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On the real quadratic fields with certain continued fraction expansions and fundamental units

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Abstract

The purpose of this paper is to investigate the real quadratic number fields $Q(\sqrt{d})$ which contain the specific form of the continued fractions expansions of integral basis element where $d \equiv 2, 3(mod4)$ is a square free positive integer. Besides, the present paper deals with determining the fundamental unit

$$\epsilon_d = \left(t_d + u_d\sqrt{d}\right)/2 \rangle 1$$

and n_d and m_d Yokoi's d-invariants by reference to continued fraction expansion of integral basis element where $\ell(d)$ is a period length. Moreover, we mention class number for such fields. Also, we give some numerical results concluded in the tables.

Keywords: Quadratic Field; Fundamental Unit; Continued Fraction Expansion; Class Number.
2010 MSC: Primary 11R11; Secondary 11R27, 11A55, 11R29.

1. Introduction and preliminaries

A quadratic field is defined as an algebraic number field $Q(\sqrt{d})$ of degree two over Q the rational numbers. $Q(\sqrt{d})$ is called real quadratic field if $d > 0$. The class number of a number field is defined by the order of the ideal class group of its ring of integers. There are infinitely many quadratic fields and all of them have class numbers such as one, two or more but it is not even known whether there are infinitely many real quadratic fields with class number one or two. So, class number problem is particularly important. There are different various methods to determine class number. One of them is Classical Dirichlet Class Number formula $h_d = \frac{\sqrt{D}.L(1,\chi_d)}{2log\epsilon_d}$ defined by discriminat, regulator and

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Dirichlet $L-$function. Dirichlet $L-$function is determined by Dirichlet character and zeta function. Also, discriminat D of the quadratic field $Q(\sqrt{d})$ is equal to d if d is congruent to 1 modulo 4 and is equal to 4 if d is congruent to 2,3 modulo 4. Besides, regulator is depend on fundamental unit ϵ_d.Moreover, in real quadratic fields, the ring of integers of $Q(\sqrt{d})$ has infinitely many units that are equal to ϵ_d^i or $-\epsilon_d^i$, where i is an arbitrary integer and ϵ_d is fundamental unit. So,it is also very significant to determine fundamental unit for studying on the class number problems, unit group and the determination of the structures of real quadratic number fields.

In [6], Kim and Ryu worked on the special circular unit ϵ_k of $k = Q(\sqrt{pq})$ and investigated unit group of such real quadratic field. Also, they proved class number of quartic field using Sinnott's index formula. Mollin and Williams determined positive square free integers d with class number $h(d) = 1$ in [9] and class number $h(d) = 2$ in [10] when ℓ period length of the continued fraction of w_d is equal or less than 25. The theorems of Friesen in [3] and Halter-Koch in [4] investigated real quadratic fields with large fundamental units by constructing a infinite family. In [19], Tomita with Yamamuro gave some results for fundamental unit by using Fibonacci sequence and continued fraction.Tomita and Kawamato obtained significant results on special real quadratic fields of minimal type in [5]. Moreover, Tomita determined the continued fraction expansion of integral basis element for period length is 3 in [18]. Sasaki in [16] and Mollin in [8] studied on lower bound of fundamental unit for real quadratic number fields and get certain important results. In [2], Benamar, Chandoul and Mkaouar gave the lower bound of the number of non-squares monic polynomials with the fixed period continued fraction expansions. Badziahin and Shallit [1], worked on special continued fraction expansions of the real numbers which are transcendental numbers. Yokoi defined several invariants but especially two of them are important for class number problem and the solutions of Pell equation ([21]-[24]).Zhang and Yue in [25] gave some congruence relations about x and y while $\epsilon_d = x + y\sqrt{d}$ and $N(\epsilon_d) = +1$ in a real quadratic field $Q(\sqrt{d})$ with odd class number. The first author obtained some results on different types of continued fraction expansion of w_d in [12],[13] and [14]. Also, we can refer to the references [7], [11],[15],[17] and [20] for readers.

Let $k = Q(\sqrt{d})$ be a real quadratic number field where $d > 0$ is a positive square-free integer. Integral basis element is denoted by $w_d = \sqrt{d}$ for $d \equiv 2, 3(mod4)$ where $\ell(d)$ is the period length in the simple continued fraction expansion of w_d. The fundamental unit ϵ_d of real quadratic number field is denoted by $\epsilon_d = \left(t_d + u_d\sqrt{d}\right)/2 \rangle 1$ where $N(\epsilon_d) = (-1)^{\ell(\mathrm{d})}$. For the set $I(d)$ of all quadratic irrational numbers in $Q(\sqrt{d})$, we say that α in $I(d)$ is reduced if $\alpha > 1$, $-1 < \alpha' < 0$ (α' is the conjugate of α with respect to Q), and denote by $R(d)$ the set of all reduced quadratic irrational numbers in $I(d)$. Then, it is well known that any number α in $R(d)$ is purely periodic in the continued fraction expansion and the denominator of its modular automorphism is equal to fundamental unit ϵ_d of $Q(\sqrt{d})$).

