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Computational method based on triangular operational matrices for solving nonlinear stochastic differential equations

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Abstract

In this article, a new numerical method based on triangular functions for solving nonlinear stochastic differential equations is presented. For this, the stochastic operational matrix of triangular functions for Itô integral are determined. Computation of presented method is very simple and attractive. In addition, convergence analysis and numerical examples that illustrate accuracy and efficiency of the method are presented.

Keywords: Brownian motion; Itô integral; Nonlinear stochastic differential equation; Stochastic operational matrix; Triangular function.
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1. Introduction

Mathematical formulation of real problems causes differential equations, integro-differential or partial differential equations involving stochastic excitations of a Gaussian white noise. Such problems are mathematically modeled by stochastic differential equations (SDE), or in more complicated cases, by nonlinear stochastic differential equations of the Itô type. Most of these equations do not have analytical solution, so it is important to find their approximate solution. In recent years, some different numerical methods for solving stochastic differential or stochastic integral equations have been presented ([12]-[9]). The topic of our study is integral form of SDE as follows

$$x(t) = x_0 + \int_0^t k_1(t,s)b(s,x(s))ds + \int_0^t k_2(t,s)\sigma(s,x(s))dB(s), \qquad t \in [0\ 1), \tag{1.1}$$

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where, x_0 is a random variable independent of $B(t)$, $B = (B(t), t \geq 0)$ is a Brownian motion and stochastic process x is a strong solution of Eq. (1.1), it is adapted to $\{\mathcal{F}_t, t \geq 0\}$, furthermore, all Lebesgue's and Itô's integrals in Eq. (1.1) are well defined [11].

Deb et al. in [6], proposed orthogonal triangular function (TF) sets which derived from the block pulse function (BPF) set. They presented the operational matrix for integration in TF and they established that the TF technique is more accurate than the BPF technique. The benefit of TF is that the TF representation does not need any integration to evaluate the coefficients, so it reduces a lot of computational cost. TF approximation has been used for the analysis of dynamic systems [7], integral equations ([13],[4]) and integro-differential equations [3].

In this paper, we derive stochastic operational matrices of TFs, useing them in reducing the nonlinear stochastic differential equation to a set of algebraic equations.

In Section 2, a brief review of TFs is presented. In Section 3, stochastic operational matrices of TFs are derived. Section 4 is devoted to the formulation of nonlinear SDE. In Section 5 convergence analysis of the method is discussed. In Section 6 some numerical examples are provided. Finally, Section 7 gives a brief conclusion.

2. TFs and their properties

Two m-set TFs presented by Deb et al. [6] are defined over the interval [0 T) as follows

$$T1_i(t) = \begin{cases} 1 - \frac{t-ih}{h} & ih \leq t < (i+1)h, \\ 0 & elsewhere, \end{cases}$$

$$T2_i(t) = \begin{cases} \frac{t-ih}{h} & ih \leq t < (i+1)h, \\ 0 & elsewhere, \end{cases}$$

where, $i = 0, \ldots, m-1$, and $h = \frac{T}{m}$. Without loss of generality, it is supposed that the interval of integration is [0 1], since any finite interval [a b] can be transformed to interval [0 1] by linear maps.

TFs, are disjoint, orthogonal, and complete [3].

We consider m-set TF vectors as

$$T1(t) = [T1_0(t), \ldots, T1_{m-1}(t)]^T, \quad T2(t) = [T2_0(t), \ldots, T2_{m-1}(t)]^T,$$

and

$$T(t) = [T1(t), T2(t)]^T.$$

A square integrable function $f(t)$ can be expanded into an m-set TF series as

$$f(t) \simeq \hat{f}(t) = F1^T T1(t) + F2^T T2(t) = F^T T(t), \quad t \in [0\ T), \tag{2.1}$$

where, $F1_i = f(ih)$ and $F2_i = f((i+1)h)$ for $i = 0, \ldots, m-1$. The vectors $F1$ and $F2$ are called the 1D-TF coefficient vectors and $2m$-vector F is defined as

$$F = [F1, F2]^T.$$

The operational matrix for integration can be obtained as [6]

$$\int_0^t T(s)ds = PT(t),$$

where,

$$P = \begin{pmatrix} P1 & P2 \\ P1 & P2 \end{pmatrix},$$

and

$$P1 = \frac{h}{2}\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{m\times m}, \quad P2 = \frac{h}{2}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}_{m\times m}.$$

