Subordination and superordination properties for convolution operator

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Abstract

In the present paper, a certain convolution operator of analytic functions is defined. Subordination and superordination- preserving properties for a useful class of analytic operators on the space of normalized analytic functions in the open unit disk are obtained. Sandwich- type results are also obtained.

Keywords: Convolution operator; differential subordination; differential superordination; best dominant; best subordinant.

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1. Introduction and preliminaries

Let $H(\Delta)$ denote the class of analytic functions in the open unit disk $\Delta = \{ z : |z| < 1 \}$, and normalized by $f(0) = f'(0) - 1 = 0$. Also let $A(p)$ be the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad p \in \mathbb{N},$$

and let $A(1) = A$. For a positive integer number $n$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{ f \in H(\Delta) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \}$$

Let $f$ and $F$ be members of the analytic function class $H(\Delta)$. The function $f$ is said to be subordinate to $F$ or $F$ is said to be superordinate of $f$, if there exist a function $w$ analytic in $\Delta$ with

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$w(0) = 0$, and $|w(z)| < 1$ such that $f(z) = F(w(z))$ and we write $f(z) \prec F(z)$ or $f \prec F$. If function $F$ is univalent, then we have $f \prec F$ if and only if $f(0) = F(0)$ and $f(\Delta) \subset F(\Delta)$.

Let $\varphi : \mathbb{C}^2 \times \Delta \to \mathbb{C}$ and $h$ be analytic in $\Delta$. If $p$ is analytic in $\Delta$ and satisfies the (first-order) differential subordination
\[
\varphi(p(z), zp'(z); z) \prec h(z),
\tag{1.1}
\]
then $p$ is called a solution of the differential subordination. The univalent function $q$ is called a dominant of the solution of the differential subordination, or dominant if $p \prec q$ for all $p$ satisfying $[1.1]$. A dominant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all dominant of $q$ of (1.1) is called the best dominant.

Let $\varphi : \mathbb{C}^2 \times \Delta \to \mathbb{C}$ and $h$ be analytic in $\Delta$. If $p$ and $\varphi(p(z), zp'(z); z)$ are univalent and $p$ satisfies the (first-order) differential superordination
\[
h(z) \prec \varphi(p(z), zp'(z); z)
\tag{1.2}
\]
then $p$ is called a solution of the differential superordination. An analytic function $q$ is called a subordinant of the solution of the differential superordinate, or more simply a subordinant if $q \prec p$ for all $q$ satisfying $[1.2]$. A univalent subordinant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all subordinant of $q$ of (1.2) is called the best subordinant.

Ali et al [2] have obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy $q_1(z) < z^{f'(z)}/f(z) < q_2(z)$, where $q_1$ and $q_2$ are given univalent functions in $\Delta$ with $q_1(0) = q_2(0) = 1$.

For two functions $f_j(z)$, $j = 1, 2$, given by
\[
f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j}z^k
\]
we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by
\[
(f_1 * f_2)(z) := z + \sum_{k=2}^{\infty} a_{k,1}a_{k,2}z^k = (f_2 * f_1)(z), \quad z \in \Delta.
\]

In terms of the Pochhammer symbol (or the shifted factorial), define $(\kappa)_n$ by
\[
(\kappa)_0 = 1, \quad \text{and} \quad (\kappa)_n = \kappa(\kappa+1)(\kappa+2)\cdots(\kappa+n-1), \quad n \in \mathbb{N} := \{1, 2, \ldots\}.
\]