Besides, Yokoi's invariants are expressed by $n_d = \left[\left[\frac{t_d}{u_d^2}\right]\right]$ and $m_d = \left[\left[\frac{u_d^2}{t_d}\right]\right]$ where $[[x]]$ represents the greatest integer not greater than x.

In this paper, we interest in the problem determining the real quadratic number fields include continued fraction expansions which have got partial quotient elements written as 6s (except the last digit of the period) for the period length.Then, we categorize them with regard to arbitrary period length.

Also, we determine the general forms of fundamental unit ϵ_d and the coefficents of fundamental units t_d , u_d by using this new formulization which have not been known yet. Lastly,we calculate class numbers for some of such fields and give results on the Yokoi's invariants with tables.

Definition 1.1. $\{Z_i\}$ sequence is defined by the recurrence relation

$$Z_i = 6Z_{i-1} + Z_{i-2}$$

for $i \geq 2$ where $Z_0 = 0$ and $Z_1 = 1$. Main results depend on the terms of this sequence.

Definition 1.2. Let $c_n = ac_{n-1} + bc_{n-2}$ be a sequence with recurrence relation of $\{c_n\}$ sequence where a, b are real numbers. The polynomial is called as a characteristic equation if it is written in the following form:

$$x^2 - ax - b = 0$$

By using the definition, we determine the characteristic equation as follows:

$$r^2 - 6r - 1 = 0$$

for $\{Z_k\}$ sequence. So, each element of sequence can be written as follows:

$$Z_k = \frac{1}{2\sqrt{10}} \left[\left(3+\sqrt{10}\right)^k - \left(3-\sqrt{10}\right)^k \right]$$

for $k \geq 0$.

Remark 1.3. If $\{Z_n\}$ is defined as in Definition 1.1 then, we specify the following equivalent.

$$Z_n \equiv \begin{cases} 1\,(mod4)\,, & n \equiv 1,3\,(mod4); \\ 2\,(mod4)\,, & n \equiv 2\,(mod4); \\ 0\,(mod4)\,, & n \equiv 0\,(mod4). \end{cases}$$

for $n \geq 0$.

Lemma 1.4. *Let d be a square-free positive integer satisfying d congruent to $2,3$ modulo 4. If we put $\omega_d = \sqrt{d}$, $a_0 = [[\omega_d]]$ into the $\omega_R = a_0 + \omega_d$ where $[[\omega_d]]$ represents the greatest integer not greater than ω_d, then we get $\omega_d \notin R(d)$, but $\omega_R \in R(d)$ holds. Moreover, for the period $l = \ell(d)$ of ω_R, we get $\omega_R = [\overline{2a_0, a_1, \ldots, a_{l-1}}]$ and $\omega_d = [a_0, \overline{a_1, \ldots, a_{l-1}, 2a_0}]$. Furthermore, let $\omega_R = \frac{(P_l\omega_R + P_{l-1})}{(Q_l\omega_R + Q_{l-1})} = [2a_0, a_1, \ldots, a_{l-1}, \omega_R]$ be a modular automorphism of ω_R, Then the fundamental unit ϵ_d of $Q\left(\sqrt{d}\right)$ is given by the following formulas:*

$$\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{\ell(d)} + Q_{\ell(d)-1}\rangle 1$$

and

$$t_d = 2a_0.Q_{\ell(d)} + 2Q_{\ell(d)-1}, \quad u_d = 2Q_{\ell(d)}.$$

where Q_i is determined by $Q_0 = 0$, $Q_1 = 1$ and $Q_{i+1} = a_iQ_i + Q_{i-1}$, $(i \geq 1)$.

Proof . Proof is omitted in the reference [18]. $\square$

Lemma 1.5. *Let d be the square free positive integer congruent to $2,3$ modulo 4 and $a_0 = \left[\left[\sqrt{d}\right]\right]$ denote the the integer part of integral basis element for d congruent to $2,3(mod4)$ .If we consider w_d which has got partial quotient elements repeated 6s in the period length $l = \ell(d)$, then we obtain continued fraction expansions as follows*

$$w_d = \sqrt{d} = \left[a_0; \overline{a_1, a_2, \ldots, a_{\ell(\mathrm{d})-1}, a_{\ell(\mathrm{d})}}\ \right] = \left[\ a_0; \overline{6, 6, \ldots, 6,\ 2a_0}\ \right]$$

for quadratic irrational numbers and $w_R = \ a_0 + \sqrt{d} = \left[\overline{2a_0, 6, \ldots, 6}\ \right]$ for reduced quadratic irrational numbers.

