Int. J. Nonlinear Anal. Appl. 8 (2017) No. 2, 251-261
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.1664.1439

(φ_1, φ_2)-variational principle

Abdelhakim Maaden*, Abdelkader Stouti

Université Sultan Moulay Slimane, Faculté des Sciences et Techniques, Laboratoire de Mathématiques et Applications, B.P. 523, Beni-Mellal 23000, Maroc

(Communicated by M. Eshaghi)

Abstract

In this paper we prove that if X is a Banach space, then for every lower semi-continuous bounded below function f, there exists a (φ_1, φ_2)-convex function g, with arbitrarily small norm, such that $f + g$ attains its strong minimum on X. This result extends some of the well-known varitional principles as that of Ekeland [On the variational principle, J. Math. Anal. Appl. 47 (1974) 323–353], that of Borwein-Preiss [A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions, Trans. Amer. Math. Soc. 303 (1987) 517–527] and that of Deville-Godefroy-Zizler [Un principe variationel utilisant des fonctions bosses, C. R. Acad. Sci. (Paris). Ser.I 312 (1991) 281–286] and [A smooth variational principle with applications to Hamilton-Jacobi equations in infinite dimensions, J. Funct. Anal. 111 (1993) 197–212].

Keywords: (φ_1, φ_2)-convex function; (φ_1, φ_2)-variational principle; Ekeland's variational principle; smooth variational principle.
2010 MSC: Primary 26A51; Secondary 52A30, 46B20.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space. Let f be a real-valued function defined on X, lower semicontinuous and bounded below. Let P be a class of functions in X which serves as a source of possible perturbations for f. By a variational principle we mean an assertion ensuring the existence of at least one perturbation g from P such that $f + g$ attains its minimum on X.

The first variational principle, based on the Bishop-Phelps lemma [3, 27], was established by Ekeland [18]. In this case, P is just the set $\{\epsilon\|x - a\|; \epsilon > 0, a \in X\}$.

*Corresponding author
Email addresses: hmaaden2002@yahoo.fr (Abdelhakim Maaden), stouti@yahoo.com (Abdelkader Stouti)

Received: October 2016 *Revised:* January 2017

If g is required to be smooth, then we speak about a smooth variational principle. The first result of this type was shown by Stegall [31, 27], where P is the elements of the dual space X^*. He proved that if X has the Radon-Nikodym property in particular; if X is reflexive; and if $dom\,(f) = \{x \in X, f(x) < +\infty\}$ is bounded and non empty, then one can take for g even a linear functional, with arbitrarily small norm. In [17], Deville-Maaden showed that if X has the Radon-Nikodym property and if the function f is lower semicontinuous and super-linear, then a variational principle holds whenever P is the set of bounded, Lipschitz, Frechet-differentiable and weakly continuous functions. However, this principle does not cover some important Banach spaces. For example the space c_0 does not have the Radon-Nikodym property while it, in fact, admits a smooth norm [5]. In this direction Borwein-Preiss [6] proved a smooth variational principle imposing only the existence of an equivalent smooth norm $\|\cdot\|$. In this case, P is the set of infinite convex combinations of translates of the square of the norm. Haydon [23] showed that there exists a Banach spaces with smooth bump function without an equivalent smooth norm (a function b is bump if it has a non empty and bounded support). So, the variational principle of Borwein-Preiss is not applicable in this space. So that, Deville et al [14, 15] extended the Borwein-Preiss variational principle to spaces with smooth bump function, with P equal to the family of Lipschitz smooth functions.

In an analytical approach we can often associate a geometrical approach to complete study of which or stimulates the analytical approach. From this perspective Browder [8] gave a geometrical result which bears at present the name of the Drop Theorem (see also [10]). Penot in [26, 21] showed that the drop theorem is a geometrical version of the Ekeland's variational principle. After this, Maaden in [25, 22] introduced and studied the notion of the smooth drop which can be seen as a geometrical version of the smooth variational principle of Borwein-Preiss.

Those variational principles are a tools that have been very important in nonlinear analysis, in that they enjoyed a big deal of applications from the geometry of Banach spaces [3, 4, 7] to the optimization theory [18, 19, 30] and of generalized differential and sub-differential calculus [1, 2, 6, 11, 13, 12, 26], calculus of variations [9, 18] up to the nonlinear semi-groups theory [7, 18] and the viscosity solutions of Hamilton-Jacobi equations [13, 12, 15].