Also, Aghalary et al [1] have defined a function $\phi_\alpha^\lambda(b, c; z)$ by
\[
\phi_\alpha^\lambda(b, c; z) := 1 + \sum_{n=1}^{\infty} \left(\frac{a}{a+n}\right)^\lambda \frac{(b)_n}{(a)_n}z^n, \quad z \in \Delta,
\tag{1.3}
\]
where $b \in \mathbb{C}$, $c \in \mathbb{C} \setminus Z_0$, $a \in \mathbb{C} \setminus Z_0$ ($Z_0 = \{0, -1, -2, \ldots\}$) and $\lambda \geq 0$. Corresponding to the function $\phi_\alpha^\lambda(b, c; z)$, given by (1.3), they have introduced the following convolution operator
\[
L_\beta^\lambda(a, b; \beta)f(z) := \phi_\alpha^\lambda(b, c; z) * \left(\frac{f(z)}{z}\right)^\beta, \quad f \in A, \quad \beta \in \mathbb{C} \setminus \{0\}.
\tag{1.4}
\]

It is easy to see that
\[
z(\phi_\alpha^\lambda(b, c; z))' = a\phi_\alpha^\lambda(b, c; z) - a\phi_\alpha^{\lambda+1}(b, c; z),
\tag{1.5}
\]
and
\[ z(L^\lambda_a(1 + b, c; \beta)_f(z))' = aL^\lambda_a(1 + b, c; \beta)_f(z) - aL^{\lambda+1}_a(1 + b, c; \beta)_f(z). \] (1.6)

The operator \( L^\lambda_a(1 + b, c; \beta)_f(z) \) includes, as its special cases, Komatu integral operator (see [4], [5], [10]), some fractional calculus operators (see [4], [12], [13]) and Carlson-Shaffer operator (see [3]).

Making use of the principle of subordinant between analytic functions Miller et al [8] obtained some interesting subordination theorems involving certain operators. Also Miller and Mocanu [7] considered subordination-preserving properties of certain integral operator investigations as the dual concept of differential subordination. In the present investigation, we obtain the subordination and superordination-preserving properties of the convolution operator \( L^\lambda_a \) defined by [1.4] with the Sandwich-type theorems.

### 2. Definitions and Preliminaries

The following definitions and Lemmas will be required in our present investigation.

**Definition 2.1.** If \( 0 \leq \alpha < 1, \lambda \geq 0 \) and \( a \in \mathbb{C} \setminus Z_0^- \) \((Z_0^- = \{0, -1, -2, \ldots\})\), let \( \mathcal{L}^\lambda_a(\alpha) \) denote the class of functions \( f \in A \) which satisfies the inequality
\[ Re[L^\lambda_a(1 + b, c; \beta)_f(z)] > \alpha \]
For \( a = 1 \), we set \( \mathcal{L}^\lambda_1(\alpha) = \mathcal{L}^\lambda(\alpha) \).

**Definition 2.2.** [6] Denote by \( Q \) the set of all functions \( q \) that are analytic and injective on \( \overline{\Delta} \setminus E(q) \) where
\[ E(q) = \{ \xi \in \Delta : \lim_{z \to \xi} q(z) = \infty \} \]
and are such that \( h'(\xi) \neq 0 \) for \( \xi \in \partial \Delta \setminus E(q) \).

**Lemma 2.3.** [6] Let \( h(z) \) be analytic and convex univalent in \( \Delta \) and \( h(0) = a \). Also \( p(z) \) be analytic in \( \Delta \) with \( p(0) = a \). If
\[ p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \quad \gamma \neq 0, \quad Re\gamma \geq 0, \]
then \( p(z) \prec q(z) \prec h(z) \), where
\[ q(z) = \frac{\gamma}{z^\gamma} \int_0^z h(t)t^{\gamma-1}dt. \]
Furthermore \( q(z) \) is a convex function and is the best dominant.

**Lemma 2.4.** [7] Let \( h(z) \) be a convex in \( \Delta \), \( h(0) = a, \gamma \neq 0 \) and \( Re\gamma \geq 0 \). Also \( p \in \mathcal{H}[a, n] \cap Q \). If \( p(z) + \frac{zp'(z)}{\gamma} \) is univalent in \( \Delta \), \( h(z) \prec p(z) + \frac{zp'(z)}{\gamma} \), and \( q(z) = \frac{\gamma}{z^\gamma} \int_0^z h(t)t^{\gamma-1}dt \) then \( q(z) \prec p(z) \), and \( q(z) \) is a convex function and is the best subordinant.