It can be concluded that

$$T(t)T^T(t)X = \tilde{X}T(t), \tag{2.2}$$

and

$$T^T(t)BT(t) = \hat{B}T(t), \tag{2.3}$$

in which B is a $2m \times 2m$ matrix, X is a $2m$-vector, $\tilde{X} = diag(X)$ and B is a $2m$ vector with elements equal to the diagonal entries of B. In addition, the integral of $f(t)$ can be approximated as follows:

$$\int_0^t f(s)ds \simeq \int_0^t F^T T(s)ds \simeq F^T PT(t).$$

A function of two variables, $k(t, s)$, can be expanded with respect to TFs as follows

$$k(t, s) \simeq \hat{k}(t, s) = T^T(t)KT(s),$$

where K is a $2m_1 \times 2m_2$ coefficient matrix of TFs. For convenience, we put $m_1 = m_2 = m$. So, K can be written as

$$K = \begin{pmatrix} K1 & K2 \\ K3 & K4 \end{pmatrix}_{2m\times 2m},$$

where $K1$, $K2$, $K3$ and $K4$ can be computed by sampling $k(t, s)$ at points s_i and t_i such that $s_i = t_i = ih$, for $i = 0, 1, \dots, m$. So, the following approximations can be obtained

$$\begin{aligned}
(K1)_{ij} &= k(s_i, t_j), \quad i = 0, 1, \dots, m-1, j = 0, 1, \dots, m-1, \\
(K2)_{ij} &= k(s_i, t_j), \quad i = 0, 1, \dots, m-1, j = 1, \dots, m, \\
(K3)_{ij} &= k(s_i, t_j), \quad i = 1, \dots, m, j = 0, 1, \dots, m-1, \\
(K4)_{ij} &= k(s_i, t_j), \quad i = 1, \dots, m, j = 1, \dots, m.
\end{aligned}$$

3. Stochastic operational matrix of TFs

In this section, we obtain stochastic operational matrix of TFs for the Itô integral. Let

$$I(T(t)) = \int_0^t T(s)dB(s) = \int_0^t \begin{pmatrix} T1(s) \\ T2(s) \end{pmatrix} dB(s) = \begin{pmatrix} \int_0^t T1(s)dB(s) \\ \int_0^t T2(s)dB(s) \end{pmatrix}, \tag{3.1}$$

therefore, we compute $\int_0^t T1_i(s)dB(s)$ and $\int_0^t T2_i(s)dB(s)$. By using definition of the unit step function we can rewrite $T1_i(t)$ and $T2_i(t)$ as follows

$$T1_i(t) = \left\{u(t - ih) - \frac{t - ih}{h}u(t - ih) + \frac{t - (i + 1)h}{h}u(t - (i + 1)h)\right\},$$

and

$$T2_i(t) = \{\frac{t-ih}{h}u(t-ih) - \frac{t-(i+1)h}{h}u(t-(i+1)h) - u(t-(i+1)h)\},$$

so

$$\int_0^t T1_i(s)dB(s) = \int_0^t \{u(s-ih) - \frac{s-ih}{h}u(s-ih) + \frac{s-(i+1)h}{h}u(s-(i+1)h)\}dB(s), \tag{3.2}$$

and

$$\int_0^t T2_i(s)dB(s) = \int_0^t \{\frac{s-ih}{h}u(s-ih) - \frac{s-(i+1)h}{h}u(s-(i+1)h) - u(s-(i+1)h)\}dB(s), \tag{3.3}$$

$u(t)$ is the unit step function. These integrations can be divided into three cases. For the case of $t \in [0\ ih)$, we have

$$\int_0^t T1_i(s)dB(s) = 0, \tag{3.4}$$

and

$$\int_0^t T2_i(s)dB(s) = 0. \tag{3.5}$$

For the case of $t \in [ih\ \ (i+1)h)$, we have

$$\begin{aligned}\int_0^t T1_i(s)dB(s) &= (i+1)\int_{ih}^t dB(s) - \frac{1}{h}\int_{ih}^t sdB(s)\\ &= (i+1)[B(t) - B(ih)] - \frac{1}{h}\int_{ih}^t sdB(s),\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\int_0^t T2_i(s)dB(s) &= \frac{1}{h}\int_{ih}^t sdB(s) - i\int_{ih}^t dB(s)\\ &= \frac{1}{h}\int_{ih}^t sdB(s) - i[B(t) - B(ih)].\end{aligned} \tag{3.7}$$

