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Soft double fuzzy semi-topogenous structures

A. Ghareeb[a,b,*], O.H. Khalil[c,d]

[a]Department of Mathematics, Colleges of Science, Al-Baha University, Al-Baha, Saudi Arabia
[b]Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt
[c]Department of Mathematics, College of Science in Al-Zulfi, Majmaah University, Al-Zulfi, Saudi Arabia
[d]Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt

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Abstract

The purpose of this paper is to introduce the concept of soft double fuzzy semi-topogenous order. Firstly, we give the definition of soft double fuzzy semi-topogenous order. Secondly, we induce a soft double fuzzy topology from a given soft double fuzzy semi-topogenous order by using soft double fuzzy interior operator.

Keywords: soft double fuzzy topology; soft double fuzzy interior operator; soft double fuzzy semi-topogenous structure.
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1. Introduction

Topology is a distinct branch of modern mathematics which is formed in the middle of the twentieth century. Due to its various applications in theoretical and applied sciences, the subject became very famous and attractive to the researchers from the around world.

There are many applications of general topology. For example, ophthalmologists have found that when the sight is restored to a blind person, the person will have a topological vision for some time. During this short time, the person who has recovered from the blindness cannot differentiate between a circle and a square and any o closed curves. He has to practice for some time to accurately describe various closed curves. Based on this idea, Zeeman has originated a topological model of the brain and the visual perception [15].

Topological psychology [5, 6] is an argumentative topic in which mathematicians have uneven opinions on it. Lewin studied topology and developed the topological notions in his theories in

*Corresponding author
Email addresses: a.ghareeb@sci.svu.edu.eg, nasserfuzt@hotmail.com (A. Ghareeb), o.khalil@mu.edu.sa (O.H. Khalil)

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psychology. He tried hard to formalize his theories into an evident form in order to avoid the rigidity and inflexibility of the results. Lewin introduced concepts incidentally and progressively developed them through experimental and observation methods.

In the real world, the exact solution cannot be found for many problems. As a solution to this problem, scientists have resorted to use approximate solutions. In 1999, Molodtsov [9] introduced a new approach for the real world problems. He proposed the soft sets as a tool for dealing with uncertainty.

In 2011, Shabir and Naz [12] introduced the notion of soft topological spaces and Min corrected some of their results [8]. Zorlutuna et al. [16] continued to study the properties of soft topological spaces by defining the concepts of soft interior point, soft interior, soft neighborhood, soft continuity and soft compactness. Later on, Nazmul et al. [10] characterized the soft neighborhoods in soft topological space. In 2012, Husain and Ahmad [2] strengthened the theory of soft topological spaces by defining it on a fixed initial universe. Varol et al. [13] presented soft Hausdorff spaces and introduced some new concepts such as convergence of sequences.

The attempts to develop the soft topology didn't stop. In 2013, Cağman et al. [1] redefined soft topological spaces by modifying the soft set. Also, Roy and Samanta [11] strengthen the definition of the soft topological spaces presented in [1] and they used the base and the subbase to characterize its properties. Recently, Zakari et al. [14] introduced the notion of soft weak structures as a generalization of soft topology. An application of soft topology in GIS has been introduced in [3].

In this paper, we will introduce and characterize soft double fuzzy semi-topogenous order. Moreover, we will investigate the relationship between it and soft double fuzzy topological spaces by using soft double fuzzy interior operator. The results which we will get in this paper are a generalization of all the corresponding notions in general topology, soft topology, fuzzy topology, double fuzzy topology and soft fuzzy topology.

2. Preliminaries

Throughout this paper, let X be a non-empty set, E be a set of attributes or parameters, $I = [0, 1]$, I^X denote the set of all fuzzy sets on X and $A \subset E$. f_A is called a fuzzy soft set [7] over X, where f is the mapping given by $f : A \to I^X$. For $e \in A$, $f(e) \equiv f_e : X \longrightarrow I$ may be considered as a fuzzy set on X. For any two fuzzy soft sets f_A and g_B defined over a common universe X, we have:

(1) $f_A \sqsubseteq g_B$ iff $A \subset B$ and $f_e \leq g_e$ for all $e \in A$.
(2) $f_A \tilde{=} g_B$ iff $f_A \sqsubseteq g_B$ and $g_B \sqsubseteq f_A$.
(3) $f_A \sqcup g_B \tilde{=} h_C$ where $C = A \cup B$ and

$$h_e = \begin{cases} f_e, & \text{if } e \in A - B, \\ g_e, & \text{if } e \in B - A, \\ f_e \vee g_e, & \text{if } e \in A \cap B. \end{cases}$$

for all $e \in C$.