**Lemma 2.5.** [11] Let \( q(z) \) be a convex univalent function in \( \Delta \) and \( \psi, \gamma \in \mathbb{C} \) with \( Re(1 + \frac{zq''(z)}{q'(z)}) > \max\{0, -Re\psi\} \), \( q(0) = a, \gamma \neq 0 \) and \( Re\gamma \geq 0 \). If \( p(z) \) is analytic in \( \Delta \) and \( \psi p(z) + \gamma zp'(z) \prec \psi q(z) + \gamma q'(z) \) then \( p(z) \prec q(z) \), and \( q(z) \) is the best dominant.

**Lemma 2.6.** [9] Let \( q(z) \) be a convex univalent function in \( \Delta \) and \( \eta \in \mathbb{C} \), assume that \( Re\eta > 0 \). If \( p(z) \in \mathcal{H}[a, n] \cap Q \) and \( p(z) + \eta zp'(z) \prec q(z) + \eta zq'(z) \) which implies that \( q(z) \prec p(z) \) and \( q(z) \) is the best subordinant.
3. Differential subordination defined by convolution operator

Theorem 3.1. If $0 \leq \alpha < 1$, $\lambda \geq 0$ and $a \in \mathbb{C}\setminus \mathbb{Z}_0^-$, then we have

\[ L_\lambda^a(\alpha) \subset L_\lambda^{a+1}(\delta), \]

where

\[ \delta(\alpha, a) = a\beta(a) + a(2\alpha - 1)\beta(a + 1), \]

and

\[ \beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} \, dt. \]

The result is sharp.

Proof. First note that $f \in L_\lambda^a(\alpha)$ and

\[ z(L_\lambda^{\lambda+1}(b,c;\beta)f(z))' = aL_\lambda^\lambda(b,c;\beta)f(z) - aL_\lambda^{\lambda+1}(b,c;\beta)f(z). \quad (3.1) \]

We define $p(z) = L_\lambda^{\lambda+1}(b,c;\beta)f(z)$. From the relation (1.1) we have

\[ L_\lambda^\lambda(b,c;\beta)f(z) = p(z) + \frac{zp'(z)}{a}. \]

Now from Lemma 2.3 for $\gamma = a$, it follows that

\[ p(z) = L_\lambda^{\lambda+1}(b,c;\beta)f(z) \preceq q(z) = \frac{a}{z^a} \int_0^z 1 + (2\alpha - 1)t \frac{t^{a-1}}{1+t} \, dt, \]

therefore we have

\[ L_\lambda^a(\alpha) \subset L_\lambda^{a+1}(\delta), \]

where

\[ \delta = \min_{|z| \leq 1} \text{Re}(z) = q(1) = a\beta(a) + a(2\alpha - 1)\beta(a + 1). \]

Furthermore $q(z)$ is a convex function and is the best dominant. □

For the class $L^\lambda$ we obtain the next corollary.

Corollary 3.2. If $0 \leq \alpha < 1$ and $\lambda \geq 0$, then we have

\[ L^\lambda(\alpha) \subset L^{\lambda+1}(\delta), \]

where

\[ \delta = \delta(\alpha) = 2\alpha - 1 + 2(1 - \alpha) \ln 2, \]

and the result is sharp.

Theorem 3.3. Let $h \in H(\Delta)$, with $h(0) = 1$ and $h'(0) \neq 0$, which verifies the inequality $\text{Re}[1 + \frac{zh''(z)}{h'(z)}] > -\frac{1}{2}$. If $f \in A$ and satisfies the differential subordination

\[ L_\lambda^a(b,c;\beta)f(z) \preceq h(z), \quad (3.2) \]

then

\[ L_\lambda^{\lambda+1}(b,c;\beta)f(z) \preceq q(z), \quad (3.3) \]

where

\[ q(z) = \frac{a}{z^a} \int_0^z h(t)t^{a-1} \, dt. \]