Finally, for the case of $t \in [(i+1)h\ \ T)$, we get

$$\begin{aligned}\int_0^t T1_i(s)dB(s) &= \int_0^{ih} T1_i(s)dB(s) + \int_{ih}^{(i+1)h} T1_i(s)dB(s) + \int_{(i+1)h}^t T1_i(s)dB(s)\\ &= \int_{ih}^{(i+1)h} T1_i(s)dB(s) = (i+1)\int_{ih}^{(i+1)h} dB(s) - \frac{1}{h}\int_{ih}^{(i+1)h} sdB(s)\\ &= (i+1)[B((i+1)h) - B(ih)] - \frac{1}{h}\big[sB(s)\big]_{ih}^{(i+1)h} - \frac{1}{h}\int_{ih}^{(i+1)h} B(s)ds\\ &= \frac{1}{h}\int_{ih}^{(i+1)h} B(s)ds - B(ih),\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\int_0^t T2_i(s)dB(s) &= \frac{1}{h}\int_{ih}^{(i+1)h} sdB(s) - i\int_{ih}^{(i+1)h} dB(s)\\ &= \frac{1}{h}\int_{ih}^{(i+1)h} sdB(s) - i[B((i+1)h) - B(ih)]\\ &= B((i+1)h) - \frac{1}{h}\int_{ih}^{(i+1)h} B(s)ds.\end{aligned} \tag{3.9}$$

The result of these three cases can be expanded in to TF series

$$I(T1_i(t)) = \int_0^t T1_i(s)dB(s) \simeq [\xi_{i0}, \ldots, \xi_{im-1}]T1(t) + [\zeta_{i0}, \ldots, \zeta_{im-1}]T2(t), \tag{3.10}$$

and

$$I(T2_i(t)) = \int_0^t T2_i(s)dB(s) \simeq [\alpha_{i0}, \ldots, \alpha_{im-1}]T1(t) + [\beta_{i0}, \ldots, \beta_{im-1}]T2(t), \tag{3.11}$$

where $\xi_{ij} = I(T1_i(jh))$, $\alpha_{ij} = I(T2_i(jh))$ and $\zeta_{ij} = I(T1_i((j+1)h))$, $\beta_j = I(T2_i((j+1)h))$ for $j = 0, 1, \ldots, m-1$. From Eqs. ((3.4)-(3.9)) we get

$$\begin{aligned}
&\xi_{ij} = \alpha_{ij} = 0, \quad j \leq i, \\
&\xi_{ij} = \frac{1}{h}\int_{ih}^{(i+1)h} B(s)ds - B(ih), \quad i < j, \\
&\alpha_{ij} = B((i+1)h) - \frac{1}{h}\int_{ih}^{(i+1)h} B(s)ds, \quad i < j,
\end{aligned}$$

and for $i = 0, \ldots, m-1$, $j = 0, \ldots, m-1$, $\zeta_{ij} = \xi_{i(j+1)}, \beta_{ij} = \alpha_{i(j+1)}$. Finally we can write

$$\int_0^t T1_i(s)dB(s) \simeq P1_s T1(t) + P2_s T2(t), \tag{3.12}$$

where, $P1_s$ and $P2_s$ are $m \times m$ stochastic operational matrices in TF domain. These matrices can be computed as follow

$$P1_s = \begin{pmatrix} 0 & \xi_{01} & \xi_{02} & \ldots & \xi_{0(m-1)} \\ 0 & 0 & \xi_{12} & \ldots & \xi_{1(m-1)} \\ 0 & 0 & 0 & \ldots & \xi_{2(m-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}_{m\times m},$$

and

$$P2_s = \begin{pmatrix} \xi_{01} & \xi_{02} & \xi_{03} & \ldots & \xi_{0m} \\ 0 & \xi_{12} & \xi_{13} & \ldots & \xi_{1m} \\ 0 & 0 & \xi_{23} & \ldots & \xi_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \xi_{(m-1)m} \end{pmatrix}_{m\times m}.$$

