On positive solutions for a class of infinite semipositone problems

M.B. Ghaemi, M. Choubin

Department of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran
Department of Mathematics, Faculty of Basic Sciences, Payame Noor University, Tehran, Iran

Abstract

We discuss the existence of a positive solution to the infinite semipositone problem

\[-\Delta u = au - bu^\gamma - f(u) - \frac{c}{u^\alpha}, \quad x \in \Omega, \quad u = 0, \quad x \in \partial \Omega,\]

where \(\Delta\) is the Laplacian operator, \(\gamma > 1, \alpha \in (0, 1)\), \(a, b\) and \(c\) are positive constants, \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with smooth boundary \(\partial \Omega\), and \(f : [0, \infty) \to \mathbb{R}\) is a continuous function such that \(f(u) \to \infty\) as \(u \to \infty\). Also we assume that there exist \(A > 0\) and \(\beta > 1\) such that \(f(s) \leq As^\beta\), for all \(s \geq 0\). 

We obtain our result via the method of sub- and supersolutions.

Keywords: Positive solution, Infinite semipositone, Sub- and supersolutions.


1. Introduction

We consider the positive solution to the boundary value problem

\[
\begin{aligned}
-\Delta u &= au - bu^\gamma - f(u) - \frac{c}{u^\alpha}, \quad x \in \Omega, \\
\quad u &= 0, \quad x \in \partial \Omega,
\end{aligned}
\]

where \(\Delta\) denotes the Laplacian operator, \(\gamma > 1, \alpha \in (0, 1)\), \(a, b\) and \(c\) are positive constants, \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with smooth boundary \(\partial \Omega\), and \(f : [0, \infty) \to \mathbb{R}\) is a continuous function. We make the following assumptions:

(H1) \(f : [0, +\infty) \to \mathbb{R}\) is continuous function such that \(\lim_{s \to +\infty} f(s) = \infty\).
There exist $A > 0$ and $\beta > 1$ such that $f(s) \leq As^\beta$, for all $s \geq 0$. In [9], the authors have studied the equation $-\Delta u = g(u) - (c/u^\alpha)$ with Dirichlet boundary conditions, where $g$ is nonnegative and nondecreasing and $\lim_{u \to \infty} g(u) = \infty$. The case $g(u) := au - f(u)$ has been study in [8], where $f(u) \geq au - M$ and $f(u) \leq Au^\beta$ on $[0, \infty)$ for some $M, A > 0, \beta > 1$ and this $g$ may have a falling zero. In this paper, we study the equation $-\Delta u = au - bu^\gamma - f(u) - (c/u^\alpha)$ with Dirichlet boundary conditions. Our result in this paper include the result of [8], where say in Remark 2.2. Let $F(u) := au - bu^\gamma - f(u) - (c/u^\alpha)$, then $\lim_{u \to 0} F(u) = -\infty$ and hence we refer to (1.1) as an infinite semipositone problem.

In recent years, there has been considerable progress on the study of semipositione problems ($F(0) < 0$ but finite)(see [2], [3], [6]). Many results have been obtained on kind of infinite semipositone problems; see for example [7], [8], [9] and [10]. One of the main tools used in these studies is the method of sub-super solutions. By a subsolution of (1.1) we mean a function $\psi \in C^2(\Omega) \cap C(\Omega)$ that satisfies

$$-\Delta \psi \leq a\psi - b\psi^\gamma - f(\psi) - \frac{c}{\psi^\alpha} \quad \text{in } \Omega$$

$$\psi \leq 0 \quad \text{on } \partial \Omega,$$

and by a supersolution of (1.1) we mean a function $Z \in C^2(\Omega) \cap C(\Omega)$ that satisfies

$$-\Delta Z \geq aZ - bZ^\gamma - f(Z) - \frac{c}{Z^\alpha} \quad \text{in } \Omega$$

$$Z \geq 0 \quad \text{on } \partial \Omega.$$ 

Then we have the following Lemma.

**Lemma 1.1 ([11, 41]).** If there exist a subsolution $\psi$ and a supersolution $Z$ of (1.1) such that $\psi \leq Z$ on $\Omega$, then (1.1) has at least one solution $u \in C^2(\Omega) \cap C(\Omega)$ satisfying $\psi \leq u \leq Z$ on $\Omega$.