In [28, 29], Pini et all defined the notion of (φ_1, φ_2)-convex functions. They say that a real valued function f defined on a non empty subset D of $\mathbb{R}^n$ is (φ_1, φ_2)-convex if $f\left(\varphi_1\left(x, y, \lambda\right)\right) \leq \varphi_2\left(x, y, \lambda, f\right)$ for all $x, y \in D$ and for all $\lambda \in [0, 1]$, where φ_1 is a function from $D \times D \times [0, 1]$ in $\mathbb{R}^n$ and φ_2 is a function from $D \times D \times [0, 1] \times F$ in $\mathbb{R}$, with F is a given vector space of real valued functions defined on the set D. In this paper we shall use the same definition of (φ_1, φ_2)-convex functions as above with using any Banach spaces instead of $\mathbb{R}^n$. In this way, we prove that under suitable choices of the functions φ_1 and φ_2 a new variational principle for the set of (φ_1, φ_2)-convex functions (see Theorem 3.1). This (φ_1, φ_2)- variational principle is providing a unified framework to deduce Ekeland's, Borwein-Preiss's and Deville's variational principles.

2. Auxiliaries results

In this section we shall give some definitions and establish some auxiliaries results which we shall use to prove our main result in this paper.

Let $(X, \|\cdot\|)$ be a Banach space. For a continuous function $f : X \longrightarrow \mathbb{R}$ we define

$$\mu\left(f\right) = \sum_{n=1}^{\infty} \frac{\|f\|_n}{2^n},$$

where

$$\|f\|_n = \sup\left\{\left|f\left(x\right)\right|; x \in X, \|x\| \leq n\right\}.$$

Let M be the set of all continuous functions f such that $\mu(f) < \infty$. It is routine to check that (M, μ) is a Banach space.

Let $\varphi_1 : X \times X \times [0,1] \longrightarrow X$ and $\varphi_2 : X \times X \times [0,1] \times F \longrightarrow \mathbb{R}$, two functions where F is a given set of real functions on X. Define,

Definition 2.1. A function $f : X \longrightarrow \mathbb{R}$ is said to be (φ_1, φ_2)-convex if

$$f(\varphi_1(x, y, \lambda)) \leq \varphi_2(x, y, \lambda, f), \forall x, y \in X, \forall \lambda \in [0,1].$$

We notice that under suitable assumptions on φ_1 and/or φ_2, the class of (φ_1, φ_2)- convex functions is a convex cone. For example:
1) If φ_2 is super-linear with respect to $f \in F$ (that φ_2 is super-additive and positively homogeneous), then the class of (φ_1, φ_2)-convex functions is a convex cone.

Indeed, let f, g are two (φ_1, φ_2)-convex functions and $\alpha > 0$. Then, for $x, y \in X$ and $\lambda \in [0,1]$ we have

$$\begin{aligned}(f+g)(\varphi_1(x, y, \lambda)) &\leq \varphi_2(x, y, \lambda, f) + \varphi_2(x, y, \lambda, g) \\ &\leq \varphi_2(x, y, \lambda, f+g)\end{aligned}$$

and

$$\begin{aligned}(\alpha f)(\varphi_1(x, y, \lambda)) &= \alpha(f(\varphi_1(x, y, \lambda))) \\ &\leq \alpha\varphi_2(x, y, \lambda, f) \\ &= \varphi_2(x, y, \lambda, \alpha f).\end{aligned}$$

2) If $\varphi_2(x, y, \lambda, f) = C((1-\lambda) f(x) + \lambda f(y))$ for some $C > 0$, the set of (φ_1, φ_2)-convex functions is a convex cone.

In all the sequel, we define the following sets:

$$\Phi = \{f \in M : f \text{ is } (\varphi_1, \varphi_2) - \text{convex and } f \geq 0\},$$

$$F = \{f \in \Phi : f(x) \longrightarrow +\infty \text{ as } \|x\| \longrightarrow +\infty\}.$$

The metric ρ on Φ is defined as:

$$\rho(f, g) = \mu(f-g) = \sum_{n \geq 1} \frac{\|f-g\|_n}{2^n} \text{ for all } f, g \in \Phi,$$

and it is easy to show that (Φ, ρ) is a complete metric space.