The function $q(z)$ is convex and is the best dominant.
Subordination and superordination properties

Proof. Let
\[ p(z) = L^{\lambda+1}_a(b,c;\beta)f(z). \]  
(3.4)
Differentiating (3.4) with respect to \( z \), we have
\[ p'(z) = (L^{\lambda+1}_a(b,c;\beta)f(z))'. \]
From the relation (1.1), we have
\[ \frac{zp'(z)}{a} + p(z) = L^{\lambda}_a(b,c;\beta)f(z). \]
Now, in view of (2.4), we obtain the following subordination
\[ \frac{zp'(z)}{a} + p(z) \prec h(z). \]
Then from Lemma 2.3 for \( \gamma = a \) we conclude that
\[ p(z) = L^{\lambda+1}_a(b,c;\beta)f(z) \prec q(z), \]
where
\[ q(z) = \frac{a}{z^a} \int_0^z h(t)t^{a-1}dt \]
and \( q(z) \) is the best dominant. □

Taking \( \lambda = 0 \) in Theorem 3.3, we arrive the following corollary.

Corollary 3.4. Let \( h \in H(\Delta) \), with \( h(0) = 1, h'(0) \neq 0 \), and \( Re \left( 1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2} \). If \( f \in A \) and satisfies \( (\frac{f(z)}{z})^\beta \prec h(z) \), then \( L_a(b,c;\beta) \prec q(z) \) where \( q(z) = \frac{a}{z^a} \int_0^z h(t)t^{a-1}dt \). The function \( q(z) \) is the best dominant.

By setting \( a = \gamma + \beta, \lambda = 0 \) and \( b = c = 1 \) in Theorem 3.3, we get the following corollary.

Corollary 3.5. Let \( h \in H(\Delta) \), with \( h(0) = 1 \) and \( h'(0) \neq 0 \), which satisfies the inequality \( Re \left( 1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2} \). If \( f \in A \) and satisfies the differential subordination \( (\frac{f(z)}{z})^\beta \prec h(z) \), then
\[ \frac{\gamma + \beta}{z^{\gamma + \beta}} \int_0^z u^{\gamma-1}(f(u))^\beta du \prec \frac{1}{z} \int_0^z h(u)du \]
The function \( \frac{1}{z} \int_0^z h(u)du \) is the best dominant.

Corollary 3.6. Let \( 0 < R \leq 1 \) and let \( h(z) \) be convex in \( \Delta \), defined by \( h(z) = 1 + Rz + \frac{Rz^2}{2 + Rz} \), with \( h(0) = 1 \). If \( f \in A \) satisfies in the following differential subordination
\[ L^\lambda_a(b,c;\beta)f(z) \prec h(z), \]
then
\[ L^{\lambda+1}_a(b,c;\beta)f(z) \prec q(z), \]
where
\[ q(z) = \frac{a}{z^a} \int_0^z \left( 1 + Rt + \frac{Rt}{2 + Rt}t^{a-1} \right) dt \]
\[ = z^{a-1} +RAFT \left( \frac{z^a}{a + 1} + \frac{M(z)}{z} \right), \]
with
\[ M(z) = \int_0^z \frac{t^a}{2 + Rt}dt. \]
The function \( q(z) \) is convex and is the best dominant.
If $a = 1$, Corollary 3.6 becomes:

**Corollary 3.7.** Let $0 < R \leq 1$ and let $h(z)$ be convex in $\Delta$, defined by $h(z) = 1 + Rz + \frac{Rz^2}{2 + Rz}$, with $h(0) = 1$. If $f \in A$ and suppose that

$$L^{\lambda}(b, c; \beta)f(z) \prec h(z),$$

then

$$L^{\lambda+1}(b, c; \beta)f(z) \prec q(z)(z \in \Delta),$$

where

$$q(z) = \frac{1}{z} \int_0^z \left( 1 + Rt + \frac{Rt}{2 + Rt} \right) dt = 2 + \frac{Rz}{2} - \frac{2}{Rz} \log(2 + Rz).$$

The function $q(z)$ is convex and is the best dominant.