In a similar manner, the Itô integration of $T2(t)$ is

$$I(T2(t)) \simeq P3_s T1(t) + P4_s T2(t), \tag{3.13}$$

where,

$$P3_s = \begin{pmatrix} 0 & \alpha_{01} & \alpha_{02} & \ldots & \alpha_{0(m-1)} \\ 0 & 0 & \alpha_{12} & \ldots & \alpha_{1(m-1)} \\ 0 & 0 & 0 & \ldots & \alpha_{2(m-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}_{m\times m},$$

and

$$P4_s = \begin{pmatrix} \alpha_{01} & \alpha_{02} & \alpha_{03} & \dots & \alpha_{0m} \\ 0 & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1m} \\ 0 & 0 & \alpha_{23} & \dots & \alpha_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_{(m-1)m} \end{pmatrix}_{m\times m}.$$

From Eqs. (3.1),(3.12) and (3.13) conclude that

$$I(T(t)) \simeq \begin{pmatrix} P1_s T1(t) + P2_s T2(t) \\ P3_s T1(t) + P4_s T2(t) \end{pmatrix} = \begin{pmatrix} P1_s & P2_s \\ P3_s & P4_s \end{pmatrix} \begin{pmatrix} T1(t) \\ T2(t) \end{pmatrix},$$

so,

$$I(T(t)) \simeq P_s T(t),$$

where P_s, stochastic operational matrix of $T(t)$ is

$$\begin{pmatrix} P1_s & P2_s \\ P3_s & P4_s \end{pmatrix}.$$

The Itô integration of $f(t)$ can be approximated as

$$\int_0^t f(s)dB(s) \simeq F^T P_s T(t). \tag{3.14}$$

4. Implementation in Stochastic integral equation

In this section we solve Eq. (1.1) by operational matrices of TFs. First, we consider $z_1(t)$ and $z_2(t)$ as

$$z_1(t) = b\big(t, x(t)\big), \qquad z_2(t) = \sigma\big(t, x(t)\big), \tag{4.1}$$

then we find the collocation approximation for them. From Eqs. (1.1) and (4.1) we get

$$x(t) = x_0 + \int_0^t k_1(t,s)z_1(s)ds + \int_0^t k_2(t,s)z_2(s)dB(s), \tag{4.2}$$

and

$$\begin{cases} z_1(t) = b\big(t, x_0 + \int_0^t k_1(t,s)z_1(s)ds + \int_0^t k_2(t,s)z_2(s)dB(s)\big), \\ z_2(t) = \sigma\big(t, x_0 + \int_0^t k_1(t,s)z_1(s)ds + \int_0^t k_2(t,s)z_2(s)dB(s)\big). \end{cases} \tag{4.3}$$

We approximate $z_1(t), z_2(t)$, and $k_i(t,s), i = 1, 2$, by TF series as follows

$$z_1(t) \simeq \hat{z}_1(t) = Z_1^T T(t) = T^T(t) Z_1, \tag{4.4}$$

$$z_2(t) \simeq \hat{z}_2(t) = Z_2^T T(t) = T^T(t) Z_2, \tag{4.5}$$

$$k_i(t,s) \simeq \hat{k}_i(t,s) = T^T(t) K_i T(s), \quad i = 1, 2, \tag{4.6}$$

such that $2m$-vectors Z_1, Z_2, and $2m \times 2m$ matrix K_i are TFs coefficients of $z_1(t)$ and $z_2(t)$ and $k_i(t,s)$, respectively. By substituting Eqs. (4.4) and (4.5) in (4.2) we get

$$\begin{aligned}
\int_0^t k_1(t,s)z_1(s)ds &\simeq \int_0^t T^T(t)K_1T(s)T^T(s)Z_1ds \\
&= T^T(t)K_1\int_0^t T(s)T^T(s)Z_1ds \\
&\simeq T^T(t)K_1\int_0^t \tilde{Z}_1T(s)ds \\
&\simeq T^T(t)K_1\tilde{Z}_1PT(t), \qquad (4.7)
\end{aligned}$$