Throughout this paper, the functions φ_1 and φ_2 satisfies the following assumptions:

(P_1) $\varphi_1(x, x, 0) = x; \forall x \in X;$
(P_2) $\varphi_1(x, y, \lambda) + \varphi_1(z, z, 0) = \varphi_1(x+z, y+z, \lambda); \forall x, y, z \in X, \forall \lambda \in [0,1];$
(P_3) $\exists C \geq 1$, such that $\varphi_2(\lambda x, \lambda x, 0, h) \leq C[(1-\lambda) h(0) + \lambda h(x)]; \forall x \in X, \forall \lambda \in [0,1], \forall h \in \Phi;$
(P_4) For $x_0 \in X$, $\varphi_2(x - x_0, y - x_0, \lambda, h) \leq \varphi_2(x, y, \lambda, h(. - x_0)); \forall x, y \in X, \forall \lambda \in [0,1]; \forall h \in \Phi;$
(P_5) The class of (φ_1, φ_2)- convex functions is a convex cone.
We will also assume that φ_1 is continuous with respect to λ.

Example 2.2. If $\varphi_1(x, y, \lambda) = \lambda x + (1-\lambda) y$ and $\varphi_2(x, y, \lambda, f) = \lambda f(x) + (1-\lambda) f(y)$ then properties $(P_1), \ldots, (P_5)$ are satisfied and in this case, a function f is (φ_1, φ_2)-convex if and only if f is convex.

We present now two preliminaries lemmas, which are useful for the proof of our principal result of this paper. In the first, we use (P_1) and (P_3) to prove the following:

Lemma 2.3. *Let $h \in \Phi$ and let $y = \lambda x$, $\lambda > 1$. Then,* $h(y) - h(0) \geq \dfrac{\lambda}{C}\left(h(x) - Ch(0)\right).$

Proof . Let $\mu = 1/\lambda$. Then $x = \mu y$. By using (P_1) and (P_3) we obtain

$$\begin{aligned} h(x) &= h(\mu y) \\ &= h(\varphi_1(\mu y, \mu y, 0)) \\ &\leq \varphi_2(\mu y, \mu y, 0, h) \\ &\leq C\left((1-\mu)h(0) + \mu h(y)\right). \end{aligned}$$

Consequently, we get

$$h(x) - Ch(0) \leq C\mu\left(h(y) - h(0)\right).$$

Since $c > 0$ and $\mu > 0$, we deduce

$$h(y) - h(0) \geq \frac{1}{C\mu}\left(h(x) - Ch(0)\right) = \frac{\lambda}{C}\left(h(x) - Ch(0)\right)$$

and the proof is complete. □

Next, by using $(P_1), (P_2)$ and (P_4) we obtain the following:

Lemma 2.4. *Let θ be a (φ_1, φ_2)-convex function and let $h(x) = \theta(x - x_0)$. Then, h is a (φ_1, φ_2)-convex function.*

Proof . Let $x, y \in X$ and $\lambda \in [0, 1]$. By using $(P_1), (P_2)$ and (P_4) we get

$$\begin{aligned} h(\varphi_1(x, y, \lambda)) &= \theta(\varphi_1(x, y, \lambda) - x_0) \\ &= \theta(\varphi_1(x, y, \lambda) + \varphi_1(-x_0, -x_0, 0)) \\ &= \theta(\varphi_1(x - x_0, y - x_0, \lambda)) \\ &\leq \varphi_2(x - x_0, y - x_0, \lambda, \theta) \\ &\leq \varphi_2(x, y, \lambda, \theta(. - x_0)) \\ &= \varphi_2(x, y, \lambda, h), \end{aligned}$$

which shows that h is a (φ_1, φ_2)-convex function. □

Corollary 2.5. Let θ be a (φ_1, φ_2)-convex function in F then $h(x) = \theta(x - x_0)$ is in F.

3. The main result

In this section we shall establish a $(\varphi_1, \varphi_2)-$variational principle. We show that the set P which is a source of perturbation for f, is a class of $(\varphi_1, \varphi_2)-$convex functions. Furthermore we can take them of C^∞ in smooth Banach spaces.

In the mathematical field of topology, a G_δ set is a subset of a topological space that is a countable intersection of open sets. In a complete metric space, a countable union of nowhere dense sets is said to be meagre; the complement of such a set is a residual set.

An element y of a Banach space X is said a strong minimum for a real function f defined on the space X, if $f(y)$ is the infimum of f and any minimizing sequence for f converges to y.