By taking $R = 1$ in Corollary 3.7 we have the following corollaries.

**Corollary 3.8.** Let $h(z)$ be convex in $\Delta$, defined by $h(z) = 1 + z + \frac{z^2}{2 + z}$, with $h(0) = 1$. If $f \in A$, satisfies in the differential subordination

$$L^{\lambda}(b, c; \beta)f(z) \prec h(z),$$

then

$$L^{\lambda+1}(b, c; \beta)f(z) \prec q(z),$$

where

$$q(z) = 2 + \frac{z}{2} - \frac{2}{z} \log(2 + z).$$

The function $q(z)$ is convex and is the best dominant.

**Corollary 3.9.** Let $h(z)$ be convex in $\Delta$, defined by $h(z) = 1 + z + \frac{z^2}{2 + z}$, with $h(0) = 1$. Suppose that $\gamma \in \mathbb{C}$, $a = \gamma + \beta$, $\lambda = 0$ and $b = c = 1$. If $f \in A$ and satisfies the differential subordination

$$(\frac{h(z)}{z})^\beta \prec h(z),$$

then

$$\frac{\gamma + \beta}{z^{\gamma + \beta}} \int_0^z u^{\gamma-1}(f(u))^\beta du \prec q(z) = 2 + \frac{z}{2} - \frac{2}{z} \log(2 + z).$$

The function $q(z)$ is convex and is the best dominant.

**Corollary 3.10.** Let $h(z) = \frac{1 + (2a - 1)z}{1 + z}$ be convex function in $\Delta$, with $h(0) = 1$. If $f \in \mathcal{L}^{\lambda}(\alpha)$ and $L^{\lambda}(b, c; \beta)f(z) \prec h(z)$ then

$$L^{\lambda+1}(b, c; \beta)f(z) \prec q(z),$$

where

$$q(z) = 2\alpha - 1 + 2(1 - \alpha) \frac{\log(1 + z)}{z}.$$

The function $q(z)$ is convex and is the best dominant.
Theorem 3.11. Let $q(z)$ be a convex function with $q(0) = 1$, and let $h$ be a function such that $h(z) = q(z) + \frac{zq'(z)}{q(z)}$. If $f \in H(\Delta)$ and satisfies the differential subordination

$$L_\lambda^\gamma(b, c; \beta) f(z) \prec h(z),$$

then

$$L_\lambda^{\gamma+1}(b, c; \beta) f(z) \prec q(z)$$

and this result is sharp.

Proof. We have

$$z(L_\lambda^{\gamma+1}(b, c; \beta) f(z))' = aL_\lambda^\gamma(b, c; \beta) f(z) - aL_\lambda^{\gamma+1}(b, c; \beta) f(z).$$

Let $p(z) = L_\lambda^{\gamma+1}(b, c; \beta) f(z)$, then from (3.5) and (3.6), we have

$$p(z) + \frac{zp'(z)}{a} < q(z) + \frac{zq'(z)}{a}.$$

An application of Lemma 2.6, we conclude that $p(z) < q(z)$ or $L_\lambda^{\gamma+1}(b, c; \beta) f(z) \prec q(z)$ and this result is sharp. □

Theorem 3.12. Let $h \in H(\Delta)$, with $h(0) = 1$, and $h'(0) \neq 0$, which satisfies in the inequality $\text{Re} \left( 1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2}$. If $f \in A$ and satisfies the differential subordination

$$(L_\lambda^{\gamma+1}(b, c; \beta) f(z))' \prec h(z),$$

then

$$\frac{L_\lambda^{\gamma+1}(b, c; \beta) f(z)}{z} < q(z),$$

where

$$q(z) = \frac{1}{z} \int_0^z h(t) t^{a-1} dt,$$

the function $q(z)$ is the best dominant.

