



Existence of common best proximity points of generalized S -proximal contractions

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Abstract

In this article, we introduce a new notion of proximal contraction, named as generalized S -proximal contraction and derive a common best proximity point theorem for proximally commuting non-self mappings, thereby yielding the common optimal approximate solution of some fixed point equations when there is no common solution. We furnish illustrative examples to highlight our results. We extend some results existing in the literature.

Keywords: common best proximity point; optimal approximate solution; proximally commuting mappings.

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1. Introduction

Fixed point theory focusses on the strategies for solving non-linear equations of the kind $Tx = x$ in which T is a self-mapping defined on a subset of a metric space, a normed linear space, a topological vector space or some pertinent framework. But, when T is not a self-mapping, it is plausible that $Tx = x$ has no solution. Consequently, one targets to determine an element x that is in some sense close to Tx .

In the case when a fixed point equation does not possess a solution, the “Best approximation pair theorems” and “Best proximity pair theorems” are explored as alternative. A best approximation theorem provides sufficient conditions to ascertain the existence of a point $x \in A$, known as a best

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approximant, such that $d(x, Tx) = d(Tx, A)$. This line of research started with the work of Ky Fan [7] and subsequently, this result was extended in various directions (see, e.g., [14, 20, 23]). On the other hand, a best proximity point theorem establishes sufficient conditions for the existence of a point $x \in A$ such that the error $d(x, Tx)$ is minimum. Essentially, the global minimization of the real valued function $x \mapsto d(x, Tx)$ is searched for. Also, best proximity point theorems evolve as a natural generalization of fixed point theorems, since a best proximity point becomes a fixed point if the given mapping is a self-mapping.

If two non-self mappings, $S, T : A \rightarrow B$ are given, the equations $Sx = x$ and $Tx = x$ are likely to have no common solution. Hence, the common best proximity point problem is posed, concerning the existence of common optimal approximate solutions. Since, given any point $x \in A$, the distances $d(x, Sx)$ and $d(x, Tx)$ are at least $d(A, B)$, a common best proximity point theorem affirms the global minimum of both functions $x \mapsto d(x, Sx)$ and $x \mapsto d(x, Tx)$ by imposing a common approximate solution of the equations $Sx = x$ and $Tx = x$ to satisfy the condition $d(x, Sx) = d(x, Tx) = d(A, B)$. There exist a lot of papers treating this problem, see, e.g. [1, 2, 3, 5, 6, 7, 11, 12, 13, 14, 17, 18, 19, 20, 21, 22, 23, 24] and the references therein. In particular, Sadiq Basha proved in [18] the existence of solution for the mentioned problem in the case of a proximally commuting pair of mappings under so-called proximal contractions.

The intention of this article is to provide a solution to a more general problem than the one just explained. Thus, we consider two non-self mappings that are proximally commuting and bring in a concept of generalized S -proximal contraction. Our results are natural extensions of the results by Sadiq Basha [18]. Further, the presented theorems contain and complement some well-known common fixed point theorems, e.g., those from [4, 8, 9, 10] for commuting self-mappings.

2. Preliminaries

Let \mathbb{N} denote the set of non-negative integers and \mathbb{N}_+ denote the set of positive integers. In what follows, unless otherwise specified, (\mathcal{X}, d) will always be a metric space, and A, B will be two non-empty subsets of \mathcal{X} . It is standard to use the following notation:

$$\begin{aligned} d(A, B) &:= \inf\{d(x, y) : x \in A \text{ and } y \in B\}, \\ A_0 &:= \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}, \\ B_0 &:= \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}. \end{aligned}$$

In the framework of normed linear spaces, if A and B are closed subsets satisfying the condition $d(A, B) > 0$, then it can be proved that A_0 and B_0 are contained in the boundaries of A and B , respectively (see [17]). On the other hand, if $A \cap B \neq \emptyset$, then $A \cap B$ is contained in both A_0 and B_0 .

Definition 2.1. 1. A point $x \in A$ is said to be a common best proximity point of non-self mappings $S : A \rightarrow B$ and $T : A \rightarrow B$ if $d(x, Sx) = d(x, Tx) = d(A, B)$.
2. A is said to be approximatively compact with respect to B if every sequence $\{x_n\}_{n \in \mathbb{N}}$ in A , satisfying the condition $d(y, x_n) \rightarrow d(y, A)$ as $n \rightarrow \infty$ for some $y \in B$, has a convergent subsequence.