The aim result in this paper is the following variational principle:

Theorem 3.1. *Let X be a Banach space. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function bounded from below. Let Y be a subset of F such that:*

i) the metric ρ_Y in Y is such that $\rho_Y(f, g) = \mu_Y(f - g) \geq \mu(f - g)$, for all $f, g \in Y$;

ii) (Y, ρ_Y) is a Baire space;

iii) there exists $\theta \in Y$ such that $\mu_Y(\theta) < +\infty, \theta(0) = 0$, there is $k \in]0, 1[$ such that for every $\|x\| \geq k$ we have $\theta(x) \geq k^2$ and $\mu_Y(\theta(. - x_0)) \leq \mu_Y(\theta) + \|\theta\|_{\|x_0\|}$.

Then the set

$$\{g \in Y : f + g \text{ attains its strong minimum on } X\}$$

is residual in Y.

Next, we shall show that Theorem 3.1 is providing a unified framework to deduce Ekeland's variational principle [18], Borwein-Preiss's [6] variational principle and Deville-Godefroy-Zizler's Variational principle [15].

Application 1. As a first application we get the Ekeland's variational principle [18].

Let $(X, \|\cdot\|)$ be a Banach space. Assume that $\varphi_1(x, y, \lambda) = \lambda x + (1 - \lambda) y$ and $\varphi_2(x, y, \lambda, f) = \lambda f(x) + (1 - \lambda) f(y)$. Then φ_1 and φ_2 satisfies $(P_1), (P_2), (P_3)$ and (P_4). Let

$$Y = \{f : X \longrightarrow \mathbb{R} : f \text{ convex, Lipschitz, } \geq 0, f \longrightarrow +\infty \text{ as } \|x\| \longrightarrow +\infty\}.$$

We define on Y the metric ρ_Y such that for $f, g \in Y$,

$$\rho_Y(f, g) = \mu_Y(f - g) = \sum_{n \geq 1} \frac{\|f - g\|_n}{2^n} + \sup\left\{\frac{|(f - g)(x) - (f - g)(y)|}{\|x - y\|}; x \neq y\right\}.$$

It is clear that (Y, ρ_Y) satisfies (P_5) and the conditions (i) and (ii) of Theorem 3.1. Also, the function $\theta = \|x\|$ satisfies the assertion (iii) of Theorem 3.1. Consequently we have the following:

Corollary 3.2. *Let $(X, \|\cdot\|)$ be a Banach space, consider a lower semi-continuous bounded below function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$. Then for each $\varepsilon > 0$, there exists $x_0 \in X$ such that*

$$f(x) + \varepsilon\|x - x_0\| \geq f(x_0).$$

Proof . From Theorem 3.1, for each $\varepsilon > 0$, there exists $g \in Y$ such that $\mu_Y(g) < \varepsilon$ and $f + g$ attains a strong minimum at x_0. Therefore, for all $x \in X$,

$$f(x) + g(x) \geq f(x_0) + g(x_0) \text{ and } \sum_{n \geq 1} \frac{\|g\|_n}{2^n} + \sup\left\{\frac{|g(x) - g(y)|}{\|x - y\|}; x \neq y\right\} < \varepsilon,$$

which implies that

$$\begin{aligned} f(x) &\geq f(x_0) + g(x_0) - g(x) \\ &\geq f(x_0) - \varepsilon\|x - x_0\|. \end{aligned}$$

□

Application 2. Let $(X, \|\cdot\|)$ be a Banach space with smooth norm. Assume that $\varphi_1(x, y, \lambda) = \lambda x + (1 - \lambda) y$ and $\varphi_2(x, y, \lambda, f) = \lambda f(x) + (1 - \lambda) f(y)$. Then φ_1 and φ_2 satisfies (P_1), (P_2), (P_3), and (P_4). Let

$$Y = \left\{f : X \longrightarrow \mathbb{R}; f \text{ is } C^1\text{-smooth, Lipschitz, convex, } \geq 0 \text{ and } f \longrightarrow +\infty \text{ as } \|x\| \longrightarrow +\infty\right\}.$$

We define the metric ρ_Y in Y by:

$$\rho_Y(f,g) = \mu_Y(f-g) = \sum_{n\geq 1} \frac{\|f-g\|_n}{2^n} + \|(f-g)'\|_\infty \text{ for all } f,g \in Y$$

where $\|f'\|_\infty := \sup_{\|x\|\leq 1} \|f'(x)\|_{X^*}$ and the space (Y, ρ_Y) satisfies (i) and (ii) of Theorem 3.1 and so also (P_5).

