
&\le [Te^{-T}+e^{-T}][Te^{-T}+e^{-T}],
\end{aligned}$$

and

$$\begin{aligned}
m(t,s)\int_0^T\int_T^\infty a(v,w)|g(t,s,v,w)|dvdw &= \frac{ts}{1+t^2s^2}\int_0^T\int_T^\infty \frac{t^2s^2}{(1+t^2)(1+s^2)}vwe^{-(v+w)}dvdw\\
&= \frac{ts}{1+t^2s^2}\frac{t^2s^2}{(1+t^2)(1+s^2)}\int_0^T\int_T^\infty ve^{-v}we^{-w}dvdw\\
&\le [Te^{-T}+e^{-T}][-Te^{-T}-e^{-T}+1].
\end{aligned}$$

From the above estimates we infer that assumption (vi) holds. Finally, let us notice that the inequality from assumption (vii), having the form

$$\varphi(r)+b(r)\overline{M}+K=\frac{3}{8}r+\frac{1}{2}\sqrt[3]{r}+0=\frac{3}{8}r+\frac{1}{2}\sqrt[3]{r}\le r.$$

It is easy to see that each number $r \ge 1$ (this estimate can be improved) satisfies the above inequality. Thus, as the number r_0 we can take $r_0 = 1$. Consequently, all assumptions in Theorem 4.1 are provided. Hence the quadratic Hammerstein integral equation (4.11) has at least one solution belonging to the ball B_1 in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+)$.

References

[1] R. Agarwal, M. Meehan and D. O'Regan, *Fixed point theory and applications*, Cambridge University Press, 2004.

[2] A. Aghajani, R. Allahyari and M. Mursaleen, *A generalization of Darbo's theorem with application to the solvability of systems of integral equations*, J. Comput. Appl. Math. 260 (2014) 68–77.

[3] A. Aghajani, J. Banaś and Y. Jalilian, *Existence of solutions for a class of nonlinear Volterra sigular integral equations*, Comput. Math. Appl. 62 (2011) 1215–1227.

[4] A. Aghajani and Y. Jalilian, *Existence and global attractivity of solutions of a nonlinear functional integral equation*, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 3306–3312.

[5] A. Aghajani and Y. Jalilian, *Existence of nondecreasing positive solution for a system of sigular integral equations*, Mediterr. J. Math. 8 (2011) 563–576.

[6] R. Allahyari, R. Arab and A. Shole Haghighi, *Construction of compact-integral operators on $BC(\Omega)$ with application to the solvability of functional integral equations*, Bull. Belg. Math. Soc. Simon Stevin 22 (2015) 761–779.

[7] R. Arab, R. Allahyari and A. Shole Haghighi, *Construction of a Measure of Noncompactness on $BC(\Omega)$ and its Application to Volterra Integral Equations*, Mediterr. J. Math. 13 (2016) 1197–1210.

[8] J. Banaś, J. Rocha Martin and K. Sadarangani, *On solutions of a quadratic integral equation of Hammerstein type*, Math. Comput. Model. 43 (2006) 97–104.

[9] J. Banaś and M. Mursaleen, *Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations*, Springer, 2014.

[10] J. Banaś, D. O'Regan and K. Sadarangani, *On solutions of a quadratic hammerstein integral equation on an unbonded interval*, Dynam. Systems Appl. 18 (2009) 251–264.

[11] J. Banaś, *Measures of noncompactness in the space of continuous tempered functions*, Demonstr. Math. 14 (1981) 127–133.

[12] J. Banaś, *On measures of noncompactness in Banach spaces*, Comment. Math. Univ. Carolin. 21 (1980) 131–143.

[13] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lect. Pure Appl. Math., 60. Marcel Dekker, Inc., New York, 1980.

[14] J. Banaś and R. Rzepka, *An application of a measure of noncompactness in the study of asymptotic stability*, Appl. Math. Lett. 16 (2003) 1–6.

[15] M. Cichon and M.M.A. Metwali, *On Monotonic Integrable Solutions for Quadratic Functional Integral Equations*, Mediterr. J. Math. 10 (2013) 909–926.

[16] M.A. Darwish and K. Sadarangani, *Nondecreasing solutions of a quadratic Abel equation with supremum in the kernel*, Appl. Math. Comput. 219 (2013) 7830–7836.

[17] M.A. Darwish, *On quadratic integral equation of fractional orders*, J. Math. Anal. Appl. 311 (2005) 112–119.

[18] B.C. Dhage and M. Imdad, *Asymptotic behaviour of nonlinear quadratic functional integral equations involving Carathéodory*, Nonlinear Anal. 71 (2009) 1285–1291.

[19] B.C. Dhage and S. S. Bellale, *Local asymptotic stability for nonlinear quadratic functional integral equations*, Electron. J. Qual. Theory Differ. Equ. 10 (2008) 1–13.

[20] B.C. Dhage, *Nonlinear quadratic first order functional integro-difierential equations with periodic boundary conditions*, Dynam. Systems Appl. 18 (2009) 303–322.

[21] J. Garca-Falset, *Existence of fixed points and measures of weak noncompactness*, Nonlinear Anal. 71 (2009) 2625–2633.

[22] M. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc. 40 (1973) 604–608.

[23] A. Hajji, *A generalization of Darbo's fixed point and common solutions of equations in Banach spaces*, Fixed Point Theory Appl. 2013 (2013):62.

[24] S. Ji and G. Li, *A unified approach to nonlocal impulsive differential equations with the measure of noncompactness*, Adv. Differ. Equ.. 182 (2012) 1–14.

[25] M. Jleli, M. Mursaleen and B. Samet, *On a class of q-integral equations of fractional orders*, Electron. J. Differ. Eq. 2016 (2016) 1–14.

[26] M. Mursaleen and S.M.H. Rizvi, *Solvability of infinite system of second order differential equations in* c_0 *and* l_1 *by Meir-Keeler condensing operators*, Proc. Amer. Math. Soc. 144 (2016) 4279–4289.

[27] M. Mursaleen, *Application of measure of noncompactness to infinite system of differential equations*, Canad. Math. Bull. 56 (2013) 388–394.

[28] M. Mursaleen and S.A. Mohiuddine, *Applications of measures of noncompactness to the infinite system of differential equations in* l_p *spaces*, Nonlinear Anal. 75 (2012) 2111–2115.

[29] C. Ravichandran, M. Rivero, J. A. Tenreiro-Machado and J. J. Trujillo, *Controllability results for impulsive mixed-type functional integro-differential evolution equations with nonlocal conditions*, Fixed Point Theory Appl. 2013 (2013):66.