2. The main result

We shall establish the following result.

**Theorem 2.1.** Let (H1) and (H2) hold. If $a > (\frac{2}{1+\alpha})\lambda_1$, then there exists positive constant $c^* := c^*(a, A, \alpha, \beta, \gamma, \Omega)$ such that for $c \leq c^*$, problem (1.1) has a positive solution, where $\lambda_1$ be the first eigenvalue of the Laplacian operator with Dirichlet boundary conditions.

**Remark 2.2.** Theorem 2.1 was established in [8] for the case $f(u) := g(u) - bu^\gamma$, where the function $g$ satisfy the following assumptions:

- $g(u) \approx bu^\theta$ for some $\theta > \gamma$.
- There exist $A > 0$ and $\beta > 1$ such that $g(u) \leq Au^\beta$, for all $u \geq 0$.
- There exist $M > 0$ such that $g(u) \geq au - M$, for all $u \geq 0$.

In fact, the function $f$ satisfy the hypotheses of Theorem 2.1 in this paper (Since $\lim_{u \to \infty} (g(u)/bu^\theta) = 1$, hence $\lim_{u \to \infty} f(u) = \infty$) and $g$ satisfy the hypotheses of Theorem 2.1 in [8], where (1.1) changes to equation $-\Delta u = au - g(u) - (c/u^\alpha)$ with Dirichlet boundary conditions.
Proof. We shall establish Theorem 2.1 by constructing positive sub-supersolutions to equation (1.1). From an anti-maximum principle (see [5, pages 155-156]), there exists \( \sigma(\Omega) > 0 \) such that the solution \( z_\lambda \) of
\[
\begin{align*}
-\Delta z - \lambda z &= -1, & x &\in \Omega, \\
z &= 0, & x &\in \partial \Omega,
\end{align*}
\]
for \( \lambda \in (\lambda_1, \lambda_1 + \sigma) \) is positive in \( \Omega \) and is such that \( \frac{\partial z}{\partial \nu} < 0 \) on \( \partial \Omega \), where \( \nu \) is outward normal vector on \( \partial \Omega \). Fix \( \lambda^* \in (\lambda_1, \min\{\lambda_1 + \sigma, (\frac{1+\alpha}{2})a\}) \) and let
\[
K := \min \left\{ \left( \frac{(2/1 + \alpha)}{2b \|z_{\lambda^*}\|^{\frac{2\alpha-\gamma+1}{1+\alpha}}} \right)^{\frac{1}{1+\alpha}}, \left( \frac{(2/1 + \alpha)}{3b \|z_{\lambda^*}\|^{\frac{2\alpha-\gamma+1}{\alpha}}} \right)^{\frac{1}{1+\alpha}}, \left( \frac{(2/1 + \alpha)}{2A \|z_{\lambda^*}\|^{\frac{2\alpha-\gamma+1}{\alpha}}} \right)^{\frac{1}{1+\alpha}} \right\}
\]
Define \( \psi = K z_{\lambda^*}^{\frac{1}{1+\alpha}} \). Then
\[
\nabla \psi = K (2 \alpha + 1) z_{\lambda^*}^{\frac{1}{1+\alpha}} \nabla z_{\lambda^*},
\]
and
\[
-\Delta \psi = - \text{div}(\nabla \psi)
\]
\[
= -K \left( \frac{2}{1+\alpha} \right) \left\{ \left( \frac{2}{1+\alpha} \right) z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}} |\nabla z_{\lambda^*}|^2 + z_{\lambda^*}^{\frac{1+\alpha}{1+\alpha}} \Delta z_{\lambda^*} \right\}
\]
\[
= -K \left( \frac{2}{1+\alpha} \right) \left\{ \left( \frac{2}{1+\alpha} \right) z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}} |\nabla z_{\lambda^*}|^2 + z_{\lambda^*}^{\frac{1+\alpha}{1+\alpha}} (1 - \lambda^* z_{\lambda^*}) \right\}
\]
\[
= K \left( \frac{2}{1+\alpha} \right) \left\{ \lambda^* z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}} - z_{\lambda^*}^{\frac{1+\alpha}{1+\alpha}} - \left( \frac{2}{1+\alpha} \right) \frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{\frac{1+\alpha}{1+\alpha}}} \right\}