Let

$$h : [0,+\infty[\longrightarrow [0,+\infty[$$
$$t \longmapsto \begin{cases} t^2 & \text{if } \quad 0 \leq t \leq 1 \\ 2t-1 & \text{if } \quad t > 1. \end{cases}$$

The function $\theta(x) = h(\|x\|) \in Y$ satisfies the assertion (iii) of Theorem 3.1.

Therefore, we have the Borwein-Preiss's variational principle [6, 27]:

Corollary 3.3. Let $(X, \|\cdot\|)$ be a Banach space with a smooth norm and consider a lower semi-continuous function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ bounded from below. Then the set

$$\{g \in Y : f + g \text{ attains its strong minimum on } X\}$$

is residual in Y.

Application 3. Let X be a Banach space admitting Lipschitz C^1-smooth bump function. According to a construction of Leduc [24], there exists a Lipschitz function $d : X \longrightarrow \mathbb{R}$ which is C^1-smooth on $X \setminus \{0\}$ and satisfies:

i) $d(\lambda x) = \lambda d(x)$ for all $\lambda > 0$ and for all $x \in X$;

ii) there exists $C \geq 1$ such that $\|x\| \leq d(x) \leq C\|x\|$ for all $x \in X$.

Moreover the function d^2 is C^1-smooth on all the space X.

Let $\varphi_1(x,y,\lambda) = \lambda x + (1-\lambda)y$ and $\varphi_2(x,y,\lambda,f) = C^2[\lambda f(x) + (1-\lambda)f(y)]$. Then φ_1 and φ_2 satisfies $(P_1), (P_2), (P_3)$ and (P_4). Let $\theta(x) = d^2(x)$. We have

$$d^2(\lambda x + (1-\lambda)y) \leq C^2\|\lambda x + (1-\lambda)y\|^2.$$

Since the function $\|\cdot\|^2$ is convex, we deduce

$$d^2(\lambda x + (1-\lambda)y) \leq C^2\left(\lambda d^2(x) + (1-\lambda)d^2(y)\right).$$

That is the function d^2 is a (φ_1, φ_2)-convex function.

Let

$$Y = \left\{f \text{ a } (\varphi_1,\varphi_2) - \text{convex}, C^1 - \text{Lipschitz}, \geq 0 \text{ and } f \longrightarrow +\infty \text{ as } \|x\| \longrightarrow +\infty\right\}$$

and so the set Y satisfies (P_5).

The metric ρ_Y on Y is such that, for $f,g \in Y$

$$\rho_Y(f,g) = \mu_Y(f-g) = \sum_{n\geq 1} \frac{\|f-g\|_n}{2^n} + \sum_{n\geq 1} \frac{\|(f-g)'\|_n}{2^n}$$

where $\left\|f^{'}\right\|_n = \sup\limits_{\|x\|\leq n} \left\|f^{'}(x)\right\|_{X^*}$.

On the other hand, let $\theta(x) = d^2(x)$. So that,

i) $\theta(0) = 0$;

ii) $\mu_Y(\theta) < \infty$;

iii) let $0 < k < 1$. Hence, for all $x \in X$ such that $\|x\| \geq k$ we have $d^2(x) \geq \|x\|^2 \geq k^2$. Therefore the function $\theta \in Y$ and satisfies (iii) of Theorem 3.1.

Thus we have the following variational principle (for unbounded functions) of Deville-Godefroy-Zizler [14, 15, 16, 20]:

Corollary 3.4. Let $(X, \|\cdot\|)$ be a Banach space admitting a C^1-Lipschitz bump function and consider a lower semi-continuous bounded below function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$. Then the set

$$\{g \in Y : f + g \text{ attains its strong minimum on } X\}$$

is residual in Y.

Now, we are ready to give the proof of Theorem 3.1.

Proof of Theorem 3.1

Proof . Following the method of [15, 20], for $n \in \mathbb{N} \setminus \{0\}$, we let

$$G_n = \{g \in Y : \exists x_0 \in X, (f+g)(x_0) < \inf\{(f+g)(x) : \|x - x_0\| \geq 1/n\}\}.$$

Claim 1. We claim that G_n is open for each n. Indeed, let $n \in \mathbb{N}$ and $g \in G_n$. So that there is x_0 in X such that

$$(f+g)(x_0) < \inf\{(f+g)(x) : \|x - x_0\| \geq 1/n\}.$$

Let $0 < \varepsilon < 1$ such that

$$(f+g)(x_0) + 2\varepsilon < \inf\{(f+g)(x) : \|x - x_0\| \geq 1/n\}. \tag{3.1}$$