(4) $f_A \sqcap g_B \tilde{=} k_D$ where $D = A \cap B$ and $k_e = f_e \wedge g_e$ for all $e \in D$.

A fuzzy soft set f_A is called a null fuzzy soft set (denoted by $\tilde{0}$) if $f_e = \underline{0}$ for all $e \in A$ and called an absolute fuzzy soft set (denoted by $\tilde{1}$) if $f_e = \underline{1}$ for all $e \in A$. If f_A is a fuzzy soft set over X and $Y \subset X$, then the fuzzy soft set ${}^Y f_A$ is defined by ${}^Y f_e = \underline{1}_Y \wedge f_e$ for each $e \in A$, i.e., ${}^Y f_A \tilde{=} \underline{1}_Y \sqcap f_A$. The relative complement of fuzzy soft set f_A (denoted by f'_A), where $f' : A \to I^X$ is given by $f'_e(x) = 1 - f_e(x)$ for each $e \in A$ and $x \in X$. It is clear that $\tilde{0}' \tilde{=} \tilde{1}$ and $\tilde{1}' \tilde{=} \tilde{0}$. A fuzzy soft set f_E on X is called α-absolute fuzzy soft set (denoted by $\tilde{1}^\alpha$), if $f_e = \underline{\alpha}$ for each $e \in E$.

Let $\phi : X \longrightarrow Y$ and $\psi : E \longrightarrow K$ be two mappings, where E and F are the attributes set for the sets X and Y, respectively. Then the image and the preimage [4] under the fuzzy soft function $\phi_\psi : (I^X)^E \longrightarrow (I^Y)^K$ will be defined as follows:

(1) The image of f_A under the soft function ϕ_ψ, denoted by $\phi_\psi(f_A)$, is the fuzzy soft set over Y defined by $\phi_\psi(f_A) = \phi(f)_\psi(A)$, where

$$\phi(f)_k(y) = \begin{cases} \bigvee_{x\in\phi^{-1}(y)} \left(\bigvee_{e\in\psi^{-1}(k)\cap A} f_e(x)\right), & \text{if } \phi^{-1}(y) \neq \emptyset,\ \psi^{-1}(k)\cap A \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

for each $k \in K$ and $y \in Y$.

(2) The pre-image of g_B under the fuzzy soft function ϕ_ψ, denoted by $\phi_\psi^{-1}(g_B)$, is the fuzzy soft set over X defined by $\phi_\psi^{-1}(g_B) = \phi^{-1}(g)_{\psi^{-1}(B)}$, where

$$\phi^{-1}(g)_e(x) = \begin{cases} g_{\psi(e)}(\phi(x)), & \text{if } \psi(e) \in B; \\ 0, & \text{otherwise.} \end{cases}$$

for each $e \in E$ and $x \in X$. If ϕ and ψ are injective (resp. surjective) functions, then ϕ_ψ is said to be injective (resp. surjective) fuzzy soft function.

For any two fuzzy soft functions $\phi_\psi : (I^X)^E \longrightarrow (I^Y)^F$ and ${\phi^*}_{\psi^*} : (I^Y)^F \longrightarrow (I^Z)^K$, we have ${\phi^*}_{\psi^*} \circ \phi_\psi : (I^X)^E \longrightarrow (I^Z)^K$ where ${\phi^*}_{\psi^*} \circ \phi_\psi \equiv (\phi^* \circ \phi)_{\psi^*\circ\psi}$.

3. Soft double fuzzy topological spaces

In this section, we introduce basic properties of soft double fuzzy topology and investigate the relationship with soft double fuzzy interior operator.

Definition 3.1. The pair of mappings $\tau, \tau^* : E \longrightarrow I^{(I^X)^E}$ is called a soft double fuzzy topology on X if it satisfies the following conditions for each $e \in E$, f_A, $g_B \in (I^X)^E$ and $\{(f_A)_i | i \in \Gamma\} \subset (I^X)^E$:

(1) $\tau_e(f_A) \leq \tau_e^*(f_A)'$.
(2) $\tau_e(f_A \sqcap g_B) \geq \tau_e(f_A) \wedge \tau_e(g_B)$ and $\tau_e^*(f_A \sqcap g_B) \leq \tau_e^*(f_A) \vee \tau_e^*(g_B)$.
(3) $\tau_e(\bigsqcup_{i\in\Gamma}(f_A)_i) \geq \bigwedge_{i\in\Gamma} \tau_e((f_A)_i)$ and $\tau_e^*(\bigsqcup_{i\in\Gamma}(f_A)_i) \leq \bigvee_{i\in\Gamma} \tau_e^*((f_A)_i)$.