\]
Let \( \delta > 0, \mu > 0, m > 0 \) be such that \( |\nabla z_{\lambda^*}|^2 \geq m \) in \( \Omega_\delta \) and \( z_{\lambda^*} \geq \mu \) in \( \Omega \setminus \overline{\Omega}_\delta \), where \( \overline{\Omega}_\delta := \{ x \in \Omega : d(x, \partial \Omega) \leq \delta \} \). Let
\[
c^* := K^{1+\alpha} \min \left\{ \left( \frac{2}{1+\alpha} \right) \left( \frac{1-\alpha}{1+\alpha} \right) m^2, \frac{1}{3} \mu^2 \left( a - \left( \frac{2}{1+\alpha} \right) \lambda^* \right) \right\}.
\]
Let \( x \in \overline{\Omega}_\delta \) and \( c \leq c^* \). Since \( \left( \frac{2}{1+\alpha} \right) \lambda^* < a \), we have
\[
K \left( \frac{2}{1+\alpha} \right) \lambda^* z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}} < a \left( K z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}} \right), \tag{2.1}
\]
From the choice of \( K \), we have
\[
\frac{1}{2} \left( \frac{2}{1+\alpha} \right) \geq b K^{\gamma-1} \|z_{\lambda^*}\|^{\frac{2\alpha-\gamma+1}{1+\alpha}} \tag{2.2}
\]
\[
\frac{1}{2} \left( \frac{2}{1+\alpha} \right) \geq AK^{\beta-1} \|z_{\lambda^*}\|^{\frac{2\beta-\alpha+1}{1+\alpha}} \tag{2.3}
\]
and by (2.2), (2.3) and (H2), we know that
\[ -\frac{1}{2} K \left( \frac{2}{1 + \alpha} \right) z_{\lambda^*}^{1 - \alpha} \leq -b \left( K z_{\lambda^*}^{2/\alpha} \right)^\gamma \]  
\[ -\frac{1}{2} K \left( \frac{2}{1 + \alpha} \right) z_{\lambda^*}^{1 - \alpha} \leq -A \left( K z_{\lambda^*}^{2/\alpha} \right)^\beta \leq -f \left( K z_{\lambda^*}^{2/\alpha} \right) \]  
\[ (2.4) \]
\[ (2.5) \]
Since \( |\nabla z_{\lambda^*}|^2 \geq m \) in \( \overline{\Omega}_\delta \), from the choice of \( c^* \) we have
\[ -K \left( \frac{2}{1 + \alpha} \right) \left( 1 - \frac{1}{1 + \alpha} \right) \frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{2/\alpha}} \leq -K \left( \frac{2}{1 + \alpha} \right) \left( 1 - \frac{1}{1 + \alpha} \right) m^2 z_{\lambda^*}^{2/\alpha} \]
\[ \leq -\frac{c}{\left( K z_{\lambda^*}^{2/\alpha} \right)^\alpha}. \]  
\[ (2.6) \]
Hence for \( c \leq c^* \), combining (2.1), (2.4), (2.5) and (2.6) we have
\[ -\Delta \psi = K \left( \frac{2}{1 + \alpha} \right) \left\{ \lambda^* z_{\lambda^*}^{2/\alpha} - z_{\lambda^*}^{1 - \alpha} - \left( 1 - \frac{1}{1 + \alpha} \right) \frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{2/\alpha}} \right\} \]
\[ = K \left( \frac{2}{1 + \alpha} \right) \lambda^* z_{\lambda^*}^{2/\alpha} - \frac{1}{2} K \left( \frac{2}{1 + \alpha} \right) z_{\lambda^*}^{1 - \alpha} \]
\[ -\frac{1}{2} K \left( \frac{2}{1 + \alpha} \right) z_{\lambda^*}^{1 - \alpha} \]
\[ -K \left( \frac{2}{1 + \alpha} \right) \left( 1 - \frac{1}{1 + \alpha} \right) \frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{2/\alpha}} \]
\[ \leq \left( K z_{\lambda^*}^{2/\alpha} \right)^\gamma - b \left( K z_{\lambda^*}^{2/\alpha} \right)^\gamma - f \left( K z_{\lambda^*}^{2/\alpha} \right) - \frac{c}{\left( K z_{\lambda^*}^{2/\alpha} \right)^\alpha} \]
\[ = a\psi - b\psi^\gamma - f(\psi) - \frac{c}{\psi^{\alpha}}, \quad x \in \overline{\Omega}_\delta. \]