Moreover, $A_k = \ a_0 Z_{k+1} + Z_k$ and $B_k = Z_{k+1}$ are determined in the continued fraction expansions where $\{A_k\}$ and $\{B_k\}$ are two sequences defined by:

$$A_{-2} = 0, \quad A_{-1} = 1, \quad A_k = a_k A_{k-1} + A_{k-2},$$

$$B_{-2} = 1, \quad B_{-1} = 0, \quad B_k = a_k B_{k-1} + B_{k-2},$$

for $k \geq 0$ and $k < \ell(d)$ where $\ell(d)$ is period length of w_d. We obtain $A_l = 2a_0^2 Z_l + 3a_0 Z_{l-1} + Z_{l-2}$ and $B_l = 2a_0 Z_l + Z_{l-1}$ for $k = \ell(d)$ too.

Also, in the continued fraction $w_R = \ a_0 + \sqrt{d} = [b_1, b_2, \ldots, b_n, \ldots\ ] = [2a_0, 6, \ldots, 6, \ldots\ ]$, $P_k = 2a_0 Z_k + Z_{k-1}$ and $Q_k = Z_k$ are determined where $\{P_k\}$ and $\{Q_k\}$ are two sequences defined by:

$$P_{-1} = 0, \quad P_0 = 1, \quad P_{k+1} = b_{k+1}.P_k + P_{k-1},$$
$$Q_{-1} = 1, \quad Q_0 = 0, \quad Q_{k+1} = b_{k+1}.Q_k + Q_{k-1},$$

for $k \geq 0$.

Proof . We can prove using induction.It is clear that assertion is true for $k = 0$.Assume that the result is true for $k < i$ and $0 < i \leq l - 1$ .Using the definition of $\{Z_i\}$ sequence, we obtain

$$\begin{aligned} A_{k+1} &= a_{k+1}.A_k + A_{k-1} = 6\left(a_0 Z_{k+1} + Z_k\right) + \left(a_0 Z_k + Z_{k-1}\right) \\ &= a_0\left(6Z_{k+1} + Z_k\right) + \ \left(6Z_k + Z_{k-1}\right) \\ &= a_0 Z_{k+2} + Z_{k+1} \end{aligned}$$

Also, we get the following result:

$$B_{k+1} = a_{k+1}.B_k + B_{k-1} = 6Z_{k+1} + Z_k = Z_{k+2}$$

Moreover,(since $a_l = 2.a_0$), we obtain $A_l = 2a_0^2 Z_l + 3a_0 Z_{l-1} + Z_{l-2}$ and $B_l = 2a_0.Z_l + Z_{l-1}$ for $k = \ell(d)$.

In the same vein, for the continued fraction of $a_0 + \sqrt{d} = [b_1, b_2, \ldots, b_n, \ldots\ ] = [2a_0, 6, \ldots, 6, \ldots\ ]$, we obtain that $P_k = 2a_0 Z_k + Z_{k-1}$ and $Q_k = Z_k$ are determined where $\{P_k\}$ and $\{Q_k\}$ are two sequences defined by:

$$P_{-1} = 0, \quad P_0 = 1, \quad P_{k+1} = b_{k+1}.P_k + P_{k-1},$$
$$Q_{-1} = 1, \quad Q_0 = 0, \quad Q_{k+1} = b_{k+1}.Q_k + Q_{k-1},$$

for $k \geq 0$.This completes the proof. $\square$

Theorem 1.6. *If $k \leq 24$ then with possibly only one more value remaining $h(d) = 1$ if and only if d is entry in Table 1.*

Proof . Proof is in the reference [9].□

Lemma 1.7. *f* $k \le 24$ *and* $D < 6.10^9$ *then with at most one possible exception we must have* $h(d) = 2$.

Proof . Proof is in the reference [10].□

Lemma 1.8. *For any* $s \ge 11.2$ *and* $d \ge e^s$ *in* d,

(1) if m_d *is different from* 0 *then* $h_d > \frac{0.3275.s^{-1}4^{\sigma_d}{}^{\frac{(s-2)}{2s}}.d^{\frac{(s-2)}{2s}}}{log(m_d+1)d}$ *holds with one possible exception of* d.

