Mazur-Ulam theorem in probabilistic normed groups

Alireza Pourmoslemi\textsuperscript{a}, Kourosh Nourouzi\textsuperscript{b,∗}

\textsuperscript{a}Department of Mathematics, Payame Noor University, Tehran, Iran
\textsuperscript{b}Faculty of Mathematics, K.N. Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran

(Communicated by C. Park)

Abstract

In this paper, we give a probabilistic counterpart of Mazur-Ulam theorem in probabilistic normed groups. We show, under some conditions, that every surjective isometry between two probabilistic normed groups is a homomorphism.

Keywords: Probabilistic normed groups; Invariant probabilistic metrics; Mazur-Ulam Theorem.

2010 MSC: Primary 54E70; Secondary 20F38.

1. Introduction and preliminaries

Mazur and Ulam showed that every bijective isometry between real normed spaces is affine\textsuperscript{[5]}. Since then it has attracted the attention of some researchers in order to generalize this result (see e.g.\textsuperscript{[8]}). In particular, the Mazur-Ulam theorem has been investigated in normed and metric groups\textsuperscript{[3, 10]} and in probabilistic and random normed spaces\textsuperscript{[1, 6]}.

In this paper we give a probabilistic counterpart of the Mazu-Ulam theorem in probabilistic normed groups introduced by the authors in\textsuperscript{[7]}. We begin with some basic notions which will be needed in this paper.

A distribution function is a function $F$ from the extended real line $[-\infty, +\infty]$ to the interval $[0, 1]$ such that $F$ is nondecreasing and left-continuous and satisfies $F(-\infty) = 0$, $F(+\infty) = 1$.\textsuperscript{[A]}

We denote the set of all distribution functions by $\Delta$. A subset of $\Delta$ consisting of all distribution functions $F$ with $F(0) = 0$ will be denoted by $\Delta^+$. The subset $D^+$ of $\Delta^+$ is defined as follows:

$$D^+ = \{ F \in \Delta^+ : l^- F(+\infty) = 1 \},$$

∗Corresponding author

Email addresses: a_pourmoslemy@pnu.ac.ir (Alireza Pourmoslemi), nourouzi@kntu.ac.ir (Kourosh Nourouzi)

Received: March 2016    Revised: October 2016
where \( \lim f(x) \) denotes the left limit of the function \( f \) at the point \( x \). For \( F, G \in \Delta^+ \) we mean \( F \leq G \) by \( F(x) \leq G(x) \), for all \( x \in \mathbb{R} \). The distribution function \( H_a \) is given by

\[
H_a(x) = \begin{cases} 
0, & \text{if } x \leq a, \\
1, & \text{if } x > a,
\end{cases}
\]

for all \( a, x \in \mathbb{R} \). The maximal element for \( \Delta^+ \) (and also for \( D^+ \)) according to the presented order is the distribution function \( H_0 \).

A triangular norm (briefly \( t \)-norm) is a binary function \( T \) from \([0,1] \times [0,1]\) to \([0,1]\) which is associative, commutative, nondecreasing in each place and \( T \) is an associative, commutative, nondecreasing for all \( F, G, H \in \Delta^+ \) and it has \( H_0 \) as unit \([4]\). A sequence \( \{F_n\} \) in \( \Delta^+ \) converges weakly to a distribution function \( F \), written by \( F_n \overset{w}{\rightarrow} F \), if and only if the sequence \( \{F_n(x)\} \) converges to \( F(x) \) at each continuity point \( x \) of \( F \) (see Definition 4.2.4. in \([9]\)). A triangle function \( \tau \) is said to be continuous if \( F_n \overset{w}{\rightarrow} F \) and \( G_n \overset{w}{\rightarrow} G \) in \( \Delta^+ \) imply that \( \tau(F_n, G_n) \rightarrow \tau(F, G) \). For example, if \( T \) is a continuous \( t \)-norm, then \( \tau_T \) is a continuous triangle function, where \( \tau_T \) is defined by

\[
\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s),G(t)),
\]

for all \( F, G \in \Delta^+ \) and every \( x, s, t \in \mathbb{R} \).