Next in \( \Omega \setminus \overline{\Omega}_\delta \), for \( c \leq c^* \) from the choice of \( c^* \) and \( K \), we know that
\[ \frac{c}{K^\alpha} \leq \frac{1}{3} K z_{\lambda^*}^2 \left( a - \left( \frac{2}{1 + \alpha} \right) \lambda^* \right), \]  
\[ (2.7) \]
and
\[ bK^{\gamma-1} z_{\lambda^*}^{2(\gamma-1)} \leq \frac{1}{3} \left( a - \left( \frac{2}{1 + \alpha} \right) \lambda^* \right) \]  
\[ (2.8) \]
\[ AK^{\beta-1} z_{\lambda^*}^{2(\beta-1)} \leq \frac{1}{3} \left( a - \left( \frac{2}{1 + \alpha} \right) \lambda^* \right). \]  
\[ (2.9) \]
By combining (2.7), (2.8) and (2.9) we have

$$-\Delta \psi = K\left(\frac{2}{1 + \alpha}\right) \left\{ \lambda^* \bar{\gamma}_{1,\alpha}^2 - \frac{1}{1 + \alpha} \left[ \frac{1 - \alpha}{1 + \alpha} \right] \frac{\|\nabla \lambda|^2}{z_{\lambda^*}} \right\}$$

$$\leq K\left(\frac{2}{1 + \alpha}\right) \lambda^* \bar{\gamma}_{1,\alpha}^2$$

$$= \frac{1}{z_{\lambda^*}} \sum_{i=1}^{3} \left( \frac{1}{3} K\left(\frac{2}{1 + \alpha}\right) \lambda^* \bar{\gamma}_{1,\alpha}^2 \right)$$

$$\leq \frac{1}{z_{\lambda^*}} \left\{ \left( \frac{1}{3} K\left(\frac{2}{1 + \alpha}\right) a - \frac{c}{K^\alpha} \right) + K^* \left( \frac{1}{3} a - b K^{-1} \bar{\gamma}_{1,\alpha}^2 \right) \right\}$$

$$\leq a K^* \bar{\gamma}_{1,\alpha}^2 - b K^{-1} \bar{\gamma}_{1,\alpha}^2 - A K^\alpha \bar{\gamma}_{1,\alpha}^2 - \frac{c}{K^\alpha}$$

$$\leq a \left( K^* \bar{\gamma}_{1,\alpha}^2 \right) - b \left( K \bar{\gamma}_{1,\alpha}^2 \right) - f \left( K \bar{\gamma}_{1,\alpha}^2 \right) - \frac{c}{\left( K \bar{\gamma}_{1,\alpha}^2 \right)^\alpha}$$

Thus $\psi$ is a positive subsolution of (1.1). From (H1) and $\gamma > 1$, it is obvious that $Z = M$ where $M$ is sufficiently large constant is a supersolution of (1.1) with $Z \geq \psi$. Thus, by Lemma 1.1 there exists a solution $u$ of (1.1) with $\psi \leq u \leq Z$. This completes the proof of Theorem 2.1.

3. An extension to system (3.1)

In this section, we consider the extension of (1.1) to the following system:

$$\begin{cases}
-\Delta u = a_1 u - b_1 u^\gamma - f_1(u) - \frac{c_1}{u^\alpha}, & x \in \Omega, \\
-\Delta v = a_2 v - b_2 v^\gamma - f_2(v) - \frac{c_2}{v^\alpha}, & x \in \Omega, \\
u = 0 = v, & x \in \partial \Omega,
\end{cases}$$

(3.1)

where $\Delta$ denotes the Laplacian operator, $\gamma > 1$, $\alpha \in (0, 1)$, $a_1, a_2, b_1, b_2, c_1$ and $c_2$ are positive constants, $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, and $f_i : [0, \infty) \to \mathbb{R}$ is a continuous function for $i = 1, 2$. We make the following assumptions:

(H3) $f_i : [0, +\infty) \to \mathbb{R}$ is continuous functions such that $\lim_{s \to +\infty} f_i(s) = \infty$ for $i = 1, 2$.