Remark 2.2. It is evident that every set is approximatively compact with respect to itself. Also, every compact set is approximatively compact with respect to any set. Further, it can be seen that if A is compact and B is approximatively compact with respect to A , then the sets A_0 and B_0 are non-empty.

Definition 2.3. [18] Let $S : A \rightarrow B$ and $T : A \rightarrow B$ be two non-self mappings.

1. S and T are said to commute proximally if they satisfy the condition

$$[d(u, Sx) = d(v, Tx) = d(A, B)] \Rightarrow Sv = Tu$$

for all x, u and v in A .

2. S and T can be swapped proximally if

$$[d(y, u) = d(y, v) = d(A, B) \text{ and } Su = Tv] \Rightarrow Sv = Tu$$

for all $u, v \in A$ and $y \in B$.

Clearly, any two self mappings on the same set can be swapped proximally.

Example 2.4. Let $\mathcal{X} = \mathbb{R}^2$ be endowed with the Euclidean metric and let $A = \{(x, y) : x \geq 1\}$ and $B = \{(x, y) : x \leq 0\}$. Define $S, T : A \rightarrow B$ by

$$S(x, y) = \left(\frac{1-x}{5}, 2y \right) \text{ and } T(x, y) = \left(\frac{1-x}{2}, 5y \right).$$

1. If

$$d((x_1, y_1), S(x_3, y_3)) = d((x_2, y_2), T(x_3, y_3)) = d(A, B) = 1$$

for some $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) in A , then we deduce that $x_1 = x_2 = x_3 = 1$, $y_1 = 2y_3$ and $y_2 = 5y_3$ and so $S(x_2, y_2) = T(x_1, y_1)$ for such (x_3, y_3) , that is, by Definition 2.3.(1), S and T commute proximally.

2. If

$$d((x_1, y_1), (x_2, y_2)) = d((x_1, y_1), (x_3, y_3)) = d(A, B) = 1 \text{ and } S(x_2, y_2) = T(x_3, y_3)$$

for some $(x_2, y_2), (x_3, y_3) \in A$ and $(x_1, y_1) \in B$, then we deduce that $x_1 = 0$ and $x_2 = x_3 = 1$, $y_1 = y_2 = y_3 = 0$, and so $S(x_3, y_3) = T(x_2, y_2)$. By Definition 2.3.(2), S and T can be swapped proximally.

Now, we define the notion of generalized proximal contraction for a pair of non-self mappings.

Definition 2.5. Let (\mathcal{X}, d) be a metric space, A and B be two non-empty subsets of \mathcal{X} , $S : A \rightarrow B$ and $T : A \rightarrow B$ be two non-self mappings. Then the pair (S, T) is said to be a generalized S -proximal contraction if there exist non-negative real numbers $\alpha, \beta, \gamma, \delta$ with $\alpha + 2\beta + 2\gamma + \delta < 1$ such that

$$\begin{aligned} d(Sx, Sy) \leq & \alpha d(Tx, Ty) + \beta [d(Tx, Sx) + d(Ty, Sy)] \\ & + \gamma [d(Sx, Ty) + d(Sy, Tx)] + \frac{\delta [1 + d(Tx, Sx)] d(Ty, Sy)}{1 + d(Tx, Ty)} \end{aligned} \quad (2.1)$$

for all $x, y \in A$.

In the next section, we will show that this definition is useful to obtain common best proximity point results. Here, we consider the following example.

Example 2.6. Let $\mathcal{X} = A = B = \mathbb{R}$ be endowed with the Euclidean metric and let $S, T : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$Sx = \begin{cases} 1/2, & \text{if } x \leq -1 \\ 0, & \text{if } x > -1 \end{cases} \quad \text{and} \quad Tx = x \text{ for all } x \in \mathbb{R}.$$

Obviously, $d(A, B) = 0$. It is clear that there exists no $\alpha \in (0, 1)$ such that the condition (2.1) is satisfied with $\beta = \gamma = \delta = 0$ (for example, consider $x = -1/2$ and $y = -1$). On the other hand, if we assume $\alpha = 0$, it is easy to show that condition (2.1) is satisfied with $\beta = \gamma = 7/32$, $\delta = 0$. It follows that our condition assures a proper extension of the corresponding condition in [18, Theorem 3.1].