A soft double fuzzy topology is called enriched if it satisfies the following condition:

(4) $\tau_e(\tilde{1}^\alpha) = 1$ and $\tau_e^*(\tilde{1}^\alpha) = 0$.

The triplet (X, τ_E, τ_E^*) is called a soft double fuzzy topological space. The values $\tau_e(f_A)$ and $\tau_e^*(f_A)$ are interpreted as the degree of openness and degree of non-openness of a fuzzy soft set f_A with respect to $e \in E$.

Let (τ_E^1, τ_E^{*1}) and (τ_E^2, τ_E^{*2}) be two soft double fuzzy topologies on X. We say that (τ_E^1, τ_E^{*1}) is finer than (τ_E^2, τ_E^{*2}) ((τ_E^2, τ_E^{*2}) is coarser than (τ_E^1, τ_E^{*1})), denoted by $(\tau_E^2, \tau_E^{*2}) \Subset (\tau_E^1, \tau_E^{*1})$, if $\tau_e^2(f_A) \leq \tau_e^1(f_A)$ and $\tau_e^{*2}(f_A) \geq \tau_e^{*1}(f_A)$ for all $e \in E$ and $f_A \in (I^X)^E$.

Definition 3.2. Let (X, τ_E, τ_E^*) and (Y, ν_F, ν_F^*) be two soft double fuzzy topological spaces. A fuzzy soft function $\phi_\psi : (I^X)^E \longrightarrow (I^Y)^F$ is called a soft double fuzzy continuous function if

$$\tau_e\left(\phi_\psi^{-1}(g_B)\right) \geq \nu_{\psi(e)}(g_B) \quad \text{and} \quad \tau_e^*\left(\phi_\psi^{-1}(g_B)\right) \leq \nu_{\psi(e)}^*(g_B),$$

for each $g_B \in (I^Y)^F$ and $e \in E$.

Proposition 3.3. *Let* $\{(\tau_E^i, {\tau^{i*}}_E) | i \in \Gamma\}$ *be a family of soft double fuzzy topologies on* X. *Then* $(\tau_E, \tau_E^*) = \left(\bigwedge_{i\in\Gamma} \tau_E^i, \bigvee_{i\in\Gamma} {\tau^{i*}}_E\right)$ *is also a soft double fuzzy topology on* X, *where*

$$\tau_e(f_A) = \bigwedge_{i\in\Gamma} (\tau^i)_e(f_A) \quad \textit{and} \quad \tau_e^*(f_A) = \bigvee_{i\in\Gamma} (\tau^{i*})_e(f_A),$$

for all $f_A \in (I^X)^E$ *and* $e \in E$.

Proof . Obvious. □

Definition 3.4. A map $\Im : E \times (I^X)^E \times I_1 \times I_0 \longrightarrow (I^X)^E$ is called a soft double fuzzy interior operator on X if and only if $\Im$ satisfies the following conditions, for all $e \in E$, $f_A, g_B \in (I^X)^E$, $\alpha, \alpha_1 \in I_1$ and $\beta, \beta_1 \in I_0$ such $\alpha + \beta \leq 1$ and $\alpha_1 + \beta_1 \leq 1$:

(1) $\Im(e, \tilde{1}, \alpha, \beta) = \tilde{1}$,
(2) $\Im(e, f_A, \alpha, \beta) \sqsubseteq f_A$,
(3) If $f_A \sqsubseteq g_B$, then $\Im(e, f_A, \alpha, \beta) \sqsupseteq \Im(e, g_B, \alpha, \beta)$,
(4) If $\alpha \leq \alpha_1$ and $\beta \geq \beta_1$, then $\Im(e, f_A, \alpha, \beta) \sqsupseteq \Im(e, f_A, \alpha_1, \beta_1)$,
(5) $\Im(e, f_A \sqcap g_B, \alpha, \beta) \tilde{=} \Im(e, f_A, \alpha, \beta) \sqcap \Im(e, g_B, \alpha, \beta)$.