(2) if $m_d = 0$ *(i.e.* n_d *is different from* 0*) then* $h_d > \frac{0.3275.s^{-1}4^{\sigma_d}{}^{\frac{(s-2)}{2s}}.d^{\frac{(s-2)}{2s}}}{log(\frac{d}{n_d}+1)}$ *holds with one possible exception of* d.

(3) if $Q(\sqrt{d})$ *is a real quadratic field of R.D.type, then* $h_d > \frac{0.3275.s^{-1}4^{\sigma_d}{}^{\frac{(s-2)}{2s}}.d^{\frac{(s-2)}{2s}}}{log3d}$ *holds with one possible exception of* d.

Proof . Proof is in the reference [24].□

Remark 1.9. In the reference [24], Yokoi gave a table for square free integers d between $d = 2$ and $d = 499$ include Yokoi's invariants and class numbers.

Definition 1.10. Let $d = n^2 + r$, $d > 5$ be a positive square free integer satisfying the conditions $4n \equiv 0(mod r)$ and $-n < r \le n$. In this case, the real quadratic field $Q(\sqrt{d})$ is called a Richaud-Degert (R-D) type real quadratic field.

2. The main results

Theorem 2.1. *Let* d *be the square free positive integer and* ℓ *be a positive integer holding that* $\ell \equiv 0(mod2)$ *and* $\ell > 1$. *We assume that parametrization of* d *is*

$$d = \frac{r^2 Z_\ell^2}{4} + (3Z_\ell + \ Z_{\ell-1})\, r + 10.$$

for $r > 0$ *positive integer. Then the following conditions hold:*

(1) If $\ell \equiv 0(mod4)$ *and* $r \equiv 1(mod4)$ *positive integer then* $d \equiv 3(mod4)$ *satisfies.*

(2) If $\ell \equiv 2(mod4)$ *and* $r \equiv 1(mod4)$ *positive integer then* $d \equiv 2(mod4)$ *satisfies.*
Also, we get

$$w_d = \left[\frac{rZ_\ell}{2} + 3; \overline{\underbrace{6, 6, \ldots, 6}_{\ell-1}, rZ_\ell + 6}\right]$$

and $\ell = \ell(d)$. *Besides, we obtain the following equalities:*

$$\epsilon_d = \left(\frac{rZ_\ell^2}{2} + 3Z_\ell + Z_{\ell-1}\right) + \ Z_\ell\sqrt{d}$$

$$t_d = rZ_\ell^2 + 6Z_\ell + 2Z_{\ell-1} \quad and \quad u_d = 2Z_\ell$$

for ϵ_d, t_d *and* u_d.

Proof . We assume that $\ell \equiv 0(mod2)$ and $\ell > 1$. Then, we have to investigate two cases as follows: If $\ell \equiv 0(mod4)$, then we get $Z_\ell \equiv 0(mod4)$, $Z_{\ell-1} \equiv 1(mod4)$. By considering $r \equiv 1(mod4)$ positive integer and substituting these equivalent and equations into the parametrization of d, then we obtain $d \equiv 3(mod4)$.
If $\ell \equiv 2(mod4)$, then we get $Z_\ell \equiv 2\,(mod4)$, $Z_{\ell-1} \equiv 1\,(mod4)$. So, we have $d \equiv 2(mod4)$ by using $r \equiv 1(mod4)$ positive integer.

We put,

$$w_R = \frac{rZ_\ell}{2} + 3 + \left[\frac{rZ_\ell}{2} + 3; \overline{\underbrace{6, 6, \ldots, 6}_{\ell-1}, rZ_\ell + 2}\right],$$

then we get

$$w_R = (rZ_\ell + 6) + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{\ddots + \cfrac{1}{6 + \frac{1}{w_R}}}}} = (rZ_\ell + 6) + \frac{1}{6} + \ldots + \frac{1}{w_R}.$$

By using Lemma 1.5 and the induction with the properties of continued fraction expansion, we obtain

$$w_R = (rZ_\ell + 6) + \frac{Z_{\ell-1}w_R + Z_{\ell-2}}{Z_\ell w_R + Z_{\ell-1}},$$

Considering Definition 1.1 with the above equality, we have

$$w_R^2 - (rZ_\ell + 6)\, w_R - (1 + rZ_{\ell-1}) = 0.$$

This requires that $w_R = \frac{rZ_\ell}{2} + 3 + \sqrt{d}$ since $w_R > 0$. If we consider Lemma 1.4., we get

$$w_d = \left[\frac{rZ_\ell}{2} + 3; \overline{\underbrace{6, 6, \ldots, 6}_{\ell-1}, rZ_\ell + 6}\right]$$

and $\ell = \ell\,(d)$. This shows that first the part of the proof is completed.