3. Common best proximity points for generalized S -proximal contractions

In our main theorem, we prove the existence of a common best proximity point for a generalized S -proximal contraction of proximally commuting non-self mappings.

Theorem 3.1. *Let (\mathcal{X}, d) be a complete metric space, A and B be two non-empty closed subsets of \mathcal{X} such that A is approximatively compact with respect to B , and A_0, B_0 are non-empty sets. Further, let $S, T : A \rightarrow B$ be two non-self mappings satisfying the following conditions:*

- (i) *the pair (S, T) is a generalized S -proximal contraction;*
- (ii) *S and T are continuous;*
- (iii) *S and T commute proximally;*
- (iv) *S and T can be swapped proximally;*
- (v) *$S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$.*

Then, there exists a point $x \in A$ such that $d(x, Sx) = d(x, Tx) = d(A, B)$. Moreover, if y is another common best proximity point of S and T , then $d(x, y) \leq 2d(A, B)$.

Proof . Let $x_0 \in A_0$. It follows from the assumption (v) that there exists $x_1 \in A_0$ such that $Sx_0 = Tx_1$. Further, one can find $x_2 \in A_0$ satisfying the condition $Sx_1 = Tx_2$, and, by induction, a sequence $\{x_n\}_{n \in \mathbb{N}}$ of points in A_0 such that $Sx_{n-1} = Tx_n$ for all $n \in \mathbb{N}_+$. Since the pair (S, T) satisfies condition (i), we have

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &\leq \alpha d(Tx_n, Tx_{n+1}) + \beta [d(Tx_n, Sx_n) + d(Tx_{n+1}, Sx_{n+1})] \\ &\quad + \gamma [d(Sx_n, Tx_{n+1}) + d(Sx_{n+1}, Tx_n)] \\ &\quad + \frac{\delta [1 + d(Tx_n, Sx_n)] d(Tx_{n+1}, Sx_{n+1})}{1 + d(Tx_n, Tx_{n+1})} \\ &= (\alpha + \beta + \gamma) d(Sx_{n-1}, Sx_n) + (\beta + \gamma + \delta) d(Sx_n, Sx_{n+1}), \end{aligned}$$

which implies that

$$d(Sx_n, Sx_{n+1}) \leq h d(Sx_{n-1}, Sx_n)$$

where $h = \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma - \delta} < 1$. It follows that $\{Sx_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence and hence converges to some $y \in B$. Consequently, the sequence $\{Tx_n\}_{n \in \mathbb{N}}$ also converges to y .

Using that $S(A_0)$ is contained in B_0 (assumption (v)), we conclude that there exist points $u_n \in A$ such that

$$d(Sx_n, u_n) = d(A, B)$$

for every $n \in \mathbb{N}$. Therefore, it follows from the choice of x_n that

$$d(Tx_n, u_{n-1}) = d(Sx_{n-1}, u_{n-1}) = d(A, B)$$

for every $n \in \mathbb{N}_+$. Now, using the assumption (iii), we get that

$$Tu_n = Su_{n-1}$$

for every $n \in \mathbb{N}_+$.

On the other hand, we have

$$\begin{aligned} d(y, A) &\leq d(y, u_n) \leq d(y, Sx_n) + d(Sx_n, u_n) \\ &= d(y, Sx_n) + d(A, B) \leq d(y, Sx_n) + d(y, A), \end{aligned}$$

which implies that $d(y, u_n) \rightarrow d(y, A)$ as $n \rightarrow \infty$. Since A is approximatively compact with respect to B , the sequence $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ converging to some $u \in A$. Again, since $d(y, u_{n_k-1}) \rightarrow d(y, A)$ as $k \rightarrow \infty$ and A is approximatively compact with respect to B , the sequence $\{u_{n_k-1}\}$ has a subsequence $\{u_{n_{k_j}-1}\}$ converging to some $v \in A$. Using the continuity of S and T (assumption (ii)), we obtain that

$$Tu = \lim_{j \rightarrow \infty} Tu_{n_{k_j}} = \lim_{j \rightarrow \infty} Su_{n_{k_j}-1} = Sv.$$

Further, we have

$$d(y, u) = \lim_{k \rightarrow \infty} d(Sx_{n_k}, u_{n_k}) = d(A, B)$$

and

$$d(y, v) = \lim_{j \rightarrow \infty} d(Tx_{n_{k_j}}, u_{n_{k_j}-1}) = d(A, B).$$

Using the assumption (iv), we deduce that $Tv = Su$.