Proof. Let us define the function $f$ by

$$f(z) = \frac{L_\lambda^{\gamma+1}(b, c; \beta) f(z)}{z}.$$  

(3.7)

Differentiating with respect to $z$ logarithmically, we have

$$\frac{zp'(z)}{p(z)} = \frac{z(L_\lambda^{\gamma+1}(b, c; \beta) f(z))'}{L_\lambda^{\gamma+1}(b, c; \beta) f(z)} - 1$$

and

$$p(z) + zp'(z) = (L_\lambda^{\gamma+1}(b, c; \beta) f(z))'.$$

Now, from (3.7) we obtain

$$p(z) + zp'(z) < h(z)$$

Then, by Lemma 2.3, for $\gamma = 1$ we have $p(z) < q(z)$ or

$$\frac{L_\lambda^{\gamma+1}(b, c; \beta) f(z)}{z} < \frac{1}{z} \int_0^z h(t) dt$$

and the function $q(z)$ is the best dominant. Therefore, we complete the proof of theorem 3.12. □

Suppose that $\lambda = 0$ and in Theorem 3.12, we have the following result.
Corollary 3.13. Let \( h \in H(\Delta) \), with \( h(0) = 1 \) and \( h'(0) \neq 0 \), which satisfies in the inequality
\[
\Re(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2}
\]
. If \( f \in A \) and \( (L_a(b, c; \beta)f(z))' < h(z) \) then \( L_a(b, c; \beta)f(z) < \frac{1}{z} \int_0^z h(t)dt \), and the function \( \frac{1}{z} \int_0^z h(t)dt \) is the best dominant.

By taking \( \gamma \in \mathbb{C}, \ a = \gamma + \beta, \ \lambda = 0, \) and \( b = c = 1 \) in the Theorem 3.12 we get the following result.

Corollary 3.14. Let \( f \in A, \ h \in H(\Delta) \) and \( h(0) = 1, h'(0) \neq 0 \). If \( \Re(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2} \)
\[
\frac{-(\gamma + \beta)}{z^{\gamma+\beta+1}} \int_0^z u^{\gamma-1}(f(u))^{\beta}du + \frac{\gamma + \beta}{z^{\beta+1}} < h(z),
\]
then
\[
\frac{\gamma + \beta}{z^{\gamma+\beta-1}} \int_0^z u^{\gamma-1}(f(u))^{\beta}du < \frac{1}{z} \int_0^z h(u)du.
\]
The function \( \frac{1}{z} \int_0^z h(u)du \) is the best dominant.

Corollary 3.15. Let \( 0 < R \leq 1 \) and let \( h(z) \) be convex in \( \Delta \), defined by \( h(z) = 1 + Rz + \frac{Rz}{2 + Rz} \), with \( h(0) = 1 \). If \( f \in A \) satisfies in the following differential subordination
\[
(L^{\lambda+1}(b, c; \beta)f(z))' < h(z),
\]
then
\[
\frac{L^{\lambda+1}(b, c; \beta)f(z)}{z} < q(z),
\]
where
\[
q(z) = \frac{1}{z} \int_0^z \left( 1 + Rt + \frac{Rt}{2 + Rt} \right) dt
\]
\[
= 1 + \frac{Rz}{2} + \frac{RM(z)}{z},
\]
with
\[
M(z) = \frac{z}{R} - \frac{2}{R^2} (\ln(2 + Rz)) - \frac{2}{R} \ln 2.
\]
The function \( q(z) \) is convex and is the best dominant.

Suppose that \( \gamma \in \mathbb{C}, \ a = \gamma + \beta, \ \lambda = 0 \) and \( b = c = 1 \) in Corollary 3.15 we have the following corollary.