Now, we have to determine ϵ_d, t_d and u_d using Lemma 1.4, we obtain values of Q_k as follows:

$$Q_1 = 1 = Z_1, \quad Q_2 = a_1.Q_1 + Q_0 \Rightarrow Q_2 = 6 = Z_2,$$
$$Q_3 = a_2Q_2 + Q_1 = 2Z_2 + Z_1 = 37 = Z_3, Q_4 = 228 = Z_4, \ldots$$

So, this implies that $Q_i = Z_i$ by using mathematical induction for $\forall i \geq 1$. If we substitute these values of sequence into the $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{l(d)} + Q_{l(d)-1}\rangle 1$ and rearranged, we have

$$\epsilon_d = \left(\frac{rZ_\ell^2}{2} + 3Z_\ell + Z_{\ell-1}\right) + Z_\ell\sqrt{d}$$

$$t_d = rZ_\ell^2 + 6Z_\ell + 2Z_{\ell-1} \quad \text{and} \quad u_d = 2Z_\ell$$

for ϵ_d, t_d and u_d. It completes the proof of Theorem 2.1. $\square$

Corollary 2.2. *Let d be a square free positive integer and ℓ be a positive integer holding that $\ell \equiv 0(mod2)$ and $\ell > 1$. We assume that the parametrization of d is*

$$d = \frac{Z_\ell^2}{4} + \left(\frac{Z_{\ell+1} + \ Z_{\ell-1}}{2}\right) + 10.$$

Then, we get $d \equiv 2, 3(mod4)$ and

$$w_d = \left[\frac{Z_\ell}{2} + 3; \overline{\underbrace{6, 6, \ldots, 6}_{\ell-1}, Z_\ell + 6}\right]$$

and $\ell = \ell(d)$. Furthermore, we obtain the following equalities:

$$\epsilon_d = \left(\frac{Z_\ell^2}{2} + 3Z_\ell + Z_{\ell-1}\right) + \ Z_\ell\sqrt{d}$$

$$t_d = Z_\ell^2 + 6Z_\ell + 2Z_{\ell-1} \quad \textit{and} \quad u_d = 2Z_\ell$$

$$m_d = \begin{cases} 1, & \textit{if } \ell = 2; \\ 3, & \textit{if } \ell \geq 4; \end{cases}$$

for ϵ_d, t_d, u_d and m_d.

Besides, we indicate the following Table 1 where fundamental unit is ϵ_d, integral basis element is w_d and Yokoi's invariant is m_d for $2 < \ell(d) \leq 10$.

Table 1: Square -free positive integers d where $\ell(d)$ is even and $2 \leq \ell(d) \leq 10$.

d	$\ell(d)$	m_d	w_d	ϵ_d
38	2	1	$[6; \overline{6, 12}]$	$37+6\sqrt{38}$
13727	4	3	$[117; \overline{6, 6, 6, 234}]$	$26713+228\sqrt{13727}$
18767630	6	3	$[4332; \overline{6, 6, 6, 6, 6, 8664}]$	$37507861+8658\sqrt{18767630}$
27024454235	8	3	$[164391; \overline{6, 6, 6, 6, 6, 6, 6, 328782}]$	$54047868769+328776\sqrt{27024454235}$
38967784512734	10	3	$[6242418; \overline{6, 6, 6, 6, 6, 6, 6, 6, 6, 12484836}]$	$77935529544949+12484830\sqrt{38967784512734}$

Proof . This result is obtained by substituting $r = 1$ into the Theorem 2.1 We have to prove that values of m_d are determined as follows:

$$m_d = \begin{cases} 1, & \text{if } \ell = 2; \\ 3, & \text{if } \ell \geq 4. \end{cases}$$

If we put t_d and u_d into the m_d and rearrange, then we obtain

$$m_d = \left[\left[\frac{u_d^2}{t_d}\right]\right] = \left[\left[\frac{4Z_\ell^2}{Z_\ell^2 + 6Z_\ell + 2Z_{\ell-1}}\right]\right]$$

By considering above equalization, we obtain $m_d = 1$ for $\ell = 2$. From the assumption (since Z_ℓ is increasing sequence) we get,

$$4 > 4.\left(1 + \frac{6}{Z_\ell} + \frac{2.Z_{\ell-1}}{Z_\ell^2}\right)^{-1} > 3,892$$

for $\ell \geq 4$. Therefore, we obtain $m_d = \left[\left[\frac{4Z_\ell^2}{Z_\ell^2+6Z_\ell+2Z_{\ell-1}} \right]\right] = 3$ for $\ell \geq 4$ due to definition of m_d and we describe m_d as follows:

$$m_d = \begin{cases} 1, & \text{if } \ell = 2; \\ 3, & \text{if } \ell \geq 4; \end{cases}$$

which completes the proof of Corollary 2.2.