Let $A = C(f+g)(x_0) + C(g(0) - \inf(f)) + (2C+3)\varepsilon$, where C is given by (P_3). Since $g \in Y$, g goes to $+\infty$ as $\|x\|$ goes to $+\infty$. This means that, there is k in $\mathbb{N}$ such that $k > \|x_0\|$ and $g(x) > A$ whenever $\|x\| \geq k$. This is equivalent to say that

$$g(x) > C(f+g)(x_0) + C(g(0) - \inf(f)) + (2C+3)\varepsilon \quad \text{whenever } \|x\| \geq k. \tag{3.2}$$

Let $h \in Y$ such that $\rho_Y(h, g) < \frac{\varepsilon}{2^k}$. We have

$$\sum_{n\geq 1} \frac{\|h-g\|_n}{2^n} \leq \rho_Y(h,g) = \mu_Y(h-g) < \frac{\varepsilon}{2^k}.$$

Thus

$$\frac{\|h-g\|_k}{2^k} < \frac{\varepsilon}{2^k}.$$

So,

$$|h(x) - g(x)| < \varepsilon \quad \text{whenever } \|x\| \leq k, \tag{3.3}$$

in particular

$$|h(x_0) - g(x_0)| < \varepsilon. \tag{3.4}$$

Combining (3.2) with (3.3) we obtain

$$h(x) > C(f+g)(x_0) + C(g(0) - \inf(f)) + (2C+2)\varepsilon > 0 \quad \text{whenever } \|x\| = k.$$

Since $C \geq 1$ and $h \geq 0$, we deduce for $\|x\| = k$ that

$$h(x) \geq \frac{h(x)}{C} > (f+g)(x_0) + (g(0) - \inf(f)) + (2 + (2/C))\varepsilon. \tag{3.5}$$

On the first hand, let $y \in X$ such that $\|y\| > k$. Then, there exist $\lambda > 1$ and $x \in X$ with $\|x\| = k$, such that $y = \lambda x$. By using Lemma 2.3, we deduce

$$h(y) - h(0) \geq \frac{\lambda}{C}(h(x) - Ch(0)) \geq \frac{1}{C}(h(x) - Ch(0)) = \frac{h(x)}{C} - h(0).$$

Combining this with (3.5) we show for $\|y\| \geq k$ that,

$$h(y) - h(0) > (f+g)(x_0) + g(0) - \inf f + \left(2 + \frac{2}{C}\right)\varepsilon - h(0). \tag{3.6}$$

Combining the fact that $h \geq 0, (3.6), (3.3)$ and (3.4) we obtain for all $x \in X$ such that $\|x\| \geq k$:

$$\begin{aligned}
(f+h)(x) &\geq \inf(f) + h(x) \\
&\geq \inf(f) + h(x) - h(0) \\
&> \inf(f) + (f+g)(x_0) + g(0) - \inf(f) + \left(2 + \frac{2}{C}\right)\varepsilon - h(0) \\
&> (f+g)(x_0) + \left(1 + \frac{2}{C}\right)\varepsilon \\
&> (f+h)(x_0) + \frac{2}{C}\varepsilon \\
&> (f+h)(x_0).
\end{aligned}$$

Therefore for all $x \in X$ such that $\|x\| \geq k$, we have

$$(f+h)(x) > (f+h)(x_0).$$

On other hand, if $\|x\| \leq k$ and $\|x - x_0\| \geq 1/n$, and combining (3.4), (3.1) and (3.3) we obtain

$$\begin{aligned}
(f+h)(x_0) &< (f+g)(x_0) + \varepsilon \\
&\leq \inf\{(f+g)(x) : \|x - x_0\| \geq 1/n\} - 2\varepsilon + \varepsilon \\
&\leq (f+g)(x) - \varepsilon \\
&< (f+h)(x).
\end{aligned}$$

Then for all x such that $\|x - x_0\| \geq 1/n$ we have

$$(f+h)(x_0) < (f+h)(x).$$

Hence $h \in G_n$ and G_n is open.