also, the Itô part of Eq. (4.2) can be written as

$$\begin{aligned}
\int_0^t k_2(t,s)z_2(s)dB(s) &\simeq \int_0^t T^T(t)K_2T(s)T^T(s)Z_2dB(s) \\
&= T^T(t)K_2\int_0^t T(s)T^T(s)Z_2dB(s) \\
&\simeq T^T(t)K_2\int_0^t \tilde{Z}_2T(s)dB(s) \\
&\simeq T^T(t)K_2\tilde{Z}_2P_sT(t), \qquad (4.8)
\end{aligned}$$

where $\tilde{Z}_1 = diag(Z_1), \tilde{Z}_2 = diag(Z_2)$. By substituting Eqs. (4.7) and (4.8) into Eq. (4.3) and replacing $\simeq$ with $=$, we obtain

$$\begin{cases} Z_1^TT(t) = b\big(t, x_0 + T^T(t)K_1\tilde{Z}_1PT(t) + T^T(t)K_2\tilde{Z}_2P_sT(t)\big), \\ Z_2^TT(t) = \big(t, x_0 + T^T(t)K_1\tilde{Z}_1PT(t) + T^T(t)K_2\tilde{Z}_2P_sT(t)\big). \end{cases} \qquad (4.9)$$

Now, we collocate Eq. (4.9) in $2m$ nodes $t_j = \frac{j}{2m+1}, j = 1, \dots, 2m$, as

$$\begin{cases} Z_1^TT(t_j) = b\big(t_j, x_0 + T^T(t_j)K_1\tilde{Z}_1PT(t_j) + T^T(t_j)K_2\tilde{Z}_2P_sT(t_j)\big), \\ Z_2^TT(t_j) = \big(t_j, x_0 + T^T(t_j)K_1\tilde{Z}_1PT(t_j) + T^T(t_j)K_2\tilde{Z}_2P_sT(t_j)\big). \end{cases} \qquad (4.10)$$

After solving nonlinear system Eq. (4.10) we obtain Z_1 and Z_2. Then we can approximate the solution of Eq. (4.2) as follows

$$x(t) \simeq x_m(t) = x_0 + T^T(t)K_1\tilde{Z}_1PT(t) + T^T(t)K_2\tilde{Z}_2P_sT(t). \qquad (4.11)$$

5. Error analysis

Assume $(C[0\ 1], \|\cdot\|)$ be the Banach space of all continous functions with the norm

$$\|f(t)\| = max_{0\le t\le 1}|f(t)|.$$

Concerning the error of TF series, the following estimate holds for all $f \in L^2([0\ 1])$, in similar fashion with Deb [7]

$$\|f(t) - \hat{f}(t)\| \le ch, \qquad (5.1)$$

where $\hat{f}(t)$ is defined in Eq. (2.1). In addition nonlinear terms satisfy in Lipschitz and linear growth condition such that

$$|b(t, x_1(t)) - b(t, x_2(t))| + |\sigma(t, x_1(t)) - \sigma(t, x_2(t))| \leq L_1|x_1 - x_2|, \tag{5.2}$$

and

$$|b(t, x(t))| + |\sigma(t, x(t))| \leq L_2(1 + |x|). \tag{5.3}$$

Theorem 5.1. *Let $x(t)$ and $x_m(t)$ be the exact solution and approximate solution of Eq.* (1.1) *respectively, furthermore, let conditions* (5.2), (5.3) *and*
i) $E|x(t)| \leq M, \quad t \in I = [0\ 1)$,
ii) $|k_i(t, s)| \leq M_i, \ (t, s) \in I \times I$, *i=1,2,*
hold then,

$$E\|x(t) - x_m(t)\| \to 0.$$

Proof . Let $e_i(t) = z_i(t) - \hat{z}_i(t)$ be the error function of approximate solution $z_m(t)$ to the exact solution $z(t)$, where $z_i(t)$ is defined in Eq. (4.1) also $\hat{z}_i(t), i = 1, 2$ is approximated form of $z_i(t)$ by TFs, i.e.,