Also, Table 1 can be obtained as a illustrate of this corollary. □

Remark 2.3. In the Table 1, $Q(\sqrt{d})$ is a R-D type real quadratic field for $d = 38 = 6^2 + 2$ but others aren't R-D type. By using the classical Dirichlet class number formula, we calculate class number $h_d = 1$ for $Q(\sqrt{38})$. Also, the field is obtained in the table of [24] and the Table 3.1 of [9]. Besides, in the same table, we can see that other class numbers such as $Q(\sqrt{13727})$ with $h_d = 8$ and $Q(\sqrt{18767630})$ with $h_d = 144$ are too bigger than class number two by using Proposition 4.1 of [24].

Corollary 2.4. *Let d be a square free positive integer and ℓ be a positive integer satisfying $\ell \equiv 0(mod2)$ and $\ell > 1$. Suppose that the parametrization of d is*

$$d = \frac{25Z_\ell^2}{4} + (15Z_\ell + \ 5Z_{\ell-1}) + 10.$$

Then, we have $d \equiv 2, 3(mod4)$ and

$$w_d = \left[\frac{5Z_\ell}{2} + 3; \overline{\underbrace{6, 6, \ldots, 6}_{\ell-1}, 5Z_\ell + 6} \right]$$

and $\ell = \ell(d)$. Moreover, we obtain the following equalities:

$$\epsilon_d = \left(5Z_\ell^2 2 + 3Z_\ell + Z_{\ell-1}\right) + \ Z_\ell\sqrt{d}$$

$$t_d = 5Z_\ell^2 + 6Z_\ell + 2Z_{\ell-1} \quad \text{and} \quad u_d = 2Z_\ell$$

$$n_d = 1$$

for ϵ_d, t_d, u_d and Yokoi's invariant n_d.
Besides, we get the following Table 2 where fundamental unit is ϵ_d, integral basis element is w_d and Yokoi's invariant is n_d for $2 < \ell(d) \leq 10$.(In this table, we rule out $\ell(d)$=4 since d is not a square free positive integer).

Table 2: Square -free positive integers d where $\ell(d)$ is even and $2 \leq \ell(d) \leq 10$ except $\ell(d) = 4$.

d	$\ell(d)$	n_d	w_d	ϵ_d
330	2	1	$[18; \overline{6, 36}]$	$109{+}6\sqrt{330}$
468642930	6	1	$[21648; \overline{6, 6, 6, 6, 6, 43296}]$	$187429789{+}8658\sqrt{468642930}$
675590562015	8	1	$[821943; \overline{6, 6, 6, 6, 6, 6, 6, 1643886}]$	$270235185121{+}328776\sqrt{675590562015}$
974193823208130	10	1	$[31212078; \overline{6, 6, 6, 6, 6, 6, 6, 6, 6, 62424156}]$	$389677489802749{+}12484830\sqrt{974193823208130}$

Proof . In a similar way of mentioned above corollary, this corollary is obtained by substituting $r = 5$ in Theorem 2.1. □

Remark 2.5. In the Table 2, there is no real quadratic field of R-D type except $Q(\sqrt{330})$. Also,we can give an example on class number as $h_d = 4$ for $Q(\sqrt{330})$.

Theorem 2.6. *Let d be a square free positive integer and ℓ be a positive integer holding that $\ell \geq 2$. Suppose that the parametrization of d is*

$$d = r^2 Z_\ell^2 + 2r(3Z_\ell + Z_{\ell-1}) + 10.$$

for $r > 1$ integer. If $r \equiv 0(mod2)$ is a positive integer then $d \equiv 2(mod4)$ holds.

So, we have

$$w_d = \left[rZ_\ell + 3; \overline{\underbrace{6, 6, \ldots, 6}_{\ell-1},\ 6 + 2rZ_\ell} \right]$$

with $\ell = \ell(d)$. Besides, we get fundamental unit ϵ_d, coefficients of fundamental unit t_d, u_d as follows:

$$\epsilon_d = (rZ_\ell + 3)\, Z_\ell + Z_{\ell-1} + Z_\ell\sqrt{d},$$

$$t_d = 2\,(rZ_\ell + 3)\, Z_\ell + 2Z_{\ell-1} \quad \textit{and} \quad u_d = 2Z_\ell.$$

Proof . If $\ell > 1$ arbitrary positive integer and r is even positive integer then we obtain $d \equiv 2(mod4)$ since $d = r^2 Z_\ell^2 + 2r(3Z_\ell + Z_{\ell-1}) + 10$.