(H4) There exist $A > 0$ and $\beta > 1$ such that $f_i(s) \leq A s^\beta$, $i = 1, 2$, for all $s \geq 0$.

We prove the following result by finding sub-super solutions to infinite semipositone system (3.1).

**Theorem 3.1.** Let (H3) and (H4) hold. If $\min \{a_1, a_2\} > \left(\frac{2}{1 + \alpha}\right) \lambda_1$, Then there exists positive constant $c^* := c^*(a_1, a_2, b_1, b_2, A, \Omega)$ such that for $c_1, c_2 \leq c^*$, problem (3.1) has a positive solution.
Proof. Let $\sigma$ be as in section 2 $\bar{a} = \min\{a_1, a_2\}$ and $\bar{b} = \max\{b_1, b_2\}$. Choice $\lambda^* \in (\lambda_1, \min\{\lambda_1 + \sigma, (\frac{1+\sigma}{2})\bar{a}\})$. Define

$$K := \min\left\{ \left( \frac{(2/1 + \alpha)}{2b \|z_{\lambda^*}\|_1^{\frac{2\alpha}{1+\alpha}}} \right)^{\frac{1}{\nu}}, \left( \frac{(2/1 + \alpha)}{3b \|z_{\lambda^*}\|_\infty^{\frac{2\alpha}{1+\alpha}}} \right)^{\frac{1}{\nu}}, \left( \frac{(2/1 + \alpha)}{2A \|z_{\lambda^*}\|_1^{\frac{2\alpha}{1+\alpha}}} \right)^{\frac{1}{\nu}}, \left( \frac{(2/1 + \alpha)}{3A \|z_{\lambda^*}\|_\infty^{\frac{2\alpha}{1+\alpha}}} \right)^{\frac{1}{\nu}} \right\},$$

and

$$c^* := K^{1+\alpha} \min\left\{ \left( \frac{2}{1+\alpha} \right) \left( \frac{1-\alpha}{1+\alpha} \right) m^2, \frac{1}{3} \mu^2 \left( \bar{a} - \frac{2}{1+\alpha} \lambda^* \right) \right\}.$$

By the same argument as in the proof of theorem 2.1, we can show that $(\psi_1, \psi_2) := (Kz_{\lambda^*}^{1+\alpha}, Kz_{\lambda^*}^{-\alpha})$ is a positive subsolution of (3.1) for $\max\{c_1, c_2\} \leq c^*$, i.e.

$$\begin{cases}
-\Delta \psi_1 \leq a_1 \psi_1 - b_1 \psi_1^\gamma - f_1(\psi_1) - c_{\psi_1}^{\alpha}, & x \in \Omega, \\
-\Delta \psi_2 \leq a_2 \psi_2 - b_2 \psi_2^\gamma - f_2(\psi_2) - c_{\psi_2}^{\alpha}, & x \in \Omega, \\
(\psi_1, \psi_2) \leq (0, 0), & x \in \partial \Omega.
\end{cases}$$

Also it is easy to check that constant function $(Z_1, Z_2) := (M, M)$ is a supersolution of (3.1) for $M$ large, i.e.

$$\begin{cases}
-\Delta Z_1 \geq a_1 Z_1 - b_1 Z_1^\gamma - f_1(Z_1) - c_{Z_1}^{\alpha}, & x \in \Omega, \\
-\Delta Z_2 \geq a_2 Z_2 - b_2 Z_2^\gamma - f_2(Z_2) - c_{Z_2}^{\alpha}, & x \in \Omega, \\
(Z_1, Z_2) \geq (0, 0), & x \in \partial \Omega.
\end{cases}$$

Further $M$ can be chosen large enough so that $(Z_1, Z_2) \geq (\psi_1, \psi_2)$ on $\Omega$. Hence for $\max\{c_1, c_2\} \leq c^*$, (3.1) has a positive solution and the proof is complete. □

References