**Definition 1.1.** \([7]\) A triple \( (G, F, \tau) \) is called a probabilistic normed group, where \( G \) is a group with identity element \( e \), \( \tau \) is a continuous triangle function and \( F \) is a mapping from \( G \) into \( \Delta^+ \) satisfying the following conditions:

- \((PGN1)\) \( F_x = H_0 \) if and only if \( x = e \),
- \((PGN2)\) \( F_{xy} \geq \tau(F_x, F_y) \), whenever \( x, y \in G \),
- \((PGN3)\) \( F_{x^{-1}} = F_x \), where \( x^{-1} \) is the inverse element of \( x \).

Then \( F \) is called a probabilistic group-norm on \( G \). The probabilistic group-norm \( F \) is called abelian if \( F_{xy} = F_{yx} \), for each \( x, y \in G \).

In a probabilistic normed group \( (G, F, \tau) \), for each \( x \) in \( G \) and \( \lambda > 0 \), the strong \( \lambda \)-neighborhood of \( x \) is the set

\[
N_x(\lambda) = \{ y \in G : F_{xy^{-1}}(\lambda) > 1 - \lambda \}.
\]

The strong neighborhood system for \( G \) is the union \( \bigcup_{x \in G} N_x \) where \( N_x = \{ N_x(\lambda) : \lambda > 0 \} \). Note that the strong neighborhood system for \( G \) determines a Hausdorff topology for \( G \) (see Theorem 12.1.2 in \([9]\)).

2. Main theorem

**Definition 2.1.** \([2]\) A group \( G \) is called divisible if for every \( g \in G \), and every positive integer \( n \) there exists \( y \in G \) such that \( y^n = g \). We say that group \( G \) is \( 2 \)-divisible if for each \( g \in G \) there exists \( y \in G \) such that \( y^2 = g \). The algebraic center of points \( x, y \in G \) is an element \( z \in G \), denoted by \( \sqrt{xy} \), such that \( z^2 = xy \).

**Definition 2.2.** Let \( (G, F, \mu) \) and \( (G', F', \tau) \) be two probabilistic normed groups. A mapping \( T : (G, F, \mu) \rightarrow (G', F', \tau) \) is called an isometry if for each \( x, y \in G \),

\[
F'_{T(x)T(y)^{-1}} = F_{xy^{-1}}.
\]
Let \((G, F, \tau)\) be a probabilistic normed group. Consider the following conditions:

(C1) There exists a constant \(c > 1\) such that \(F_x(t) \leq F_x(\frac{t}{c})\), for all \(x \in G\) and \(t > 0\).
(C2) \(F_x \in D^+\), for all \(x \in G\).
(C3) \(\tau(D^+ \times D^+) \subseteq D^+\).

The following example gives a probabilistic normed group satisfying the conditions (C1),(C2) and (C3).

**Example 2.3.** Consider the probabilistic normed group \((\mathbb{R}, F, \tau_T)\), where \(\mathbb{R}\) is the additive group of real numbers and \(F_x = \mathcal{H}_{|x|}\), for all \(x \in \mathbb{R}\). We have \(F_{x^n} = \mathcal{H}_{n|x|}\), for each \(n \in \mathbb{N}\) and each \(x \in \mathbb{R}\). Therefore

\[
F_{x^n}(t) = \begin{cases} 
0, & \text{if } t \leq n \vert x \vert \\
1, & \text{if } t > n \vert x \vert
\end{cases} = \begin{cases} 
0, & \text{if } \frac{t}{n} \leq \vert x \vert \\
1, & \text{if } \frac{t}{n} > \vert x \vert
\end{cases} = F_x(\frac{t}{n}),
\]

for each \(x, t \in \mathbb{R}\) and every \(n \in \mathbb{N}\). Now for \(n \geq 2\), choosing \(1 < c \leq n\) we get

\[
F_{x^n}(t) = F_x(\frac{t}{n}) \leq F_x(\frac{t}{c}),
\]

for each \(x, t \in \mathbb{R}\). Particularly, for \(n = 2\) putting \(1 < c \leq 2\), we get

\[
F_{x^2}(t) \leq F_x(\frac{t}{c}),
\]

for all \(x, t \in \mathbb{R}\). It is obvious that for every \(x \in \mathbb{R}\), \(F_x = \mathcal{H}_{|x|} \in D^+\). Since \(\tau_T(\mathcal{H}_{|x|}, \mathcal{H}_{|y|}) = \mathcal{H}_{|x|+|y|}\), for all \(x, y \in \mathbb{R}\), we get

\[
\tau_T(F_x, F_y) \in D^+.
\]