Now, using the inequality (2.1), we have

$$\begin{aligned} d(Su, Sv) &\leq \alpha d(Tu, Tv) + \beta [d(Tu, Su) + d(Tv, Sv)] \\ &\quad + \gamma [d(Su, Tv) + d(Sv, Tu)] + \frac{\delta [1 + d(Tu, Su)] d(Tv, Sv)}{1 + d(Tu, Tv)} \\ &= (\alpha + 2\beta) d(Su, Sv), \end{aligned}$$

which implies that $Su = Sv$ and hence $Tu = Su$. Using again that $S(A_0) \subseteq B_0$, we obtain that there exists a point $x \in A$ such that

$$d(x, Tu) = d(A, B) \text{ and } d(x, Su) = d(A, B).$$

Applying assumption (iii), we conclude that $Sx = Tx$. Again from (2.1), we have

$$\begin{aligned} d(Su, Sx) &\leq \alpha d(Tu, Tx) + \beta [d(Tu, Su) + d(Tx, Sx)] \\ &\quad + \gamma [d(Su, Tx) + d(Sx, Tu)] + \frac{\delta [1 + d(Tu, Su)] d(Tx, Sx)}{1 + d(Tu, Tx)} \\ &= \alpha d(Su, Sx), \end{aligned}$$

which implies that $Su = Sx$ and hence $Tu = Tx$. It follows that

$$d(x, Tx) = d(x, Tu) = d(A, B) \text{ and } d(x, Sx) = d(x, Su) = d(A, B).$$

Thus, x is a common best proximity point of S and T .

Suppose now that y is another common best proximity point of S and T , i.e.,

$$d(y, Sy) = d(A, B) \text{ and } d(y, Ty) = d(A, B).$$

Since S and T commute proximally, we have that $Sx = Tx$ and $Sy = Ty$. By (2.1), we have

$$\begin{aligned} d(Sx, Sy) &\leq \alpha d(Tx, Ty) + \beta [d(Tx, Sx) + d(Ty, Sy)] \\ &\quad + \gamma [d(Sx, Ty) + d(Sy, Tx)] + \frac{\delta [1 + d(Tx, Sx)] d(Ty, Sy)}{1 + d(Tx, Ty)} \\ &= \alpha d(Sx, Sy), \end{aligned}$$

which implies that $Sx = Sy$. Therefore, we deduce that

$$d(x, y) \leq d(x, Sx) + d(Sx, Sy) + d(y, Sy) = 2d(A, B)$$

and the proof is completed. \square

Remark 3.2. If $\beta = \gamma = \delta = 0$ in Theorem 3.1, then we obtain Theorem 3.1 of Sadiq Basha [18].

If S and T are commuting self mappings, then Theorem 3.1 gives us the following common fixed point theorem, that extends and complements the results of Chatterjea [4], Hardy [8], Jungck [9], Kannan [10], Reich [15, 16] and hence the Banach's contraction principle.

Theorem 3.3. *Let (\mathcal{X}, d) be a complete metric space. Assume that $S, T : \mathcal{X} \rightarrow \mathcal{X}$ are two self mappings satisfying the following conditions:*

(i) *there are non-negative real numbers $\alpha, \beta, \gamma, \delta$ with $\alpha + 2\beta + 2\gamma + \delta < 1$ such that*

$$\begin{aligned} d(Sx, Sy) &\leq \alpha d(Tx, Ty) + \beta [d(Tx, Sx) + d(Ty, Sy)] \\ &\quad + \gamma [d(Sx, Ty) + d(Sy, Tx)] + \frac{\delta [1 + d(Tx, Sx)] d(Ty, Sy)}{1 + d(Tx, Ty)} \end{aligned}$$

for all $x, y \in \mathcal{X}$;

(ii) *S and T are continuous;*

(iii) *S and T commute;*

(iv) *$S(\mathcal{X}) \subseteq T(\mathcal{X})$.*

Then, S and T have a unique common fixed point.