If we put

$$w_R = (rZ_\ell + 3) + \left[rZ_\ell + 3; \overline{\underbrace{6, 6, \ldots, 6}_{\ell-1},\ 6 + 2rZ_\ell} \right]$$

then we have

$$w_R = (2rZ_\ell + 6) + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{\ddots + \cfrac{1}{6 + \cfrac{1}{w_R}}}}} = (2rZ_\ell + 6) + \frac{1}{6} + \ldots + \frac{1}{w_R}.$$

By a straight forward induction argument, we get

$$w_R = (2rZ_\ell + 6) + \frac{Z_{\ell-1} w_R + Z_{\ell-2}}{Z_\ell w_R + Z_{\ell-1}}.$$

Moreover, by using Definition 1.1 with the above equality, we obtain

$$w_R^2 - (2rZ_\ell + 6)\, w_R - (1 + 2rZ_{\ell-1}) = 0.$$

This requires that $w_R = (rZ_\ell + 3) + \sqrt{d}$ since $w_R > 0$. If we consider Lemma 1.5, we get

$$w_d = \left[rZ_\ell + 3; \overline{\underbrace{6, 6, \ldots, 6}_{\ell-1},\ 6 + 2rZ_\ell} \right]$$

and $\ell = \ell(d)$.

Now, we have to determine ϵ_d, t_d and u_d using Lemma 1.4, we get

$$Q_1 = 1 = Z_1, Q_2 = a_1.Q_1 + Q_0 \Rightarrow Q_2 = 6 = Z_2$$

$$Q_3 = a_6 Q_2 + Q_1 = 6Z_2 + Z_1 = 37 = Z_3, Q_4 = 228 = Z_4.$$

So, this implies that $Q_i = Z_i$ by using mathematical induction for $\forall i \geq 0$. If we substitute these values of the sequence into the $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{l(d)} + Q_{l(d)-1}\rangle 1$ and rearrange, we have

$$\epsilon_d = (rZ_\ell + 3)\, Z_\ell + Z_{\ell-1} + Z_\ell\sqrt{d},$$

$$t_d = 2\,(rZ_\ell + 3)\, Z_\ell + 2Z_{\ell-1} \quad \text{and} \quad u_d = 2Z_\ell.$$

□

Corollary 2.7. *Suppose that the parametrization of d is*

$$d = 4Z_\ell^2 + 2(Z_{\ell+1} + Z_{\ell-1}) + 10.$$

where d is a square free positive integer and $\ell \geq 2$ is a positive integer. Then, we have $d \equiv 2(mod4)$ and

$$w_d = \left[2Z_\ell + 3; \overline{\underbrace{6, 6, \ldots, 6}_{\ell-1},\ 6 + 4Z_\ell}\right]$$

with $\ell = \ell(d)$. Moreover, we can get fundamental unit ϵ_d, coefficients of fundamental unit t_d, u_d as follows:

$$\epsilon_d = (2Z_\ell + 3)\, Z_\ell + Z_{\ell-1} + Z_\ell\sqrt{d},$$

$$t_d = 2\,(2Z_\ell + 3)\, Z_\ell + 2Z_{\ell-1} \quad \textit{and} \quad u_d = 2Z_\ell.$$

Also, we have Yokoi's d- invariant value $n_d = 1$ for $\ell \geq 2$.
Furthermore, we have the following Table 3 where fundamental unit is ϵ_d, integral basis element is w_d and and Yokoi's invariant is n_d for $2 \leq \ell(d) \leq 8$.

Table 3: Square -free positive integers d with $2 \leq \ell(d) \leq 8$.

d	$\ell(d)$	n_d	w_d	ϵ_d
230	2	1	$[15; \overline{6, 30}]$	$91+6\sqrt{230}$
5954	3	1	$[77; \overline{6, 6, 154}]$	$2855+37\sqrt{5954}$
210830	4	1	$[459; \overline{6, 6, 6, 918}]$	$104689+228\sqrt{210830}$
7913882	5	1	$[2813; \overline{6, 6, 6, 6, 5626}]$	$3952493+1405\sqrt{7913882}$
299953382	6	1	$[17319; \overline{6, 6, 6, 6, 6, 34638}]$	$149949307+8658\sqrt{299953382}$
11386845314	7	1	$[106709; \overline{6, 6, 6, 6, 6, 6, 213418}]$	$5693253935+53353\sqrt{11386845314}$
432378791438	8	1	$[657555; \overline{6, 6, 6, 6, 6, 6, 6, 1315110}]$	$216188356033+328776\sqrt{432378791438}$

Proof . If we substitute $r = 2$ in Theorem 2.6, this corollary is got .We demonstrate that the value of Yokoi's d-invariant is $n_d = 1$ for $\ell \geq 2$.