$$\hat{z}_1(s) = \hat{b}\big(s, x_m(s)\big)$$

and

$$z_1^m(s) = b\big(s, x_m(s)\big)$$

and similarly for $\hat{z}_2(s)$ and $z_2^m(s)$. We get

$$\begin{aligned} E\|z_i(t) - \hat{z}_i(t)\| &\leq E\|z_i(t) - z_i^m(t)\| + E\|\hat{z}_i(t) - z_i^m(t)\| \\ &\leq LE\|x(t) - x_m(t)\| + c_i h, \end{aligned} \tag{5.4}$$

where $i = 1, 2$. For $e_m(t) = x(t) - x_m(t)$ we can write

$$\|e_m(t)\| \leq \|I_1\| + \|I_2\|, \tag{5.5}$$

where

$$\begin{aligned} I_1 &= \int_0^t [k_1(t, s)z_1(s) - \hat{k}_1(t, s)\hat{z}_1(s)]ds, \\ I_2 &= \int_0^t [k_2(t, s)z_2(s) - \hat{k}_2(t, s)\hat{z}_2(s)]dB(s). \end{aligned} \tag{5.6}$$

For I_1 we get

$$\begin{aligned} E\|I_1\| \leq \int_0^t E(\|[k_1(t, s)\|\|z_1(s) - \hat{z}_1(s)\|)ds &+ \int_0^t E(\|\hat{z}_1(s)\|\|k_1(t, s) - \hat{k}_1(t, s)\|)ds, \\ &\leq M_1(L\int_0^t E\|e_m(s)\|ds + c_1 h) \\ &\quad + c_3 h\Big(\int_0^t E\|z_1(s) - \hat{z}_1(s)\|ds + \int_0^t E\|z_1(s)\|ds\Big) \\ &\leq M_1 L(1 + c_3 h)\int_0^t E\|e_m(s)\|ds + O(h), \end{aligned} \tag{5.7}$$

similarly for I_2,

$$\begin{aligned}
E\|I_2\| &\leq E\|\int_0^t [k_2(t,s)z_2(s) - \hat{k}_2(t,s)\hat{z}_2(s)]dB(s)\| \\
&\leq \int_0^t E\|k_2(t,s)z_2(s) - \hat{k}_2(t,s)\hat{z}_2(s)\|ds \\
&\leq \int_0^t E(\|k_2(t,s)\|\|z_2(s) - \hat{z}_2(s)\|)ds + \int_0^t E(\|\hat{z}_2(s)\|\|k_2(t,s) - \hat{k}_2(t,s)\|)ds, \\
&\leq M_2(L\int_0^t E\|e_m(s)\|ds + c_2h) + c_4h\Big(\int_0^t E\|z_2(s) - \hat{z}_2(s)\|ds + \int_0^t E\|z_2(s)\|ds\Big) \\
&\leq M_2L(1 + c_4h)\int_0^t E\|e_m(s)\|ds + O(h). \qquad (5.8)
\end{aligned}$$

From Eqs. (5.7), (5.8) and (5.5) we conclude

$$E\|e_m(t)\| \leq \alpha \int_0^t E\|e_m(s)\|ds + O(h), \qquad (5.9)$$

where $\alpha = M_1L(1 + c_3h) + M_2L(1 + c_4h)$. Hence from Eq. (5.9) and Gronwall inequality we get

$$E\|e_m(t)\| \leq O(h)(1 + \alpha \int_0^t e^{\alpha(t-s)}ds), \quad t \in [0\ 1),$$

for $h = \frac{1}{m}$, by increasing m, it implies $\|e_m(t)\| \to 0$ as $m \to \infty$. $\square$

6. Numerical examples

To illustrate efficiency and accuracy of presented method we solve below examples. Let X_i denote the TF coefficient of exact solution and Y_i be the TF coefficient of computed solutions by presented method. The error is defined as

$$\|E\|\infty = max_{1\leq i\leq m}|X_i - Y_i|.$$

Example 6.1. Consider the nonlinear stochastic integral equation as follows (population growth problem [12])

$$x(t) = x_0 + \int_0^t x(s)(\lambda - x(s))ds + \sigma \int_0^t x(s)dB(s), \quad t \in [0\ 1), \qquad (6.1)$$

with the exact solution

$$x(t) = \frac{x_0 e^{(\lambda - \frac{1}{2}\sigma^2)t + \sigma B(t)}}{1 + x_0 \int_0^t e^{(\lambda - \frac{1}{2}\sigma^2)s + \sigma B(s)}ds}.$$

The numerical results are shown in Table 1. $\overline{x}_E$ is the errors mean and s_E is the standard deviation of errors in k iteration. In addition, we consider $x_0 = 0.5,\ \lambda = 1,\ \sigma = 0.25$. The accuracy is good in comparison with [2] and [1].