Corollary 3.16. Let \( h(z) \) be convex in \( \Delta \), defined by \( h(z) = 1 + z + \frac{z}{2+z}, \) with \( h(0) = 1 \). If \( f \in A \), satisfies in the differential subordination
\[
\frac{-(\gamma + \beta)}{z^{\gamma+\beta+1}} \int_0^z u^{\gamma-1}(f(u))^{\beta}du + \frac{\gamma + \beta}{z^{\beta+1}} < h(z),
\]
then
\[
\frac{\gamma + \beta}{z^{\gamma+\beta-1}} \int_0^z u^{\gamma-1}(f(u))^{\beta}du < \frac{1}{z} \int_0^z h(u)du,
\]
where
\[
q(z) = 2 + \frac{z}{2} - \frac{2}{z} \log(2 + z).
\]
The function \( q(z) \) is convex and is the best dominant.
Corollary 3.17. Let \( h(z) = \frac{1+(2\alpha-1)z}{1+z} \) be convex function in \( \Delta \), with \( h(0) = 1 \). If \( f \in \mathcal{L}^\lambda(\alpha) \) and \( (L^\lambda+1(b,c;\beta)f(z))' \prec h(z) \), then
\[
\frac{L^\lambda+1(b,c;\beta)f(z)}{z} \prec q(z),
\]
where
\[
q(z) = 2\alpha - 1 + 2(1-\alpha)\log(1+z).
\]
The function \( q(z) \) is convex and is the best dominant.

Theorem 3.18. Let \( q(z) \) be a convex function in \( \Delta \), \( q(0) = 1 \) and \( h(z) = q(z) + \frac{zq'(z)}{q(z)} \). If \( f \in H(\Delta) \) and satisfies the differential subordination
\[
(L^\lambda+1(a;\beta)f(z))' \prec h(z),
\]
then
\[
\frac{L^\lambda+1(a;\beta)f(z)}{z} \prec q(z)
\]
and this result is sharp.

Proof. Let
\[
p(z) = \frac{L^\lambda+1(a;\beta)f(z)}{z}.
\]
Logarithmic differentiation of (3.9) and through a little simplification we obtain
\[
p(z) +zp'(z) = (L^\lambda+1(a;\beta)f(z))'.
\]
Now by using Lemma 2.6 we conclude that the differential equation
\[
\frac{L^\lambda+1(a;\beta)f(z)}{z} \prec q(z)
\]
and this result is sharp. □

4. Differential superordination defined by convolution operator

The results this section are obtained with differential superordination method.

Theorem 4.1. Let \( h \) be convex function in \( \Delta \), with \( h(0) = 1 \), and \( f \in A \). Assume that \( L^\lambda(a;\beta)f(z) \) is univalent with \( L^\lambda+1(a;\beta)f(z) \in H[1,n] \cap \mathcal{Q} \). If \( h(z) \prec L^\lambda(a;\beta)f(z) \) then
\[
q(z) \prec L^\lambda+1(a;\beta)f(z),
\]
where
\[
q(z) = \frac{a}{z^a} \int_0^z h(t)t^{a-1}dt.
\]
The function \( q(z) \) is the best subordinant.
Proof. If we let
\[ p(z) = L_a^{\lambda+1}(b, c; \beta) f(z), \]
then from the relation \((1.6)\) we have \(p(z) + \frac{zp'(z)}{a} = L_a^\lambda(b, c; \beta) f(z)\). Now according to Lemma \(2.4\) we get the desired result \((4.1)\). □

Corollary 4.2. Suppose that \(\gamma \in \mathbb{C}, a = \gamma + \lambda = 0\) and \(b = c = 1\). Let \(h \in H(\Delta)\) be convex function in \(\Delta\), with \(h(0) = 1\), and \(f \in A\). Assume that \((\frac{f(z)}{z})^\beta\) is univalent with \(\frac{1}{z^{\gamma+1}} \int_0^z u^{\gamma-1}(f(u))^{\beta} du \in H[1, n] \cap Q\). If \(h(z) < (\frac{f(z)}{z})^\beta\) then
\[ \frac{1}{z} \int_0^z h(u) du < \gamma + \beta \int_0^z u^{\gamma-1}(f(u))^{\beta} du \]
and \(\frac{1}{z} \int_0^z h(u) du\) is the best subordinant.