We know that $n_d = \left[\left[\frac{t_d}{u_d^2} \right]\right]$ considering Yokoi's references. If we substitute t_d and u_d into the n_d, then we get

$$n_d = \left[\left[\frac{t_d}{u_d^2} \right]\right] = \left[\left[\frac{4Z_\ell^2 + 6Z_\ell + 2Z_{\ell-1}}{4Z_\ell^2} \right]\right] = 1 + \left[\left[\frac{3}{2Z_\ell} + \frac{Z_{\ell-1}}{2Z_\ell^2} \right]\right] = 1,$$

since Z_ℓ is increasing and $0 < \frac{3}{2Z_\ell} + \frac{Z_{\ell-1}}{2Z_\ell^2} < 0,264$ for $\ell \geq 2$. Therefore, we obtain $n_d = 1$ for $\ell \geq 2$. This completes the proof of Corollary 2.7. Besides, Table 3 is created as numerical example. □

Remark 2.8. In the Table 3, there isn't any R-D type of real quadratic field excluding $Q(\sqrt{230})$ with $h_d = 2$ which is also obtained in the Table 2.1 of [10] and the table of [24]. Additionally, we can calculate values of class numbers for some real quadratic fields as follows:
$Q(\sqrt{5954})$ with class number h_d=12, $Q(\sqrt{210830})$, with class number h_d=48, $Q(\sqrt{7913882})$ with class number h_d=138.

Corollary 2.9. Let d be a square free positive integer and ℓ be a positive integer satisfying $\ell \geq 2$. Suppose that parametrization of d is

$$d = 16Z_\ell^2 + 4(Z_{\ell+1} + Z_{\ell-1}) + 10.$$

Then, we have $d \equiv 2(mod4)$ and

$$w_d = \left[4Z_\ell + 3; \overline{\underbrace{6, 6, \ldots, 6}_{\ell-1},\ 6 + 8Z_\ell}\right]$$

with $\ell = \ell(d)$. Additionally, we get fundamental unit ϵ_d, coefficients of fundamental unit t_d, u_d as follows:

$$\epsilon_d = (4Z_\ell + 3)\, Z_\ell + Z_{\ell-1} + Z_\ell\sqrt{d},$$
$$t_d = 2\,(4Z_\ell + 3)\, Z_\ell + 2Z_{\ell-1} \quad \text{and} \quad u_d = 2Z_\ell.$$

and Yokoi's d- invariant value $n_d = 2$ for $\ell \geq 2$.

Besides, we indicate the Table 4 where fundamental unit is ϵ_d, integral basis element is w_d and Yokoi's invariant is n_d for $2 \leq l(d) \leq 8$ (In this table, we rule out $\ell(d)$=2,3,4,6,8 since d is not a square free positive integer).

Table 4: Square -free positive integers d with $\ell(d) = 5$ or $\ell(d) = 7$.

d	$\ell(d)$	n_d	w_d	ϵ_d
31619954	5	2	$[5623; \overline{6, 6, 6, 6, 11246}]$	$7900543{+}1405\sqrt{31619954}$
45546031490	7	2	$[213415; \overline{6, 6, 6, 6, 6, 6, 426830}]$	$11386339153{+}53353\sqrt{45546031490}$

Proof . The result is got if we substitute $r = 4$ in Theorem 2.6 The value of Yokoi's d-invariant is obtained as $n_d = 2$ for $\ell \geq 2$.

We know that $n_d = \left[\left[\frac{t_d}{u_d^2}\right]\right]$ from H. Yokoi's references. If we substitute t_d and u_d into the n_d, then we get

$$n_d = \left[\left[\frac{t_d}{u_d^2}\right]\right] = \left[\left[\frac{8Z_\ell^2 + 6Z_\ell + 2Z_{\ell-1}}{4Z_\ell^2}\right]\right] = 2 + \left[\left[\frac{3}{2Z_\ell} + \frac{Z_{\ell-1}}{2Z_\ell^2}\right]\right] = 2,$$

since (Z_ℓ) is increasing and $0 < \frac{3}{2Z_\ell} + \frac{Z_{\ell-1}}{2Z_\ell^2} < 0,0012$ for $\ell \geq 2$. Therefore, we obtain $n_d = 2$ for $\ell \geq 2$. This completes the proof of Corollary 2.9. For numerical examples, we tabulate the Table 4. □

Remark 2.10. There isn't any R-D type of real quadratic field in the Table 4. As an illustration, we can say that the class number is $h_d = 324$ for $Q(\sqrt{31619954})$.

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