Table 1: Mean, standard deviation and Confidence Interval for error mean, $m = 32,\ k = 500$.

t_i	$\overline{x}_E$	s_E	0.95 Confidence Interval	
			Lowerbound	*Upperbound*
0	2.04×10^{-4}	2.10×10^{-5}	1.99×10^{-4}	1.39×10^{-3}
0.1	4.13×10^{-3}	3.50×10^{-4}	4.06×10^{-3}	4.20×10^{-3}
0.2	6.35×10^{-3}	4.67×10^{-4}	6.26×10^{-3}	6.44×10^{-3}
0.3	1.07×10^{-2}	7.10×10^{-4}	1.06×10^{-2}	1.08×10^{-2}
0.4	4.30×10^{-2}	1.61×10^{-3}	4.26×10^{-2}	4.33×10^{-2}
0.5	4.65×10^{-2}	1.10×10^{-3}	4.62×10^{-2}	4.67×10^{-2}
0.6	8.07×10^{-2}	1.81×10^{-3}	1.40×10^{-1}	1.42×10^{-1}
0.7	1.22×10^{-2}	4.66×10^{-3}	1.21×10^{-1}	1.23×10^{-1}
0.8	1.77×10^{-1}	7.33×10^{-3}	1.75×10^{-1}	1.78×10^{-1}
0.9	2.01×10^{-1}	1.00×10^{-2}	1.99×10^{-1}	2.03×10^{-1}

Example 6.2. The following nonlinear Stochastic differential equation is considered [12]

$$dx(t) = \frac{1}{2}a^2 n[x(t)]^{2n-1}dt + a[x(t)]^n dB(t), \quad t \in [0\ 1), \tag{6.2}$$

with the exact solution $x(t) = \left(x_0^{1-n} - a(n-1)B(t)\right)^{\frac{1}{1-n}}$.

The numerical results are shown in Table 2 for $n = a = 2$. $\overline{x}_E$ is the errors mean and s_E is the standard deviation of errors in k iteration.

Table 2: Mean, standard deviation and Confidence Interval for error mean, $m = 32,\ k = 500$.

t_i	$\overline{x}_E$	s_E	0.95 Confidence Interval	
			Lowerbound	*Upperbound*
0	2.30×10^{-3}	3.11×10^{-4}	2.23×10^{-3}	2.36×10^{-3}
0.1	5.13×10^{-3}	5.52×10^{-3}	4.04×10^{-3}	6.21×10^{-3}
0.2	9.05×10^{-3}	4.87×10^{-3}	8.09×10^{-3}	1.00×10^{-2}
0.3	1.07×10^{-2}	9.10×10^{-3}	8.91×10^{-3}	1.24×10^{-2}
0.4	5.41×10^{-2}	1.11×10^{-2}	5.19×10^{-2}	5.62×10^{-2}
0.5	7.66×10^{-2}	5.24×10^{-2}	6.63×10^{-2}	8.68×10^{-2}
0.6	9.97×10^{-2}	7.03×10^{-2}	8.45×10^{-2}	1.11×10^{-1}
0.7	1.10×10^{-1}	9.06×10^{-2}	1.07×10^{-1}	1.11×10^{-1}
0.8	5.77×10^{-1}	1.00×10^{-1}	3.81×10^{-1}	7.73×10^{-1}
0.9	9.01×10^{-1}	3.17×10^{-1}	8.38×10^{-1}	9.63×10^{-1}

7. Conclusion

The aim of present work is to apply a method for solving nonlinear stochastic differential equations. The properties of TFs with the collocation method are used to reduce the problem to a system of nonlinear algebraic equations. The benefit of this method is low cost of setting up the equations due to properties of TFs. Also this method can be easily applied to nonlinear SVD equations. For showing efficiency, the method is applied for SVD equation that arises from population model. The results show good accuracy of method.

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