Corollary 4.3. Let \(h(z)\) be a convex mapping in \(\Delta\), defined by \(h(z) = 1 + z + \frac{z}{2+z}\), with \(h(0) = 1\). Suppose that \(\gamma \in \mathbb{C}, a = \gamma + \lambda = 0, b = c = 1,\) and \(f \in A\) and \((\frac{f(z)}{z})^\beta\) is univalent with \(\frac{1}{z^{\gamma+1}} \int_0^z u^{\gamma-1}(f(u))^{\beta} du \in H[1, n] \cap Q\). If \(h(z) < (\frac{f(z)}{z})^\beta\) then \(q(z) < \gamma + \beta \int_0^z u^{\gamma-1}(f(u))^{\beta} du\), where \(q(z) = 2 + \frac{z}{2} - \frac{z}{2} \log(2+z)\). The function \(q(z)\) is the best subordinant.

Corollary 4.4. Let \(h(z) = \frac{1+2z}{1+z}\) be a convex function in \(\Delta\) with \(h(0) = 1\). Assume that \(f \in L^{\lambda+1}(\alpha)\) and \(L^\lambda(b, c; \beta) f(z)\) is univalent with \(L^{\lambda+1}(b, c; \beta) f(z) \in H[1, n] \cap Q\). If \(h(z) < L^\lambda(b, c; \beta) f(z)\) then
\[ q(z) < L_a^{\lambda+1}(b, c; \beta) f(z), \]
where
\[ q(z) = 2\alpha - 1 + 2(1-\alpha) \frac{\log(1+z)}{z}. \]
The function \(q(z)\) is the best subordinant.

Theorem 4.5. Let \(h\) be a convex function in \(\Delta\), with \(h(0) = 1,\) and \(f \in A\). Assume that \((L_a^{\lambda+1}(b, c; \beta) f(z))'\) is univalent with \(L_a^{\lambda+1}(b, c; \beta) f(z) \in H[1, n] \cap Q\). If \(h(z) < (L_a^{\lambda+1}(b, c; \beta) f(z))'\) then
\[ q(z) < L_a^{\lambda+1}(b, c; \beta) f(z) \]
where
\[ q(z) = \frac{1}{z} \int_0^z h(t) dt. \]
The function \(q(z)\) is the best subordinant.

5. Sandwich results
Combining results of differential subordinations and superordinations, we arrive at the following "Sandwich results".

Theorem 5.1. Let \(q_1(z)\) be convex univalent in the open unit disk \(\Delta\), and \(q_2(z)\) be univalent in the open unite disk \(\Delta\) and \(f \in A\). Also let \(L_a^\lambda(b, c; \beta) f(z)\) be univalent with \(L_a^{\lambda+1}(b, c; \beta) f(z) \in H[1, n] \cap Q\). The following subordinate relationship \(q_1(z) < L_a^\lambda(b, c; \beta) f(z) < q_2(z)\) implies \(q_1(z) < L_a^{\lambda+1}(b, c; \beta) f(z) < q_2(z)\). Moreover the functions \(q_1(z)\) and \(q_2(z)\) are the best subordinant and the best dominant respectively.
Theorem 5.2. Suppose that $q_1(z)$ is convex univalent, and let $q_2(z)$ be univalent in $\Delta$ and $f \in A$. Let $(L^{\lambda+1}_a(b,c;\beta)f(z))'$ be univalent with $\frac{L^{\lambda+1}_a(b,c;\beta)f(z)}{z} \in H[1,n] \cap Q$. If $q_1(z) \prec (L^{\lambda+1}_a(b,c;\beta)f(z))' \prec q_2(z)$ then $q_1(z) \prec \frac{L^{\lambda+1}_a(b,c;\beta)f(z)}{z} \prec q_2(z)$. Moreover the functions $q_1(z)$ and $q_2(z)$ are the best subordinant and the best dominant respectively.

References