Now consider the probabilistic normed group \((\mathbb{R}^+, F, \tau_T)\), where \(\mathbb{R}^+\) is the multiplicative group with \(e = 1\). Let \(F_h = \mathcal{H}_{|\log(h)|}\), for all \(h \in \mathbb{R}^+\). We have

\[
F_{h^2}(t) = \mathcal{H}_{|\log(h^2)|}(t) = \mathcal{H}_{\frac{1}{2}|\log(h)|}(t) = \mathcal{H}_{|\log(h)|(\frac{t}{2})},
\]

for each \(t, h \in \mathbb{R}^+\). Putting \(1 < c \leq 2\), we have \(F_{h^2}(t) \leq F_h(\frac{t}{c})\).

**Theorem 2.4.** Let \((G, F, \mu)\) and \((G', F', \tau)\) be two probabilistic normed groups such that both \(G, G'\) are uniquely 2-divisible abelian groups, and conditions (C1), (C2) and (C3) hold for both \((G', F', \tau)\) and \((G, F, \mu)\). If \(T : G \to G'\) is a surjective isometry, then

\[
F'_{T(\sqrt{xy})(\sqrt{T(x)T(y)})^{-1}} = \mathcal{H}_0,
\]

for all \(x, y \in G\).

**Proof.** Let \(x, y \in G\) and set

\[
a = \sqrt{xy}, \quad b = \sqrt{T(x)T(y)}, \quad E = F'_{\sqrt{T(x)T(y)}^{-1}}.
\]

Let \(\{q_n\}\) be a sequence of maps from \(G'\) to itself, defined for each \(z \in G'\) by

\[
q_0(z) = T(a^2(T^{-1}(z))^{-1}), \quad q_1(z) = b^2z^{-1},
\]
and for \( n \in \mathbb{N} \),
\[ q_{n+1} = q_{n-1} \circ q_n \circ q_{n-1}^{-1}. \]

For \( n \in \mathbb{N} \) define \( \{p_n\} \), a sequence of points in \( G' \), by
\[ p_1 = b, \quad p_{n+1} = q_{n-1}(p_n). \]

By induction, one can see that for all \( n \in \mathbb{N}_0 \) we have
\[ q_n(T(x)) = T(y), \quad q_n(T(y)) = T(x). \tag{2.1} \]

We show that for each \( u,v \in G' \) and all \( n \in \mathbb{N}_0 \),
\[ F'_{q_n(u)q_n(v)^{-1}} = F'_{uv^{-1}}. \]

For \( n = 0 \),
\[
F'_{q_0(u)q_0(v)^{-1}} = F'_{T(a^2(T^{-1}(u))^{-1})(a^2(T^{-1}(v))^{-1})^{-1}}
= F_{a^2(T^{-1}(u))^{-1}(a^2(T^{-1}(v))^{-1})^{-1}} = F_{a^2a^{-2}(T^{-1}(u))^{-1}T^{-1}(v)}
= F_{T^{-1}(u)^{-1}T^{-1}(v)} = F_{T^{-1}(u)^{-1}T^{-1}(v)}
= F'_{T^{-1}(u))(T^{-1}(v)^{-1})
= F'_{uv^{-1}}.
\]

Suppose that the statement holds for some \( n \in \mathbb{N} \). Then we get
\[
F'_{q_{n+1}(u)q_{n+1}(v)^{-1}} = F'_{q_{n-1}q_nq_{n-1}^{-1}(u)(q_{n-1}q_nq_{n-1}^{-1}(v))^{-1}}
= F'_{q_nq_{n-1}^{-1}(u)(q_{n-1}q_n^{-1}(v))^{-1}}
= F'_{q_{n-1}^{-1}(u)(q_{n-1}^{-1}(v))^{-1}}
= F'_{q_{n-1}^{-1}(u)(q_{n-1}^{-1}(v))^{-1}}
= F'_{uv^{-1}}.
\]