Claim 2. The set G_n is dense in Y. Indeed, let $g \in Y$ and $0 < \varepsilon < 1$. Let $c > 0$ be such that

$$(f+g)(x) > \inf(f+g) + 1 \text{ whenever } \|x\| > c.$$

Let $1 > \delta > 0$ be such that $\delta \left(\mu_Y(\theta) + \|\theta\|_c\right) < \varepsilon$. Let $x_0 \in X$ be such that

$$(f+g)(x_0) < \inf (f+g) + \frac{\delta}{n^2}. \tag{3.7}$$

Since $\dfrac{\delta}{n^2} < 1$, we deduce

$$\|x_0\| \leq c. \tag{3.8}$$

Let $h(x) = \delta\theta(x - x_0)$. Now Corollary 2.5 ensure that h is a (φ_1, φ_2)-convex function in F. From the hypothesis (iii) of Theorem 3.1 and (3.8), we get

$$\rho_Y(h, 0) = \mu_Y(h) = \delta\mu_Y(\theta(\cdot - x_0)) \leq \delta\mu_Y(\theta) + \delta\|\theta\|_{\|x_0\|} \leq \delta\left(\mu_Y(\theta) + \|\theta\|_c\right) < \varepsilon.$$

Now if $\|x - x_0\| \geq 1/n$, and by (iii) of Theorem 3.1 we deduce

$$h(x) = \delta\theta(x - x_0) \geq \frac{\delta}{n^2}.$$

By using (3.7), we get

$$\begin{aligned}
\inf\{f + g + h : \|x - x_0\| \geq 1/n\} &\geq \inf\{f + g : \|x - x_0\| \geq 1/n\} + \frac{\delta}{n^2} \\
&\geq \inf\{f + g\} + \frac{\delta}{n^2} \\
&> (f+g)(x_0) - \frac{\delta}{n^2} + \frac{\delta}{n^2}.
\end{aligned}$$

Moreover $h(x_0) = \delta\theta(0) = 0$, then,

$$\inf\{f + g + h : \|x - x_0\| \geq 1/n\} > (f+g)(x_0) = (f+g+h)(x_0).$$

Thus $(g+h) \in G_n$ and G_n is a dense subset in Y.

Therefore the set $G := \bigcap_{n\geq 1} G_n$ is residual in Y. Following the proof of [15], we can show $f+g$ attains its strong minimum on X for each $g \in G$. To convince the reader we shall present their proof. So, for each $n \geq 1$, there exists $x_n \in X$ such that

$$(f+g)(x_n) < \inf\left\{(f+g)(x); \|x - x_n\| \geq \frac{1}{n}\right\}.$$

We have for each $p > n$, $\|x_p - x_n\| < \frac{1}{n}$ (otherwise, by the definition of x_n, $(f+g)(x_p) > (f+g)(x_n)$ and since $\|x_n - x_p\| \geq \frac{1}{n} \geq \frac{1}{p}$, by the definition of x_p, $(f+g)(x_n) > (f+g)(x_p)$, a contradiction). Thus (x_n) is a Cauchy sequence converging to some $x_\infty \in X$ and we claim that x_∞ is a strong minimum for $f+g$. Indeed, since f is lower semi-continuous,

$$\begin{aligned}
(f+g)(x_\infty) &\leq \liminf (f+g)(x_n) \\
&\leq \liminf \inf\left[\left\{(f+g)(x); \|x - x_n\| \geq \frac{1}{n}\right\}\right] \\
&\leq \inf\{(f+g)(x); x \in X \setminus \{x_\infty\}\}.
\end{aligned}$$

Moreover, let (y_n) be a sequence in X such that $((f+g)(y_n))$ converges to $(f+g)(x_\infty)$. Let us assume that (y_n) does not converge to x_∞. Extracting if necessary a subsequence, we can assume that there exists $\varepsilon > 0$ such that for all n, $\|y_n - x_\infty\| \geq \varepsilon$. Thus there exists an integer p such that $|x_p - y_n| \geq \frac{1}{p}$ for all n. Consequently

$$\begin{aligned}(f+g)(x_\infty) &\leq (f+g)(x_p)\\ &< \inf\left\{(f+g)(x); \|x - x_p\| > \frac{1}{p}\right\}\\ &\leq (f+g)(y_n)\end{aligned}$$

for all n, and this contradicts the convergence of $(f+g)(y_n)$ to $(f+g)(x_\infty)$. □

Acknowledgement.

The authors will thank the referee for his/her valuable suggestions and comments.