4. Illustrative examples

To illustrate our Theorem 3.1, we give the following example.

Example 4.1. Let \mathcal{X}, A, B, S and T be defined as in Example 2.4. Then, $d(A, B) = 1$, $A_0 = \{(1, y) : y \in \mathbb{R}\}$ and $B_0 = \{(0, y) : y \in \mathbb{R}\}$. Now, we have

$$\begin{aligned} d^2(S(x_1, y_1), S(x_2, y_2)) &= \frac{1}{25}(x_1 - x_2)^2 + 4(y_1 - y_2)^2 \\ &= \frac{4}{25} \left[\frac{1}{4}(x_1 - x_2)^2 + 25(y_1 - y_2)^2 \right] \\ &= \left(\frac{2}{5} \right)^2 [d(T(x_1, y_1), T(x_2, y_2))]^2 \end{aligned}$$

for all $(x_1, y_1), (x_2, y_2) \in A$. Therefore, the pair (S, T) is a generalized S -proximal contraction with $\alpha = \frac{2}{5}$ and $0 \leq \beta, \gamma < \frac{1}{10}$, $0 \leq \delta < \frac{1}{5}$. It is easy to see that all other hypotheses of Theorem 3.1 are satisfied and $(1, 0)$ is a common best proximity point of S and T .

Example 4.2. Let \mathcal{X} , S and T be defined as in Example 2.6. It is easy to show that all the conditions of Theorem 3.3 are satisfied and 0 is the unique common fixed point of S and T .

In the following example, we show that all the conditions of Theorem 3.1 are crucial for its validity.

Example 4.3. Let \mathbb{R}^2 be endowed with the Euclidean metric and let $A = \{(x, -1) : x \in \mathbb{R}\}$, $B = \{(x, 1) : x \in \mathbb{R}\}$. Then $d(A, B) = 2$, $A_0 = A$ and $B_0 = B$. Define mappings $S, T : A \rightarrow B$ by

$$S(x, -1) = \left(\frac{1}{4 + |x|}, 1 \right), \quad T(x, -1) = \left(\frac{2}{2 + |x|}, 1 \right).$$

Then, obviously, $S(A_0) \subseteq B_0$ and $T(A_0) \subseteq B_0$. Note that, for $x_1, x_2 \in \mathbb{R}$,

$$d(S(x_1, -1), S(x_2, -1)) = \left| \frac{1}{4 + |x_1|} - \frac{1}{4 + |x_2|} \right| = \frac{||x_2| - |x_1||}{(4 + |x_1|)(4 + |x_2|)},$$

$$d(T(x_1, -1), T(x_2, -1)) = \left| \frac{2}{2 + |x_1|} - \frac{2}{2 + |x_2|} \right| = 2 \frac{||x_2| - |x_1||}{(2 + |x_1|)(2 + |x_2|)}.$$

Hence, the inequality (2.1) is satisfied for all $x_1, x_2 \in \mathbb{R}$ if $\alpha, \beta, \gamma, \delta \geq 0$ are chosen such that $\alpha = \frac{1}{2}$ and $2\beta + 2\gamma + \delta < \frac{1}{2}$. Thus, the pair (S, T) is a generalized S -proximal contraction.

In order to check the condition of Definition 2.4.2., suppose that $y, u, v \in \mathbb{R}$ are such that $d((y, 1), (u, -1)) = d((y, 1), (v, -1)) = 2$. It follows that $y = u = v$. But then $S(u, -1) = T(v, -1)$ would imply that $\frac{1}{4+|u|} = \frac{2}{2+|u|}$, which is impossible. Thus, the mappings S and T can (trivially) be swapped proximally.

Suppose now that $d((u, -1), S(x, -1)) = d((v, -1), T(x, -1)) = 2$. Then $u = \frac{1}{4+|x|}$ and $v = \frac{2}{2+|x|}$, but

$$S(v, -1) = \left(\frac{2 + |x|}{10 + 4|x|}, 1 \right) \neq \left(\frac{2(4 + |x|)}{5 + |x|}, 1 \right) = T(u, -1).$$

Hence, S and T do not commute proximally.

All the conditions of Theorem 3.1 are satisfied except proximal commutativity of the given mappings and they have no common best proximity points.

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