So
\[ F'_{q_n(u)q_n(v)^{-1}} = F'_{uv^{-1}}, \]
for each \( u,v \in G' \) and all \( n \in \mathbb{N}_0 \). Now by induction we are going to show that
\[ F'_{p_nT(x)^{-1}} = E, \quad F'_{p_nT(y)^{-1}} = E, \tag{2.2} \]
for \( n \in \mathbb{N} \). For \( n = 1 \), we have
\[ F'_{p_1T(x)^{-1}} = F'_{\sqrt{T(x)T(y)T(x)^{-1}}} = F'_{\sqrt{T(y)T(x)^{-1}}} = E. \]

(Note that in the above equation we use the fact that if \( s^2 = tr \) and \( v^2 = mn \), then \( s^2v^2 = (sv)^2 \) and \( sv = \sqrt{tmn} = \sqrt{tr} \sqrt{mn} \), for all \( s,v,r,t,m,n \in G' \).)

Likewise,
\[ F'_{p_1T(y)^{-1}} = F'_{\sqrt{T(y)T(x)^{-1}}} = E. \]
Hence (2.2) holds for \( n = 1 \). Suppose that (2.2) holds for some \( n \in \mathbb{N} \). Then by using (2.1) and the induction hypothesis we get

\[
F'_{p_{n+1}T(x)^{-1}} = F'_{q_{n-1}(p_n)(q_{n-1}(T(y)))^{-1}} = F'_{p_nT(y)^{-1}} = E.
\]

Similarly we have

\[
F'_{p_{n+1}T(y)^{-1}} = E.
\]

Now by (2.2) for \( n \geq 2 \),

\[
F'_{p_{n}p_{n-1}^{-1}} = F'_{p_{n}T(x)^{-1}T(x)p_{n-1}^{-1}} \geq \tau(F'_{p_{n}T(x)^{-1}}, F'_{T(x)p_{n-1}^{-1}}) = \tau(E, E).
\]  

(2.3)

Again, by induction we prove that there is constant \( c > 1 \) such that

\[
F'_{q_{n}(z)^{-1}}(t) \leq F'_{p_{n}z^{-1}}(\frac{t}{c}),
\]

(2.4)

for each \( z \in G', t > 0 \) and \( n \in \mathbb{N} \). For \( n=1 \), we have

\[
F'_{q_{1}(z)^{-1}} = F'_{b^{2}z^{-1}z^{-1}} = F'_{b^{2}(z^{-1})^{2}} = F'_{(b^{2}-1)^{2}}.
\]

By the condition (C1), there exists constant \( c > 1 \) such that

\[
F'_{(b^{2}-1)^{2}}(t) \leq F'_{b^{2}-1}(\frac{t}{c}),
\]

for each \( z \in G' \) and \( t > 0 \). Hence

\[
F'_{q_{1}(z)^{-1}}(t) \leq F'_{p_{1}z^{-1}}(\frac{t}{c}),
\]

for each \( z \in G' \) and \( t > 0 \). Now suppose that the statement holds for some natural number \( n \). Then for each \( z \in G' \) and \( t > 0 \),

\[
F'_{q_{n+1}(z)^{-1}}(t) = F'_{q_{n-1}q_{n-1}^{-1}(z)(q_{n-1}q_{n-1}^{-1}(z))^{-1}}(t)
\]

\[
= F'_{q_{n}q_{n}^{-1}(z)(q_{n}^{-1}(z))^{-1}}(t)
\]

\[
\leq F'_{p_{n}(q_{n}^{-1}(z))^{-1}}(\frac{t}{c})
\]

\[
= F'_{q_{n}^{-1}(p_{n}q_{n}^{-1}(z))^{-1}}(\frac{t}{c})
\]

\[
= F'_{q_{n}^{-1}(p_{n}q_{n}^{-1}(z))^{-1}}(\frac{t}{c})
\]

\[
= F'_{p_{n+1}z^{-1}}(\frac{t}{c}).
\]