References

[1] J.P. Aubin, *Lipshits behavior of solutions to convex minimization problems,* Oper. Res. 9 (1984) 87–111.
[2] J.P. Aubin and I. Ekeland, *Applied Nonlinear Analysis,* Wiley-Interscience, New York, 1984.
[3] E. Bishop and R.R. Phelps, *The Support Functional of Convex Set,* Selected Papers of Errett Bishop, pp. 293–301, 1986.
[4] E. Bishop and R.R. Phelps, *The support cones in Banach spaces and their applications,* Adv. Math. 13 (1974) 1–19.
[5] N. Bonic and J. Frampton, *Smooth functions on Banach manifolds,* J. Math. Mech. 15 (1966) 877–898.
[6] J. Borwein and D. Preiss, *A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions,* Trans. Amer. Math. Soc. 303 (1987) 517–527.
[7] H. Brezis and F. Browder, *A general ordering principle in nonlinear functional analysis,* Adv. In Math. 21 (1976) 355–364.
[8] F.E. Browder, *Normal solvability and the Fredholm altertive for mapping into infinite dimensional manifolds,* J. Funct. Anal. 8 (1971) 250–274.
[9] F.H. Clarke, *Optimization and Nonsmooth Analysis,* Wiley-Interscience, New York, 1983.
[10] J. Danes, *A geometric theorem useful in non-linear functional analysis,* Boll. Un. Math. Ital. 6 (1972) 369–375.
[11] R. Deville, *Stability of subdifferentials of nonconvex functions in Banach spaces,* Set-Valued Anal. 2 (1994) 141–157.
[12] R. Deville and E.M. El Haddad, *The subdifferential of the sum of two functions in Banach spaces. I. First order case,* B. Aust. Math. Soc. 51 (1996) 235–248.
[13] R. Deville and E.M. El Haddad, *The subdifferential of the sum of two functions in Banach spaces. II. Second order case,* J. Convex Anal. 3 (1995) 295–308.
[14] R. Deville, G. Godefroy and V. Zizler, *Un principe variationel utilisant des fonctions bosses,* C. R. Acad. Sci. (Paris). Ser.I 312 (1991) 281–286.
[15] R. Deville, G. Godefroy and V. Zizler, *A smooth variational principle with applications to Hamilton-Jacobi equations in infinite dimensions,* J. Funct. Anal. 111 (1993) 197–212.
[16] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces,* Longman Scientific and Technical, Pitman Monographs and surveys in Pure and Applied Math. 64, 1993.
[17] R. Deville and A. Maaden, *Smooth variational principles in Radon-Nikodym spaces,* B. Aust. Math. Soc. 60 (1999) 109–118.
[18] I. Ekeland, *On the variational principle,* J. Math. Anal. Appl. 47 (1974) 323–353.
[19] I. Ekeland, *Nonconvex minimization problems,* B. Am. Math. Soc. (New Series) I (1979) 443–474.
[20] M. Fabian, P. Hajeik and J. Vanderwerff, *On smooth variational principles in Banach spaces,* J. Math. Anal. Appl. 197 (1996) 153–172.
[21] P. Georgiev, *The strong Ekeland variational principle, the strong drop theorem and applications,* J. Math. Anal. Appl. 131 (1988) 1–21.

[22] P. Georgiev, D. Kutzarova and A. Maaden, *On the smooth drop property,* Nonlinear Anal. Theo. Meth. Appl. 26 (1996) 595–602.

[23] R. Haydon, *A counterexample in several questions about scattered compact spaces,* B. London. Math. Soc. 22 (1990) 261–268.

[24] M. Leduc, *Densité de certaines familles d'hyperplans tangents,* C. R. Acd. Sci (Paris). Ser. A. 270 (1970) 326–328.

[25] A. Maaden, *Théorème de la goutte lisse,* Rocky Mountain J. Math. 25 (1995) 1093–1101.

[26] J.P. Penot, *The drop theorem, the petal theorem and the Ekeland's variational principle,* J. Nonlinear Anal. Appl. 10 (1986) 813–822.

[27] R.R. Phelps, *Convex functions, monotone operators and differentiability,* Lecture notes in Math. 1364 (1989): 378.

[28] R. Pini and C. Singh, (φ_1, φ_2)*-convexity,* Optim. 40 (1997) 103–120.

[29] R. Pini and C. Singh, (φ_1, φ_2)*-optimality and duality under differentiability,* Optim. 41 (1997) 101–116.

[30] H. Riahi, *Application du principe variationnel d'Ekeland à l'existence d'optima de Pareto,* Extr. Math. 7 (1992) 42–45.

[31] C. Stegall, *Optimization of functions on certain subset of Banach spaces,* Math. Ann. 236 (1978) 171–176.