In the inequality (2.4) replace \( z \) by \( p_{n+1} \). Then for \( n \in \mathbb{N} \) and \( t > 0 \), we obtain

\[
F'_{q_{n}(p_{n+1})^{-1}p_{n+1}^{-1}}(t) \leq F'_{p_{n}p_{n+1}^{-1}}(\frac{t}{c}) = F'_{(p_{n}p_{n+1})^{-1}}(\frac{t}{c}).
\]

Therefore

\[
F'_{p_{n+2}p_{n+1}^{-1}}(t) \leq F'_{p_{n+1}p_{n+1}^{-1}}(\frac{t}{c}),
\]

and for \( n \geq 3 \) and each \( t > 0 \), we have

\[
F'_{p_{n}p_{n-1}^{-1}}(t) \leq F'_{p_{n-1}p_{n-2}^{-1}}(\frac{t}{C}) \leq \cdots \leq F'_{p_{2}p_{1}^{-1}}(\frac{t}{C^{n-2}}).
\]

(2.5)
By (2.3) and (2.5) for $n \geq 3$ we get
\[ \tau(E,E)(t) \leq F_{p_2p_1}'(\frac{t}{c_{n-2}}). \] (2.6)

On the other hand, there is $c_1 > 1$ such that
\[
F_{p_2p_1}'(t) = F_{T(a^{2(T^{-1}(b))^{-1}}(TT^{-1}(b))^{-1})(T^{-1}(b))^{-1}}(t)
\leq F_{a(T^{-1}(b))^{-1}}(t) = F_{a(T^{-1}(b))^{-1}}(\frac{t}{c_1})
= F_{T(a)(TT^{-1}(b))^{-1}}(\frac{t}{c_1})
= F_{T(a)b^{-1}}(\frac{t}{c_1}).
\]
for each $t > 0$. Consequently,
\[
\tau(E,E)(c_1c^{n-2}t) \leq F_{p_2p_1}'(c_1t) \leq F_{T(a)b^{-1}}(t),
\]
for each $t > 0$. Since $F'_z \in D^+$ for each $z \in G'$, and $\tau(D^+ \times D^+) \subseteq D^+$ we have
\[
\lim_{n \to +\infty} \tau(E,E)(c_1c^{n-2}t) = 1,
\]
for each $t > 0$. But $\mathcal{H}_0$ is a maximal element of $D^+$, therefore
\[
F_{T(a)b^{-1}} = \mathcal{H}_0.
\]

\hfill \Box

**Theorem 2.5.** Suppose that $(G,F,\mu)$ and $(G',F',\tau)$ are two probabilistic normed groups such that both $G,G'$ are uniquely 2-divisible abelian groups. Let the conditions (C1), (C2) and (C3) hold for both $(G',F',\tau)$ and $(G,F,\mu)$. If $U : (G,F,\mu) \to (G',F',\tau)$ is a surjective isometry with $U(e) = e$, then $U$ is a homomorphism.

**Proof.** We can apply Theorem 2.4 for surjective isometry $U$. For each $x,y \in G$ we have
\[
F'_{U(\sqrt{xy})(\sqrt{U(x)U(y)})^{-1}} = \mathcal{H}_0.
\]
Thus
\[
U(\sqrt{xy})(\sqrt{U(x)U(y)})^{-1} = e,
\]
for each $x,y \in G$. That is,
\[
U(\sqrt{xy}) = \sqrt{U(x)U(y)}, \tag{2.7}
\]
for each $x,y \in G$. In the equation (2.7), let $y = e$. Since $U(e) = e$, we have
\[
U(\sqrt{x}) = \sqrt{U(x)},
\]
for each $x \in G$. Now for arbitrary $x,y \in G$ we get
\[
U(xy) = (U(\sqrt{xy}))^2 = (\sqrt{U(x)U(y)})^2 = U(x)U(y),
\]
i.e., $U$ is a homomorphism. \hfill \Box

**Acknowledgments**

The authors would like to thank the reviewers for their helpful comments to improve the paper.
References