The pair $(X, \Im_E)$ is called a soft double fuzzy interior space. A soft double fuzzy interior operator $\Im_E$ is called topological provided that:

$$\Im(e, \Im(e, f_A, \alpha, \beta), \alpha, \beta) \tilde{=} \Im(e, f_A, \alpha, \beta).$$

Theorem 3.5. *Let* (X, τ_E, τ_E^*) *be a soft double fuzzy topological space. Define the map* $\Im_{(\tau_E, \tau_E^*)} : E \times (I^X)^E \times I_0 \times I_1 \longrightarrow (I^X)^E$ *by*

$$\Im_{(\tau_E,\tau_E^*)}(e, f_A, \alpha, \beta) = \bigsqcup \{g_B \in (I^X)^E : \; g_B \sqsubseteq f_A, \; \tau_e(g_B) \geq \alpha, \; \tau_e^*(g_B) \leq \beta\}$$

for each $e \in E$, $f_A \in (I^X)^E$, $\alpha \in I_0$ *and* $\beta \in I_1$. *Then* $\Im_{(\tau_E,\tau_E^*)}$ *is a topological soft double fuzzy interior operator.*

Proof .(1), (2) and (3) are obvious. We will prove (4) and (5) as follows:

(4) Let $\alpha \leq \alpha_1$ and $\beta \geq \beta_1$. Then

$$\begin{aligned}\Im_{(\tau_E,\tau_E^*)}(e, f_A, \alpha, \beta) &\tilde{=} \bigsqcup \{g_B \in (I^X)^E : g_B \sqsubseteq f_A, \tau_e(g_B) \geq \alpha, \tau_e^*(g_B) \geq \beta\} \\ &\sqsupseteq \bigsqcup \{g_B \in (I^X)^E : g_B \sqsubseteq f_A, \tau_e(g_B) \geq \alpha_1, \tau_e(g_B) \leq \beta_1\} \\ &\tilde{=} \Im_{(\tau_E,\tau_E^*)}(e, f_A, \alpha_1, \beta_1),\end{aligned}$$

for each $e \in E$, $f_A \in (I^X)^E$, α, $\alpha_1 \in I_1$, β, $\beta_1 \in I_0$.

(5) Since $f_A \sqcap g_B \sqsubseteq f_A$ and $f_A \sqcap g_B \sqsubseteq g_B$, then we have

$$\Im_{(\tau_E,\tau_E^*)}(e, f_A \sqcap g_B, \alpha, \beta) \sqsubseteq \Im_{(\tau_E,\tau_E^*)}(e, f_A, \alpha, \beta)$$

and $\Im_{(\tau_E,\tau_E^*)}(e, f_A \sqcap g_B, \alpha, \beta) \sqsubseteq \Im_{(\tau_E,\tau_E^*)}(e, g_B, \alpha, \beta)$. Thus

$$\Im_{(\tau_E,\tau_E^*)}(e, f_A \sqcap g_B, \alpha, \beta) \sqsubseteq \Im_{(\tau_E,\tau_E^*)}(e, f_A, \alpha, \beta) \sqcap \Im_{(\tau_E,\tau_E^*)}(e, g_B, \alpha, \beta),$$

for each $e \in E$, f_A, $g_B \in (I^X)^E$, $\alpha \in I_1$ and $\beta \in I_0$.
Conversely, since $\mathfrak{I}_{(\tau_E,\tau_E^*)}(e, f_A, \alpha, \beta) \sqcap \mathfrak{I}_{(\tau_E,\tau_E^*)}(e, g_B, \alpha, \beta) \sqsubseteq f_A \sqcap g_B$,

$$\begin{aligned}\tau_e(\mathfrak{I}_{(\tau_E,\tau_E^*)}(e, f_A, \alpha, \beta) \sqcap \mathfrak{I}_{(\tau_E,\tau_E^*)}(e, g_B, \alpha, \beta)) &\geq \tau_e(\mathfrak{I}_{(\tau_E,\tau_E^*)}(e, f_A, \alpha, \beta)) \wedge \tau_e(\mathfrak{I}_{(\tau_E,\tau_E^*)}(e, g_B, \alpha, \beta)) \\ &\geq \alpha \wedge \alpha = \alpha,\end{aligned}$$

and

$$\begin{aligned}\tau_e^*(\mathfrak{I}_{(\tau_E,\tau_E^*)}(e, f_A, \alpha, \beta) \sqcap \mathfrak{I}_{(\tau_E,\tau_E^*)}(e, g_B, \alpha, \beta)) &\leq \tau_e^*(\mathfrak{I}_{(\tau_E,\tau_E^*)}(e, f_A, \alpha, \beta)) \vee \tau_e(\mathfrak{I}_{(\tau_E,\tau_E^*)}(e, g_B, \alpha, \beta)) \\ &\leq \beta \vee \beta = \beta,\end{aligned}$$

then

$$\mathfrak{I}_{(\tau_E,\tau_E^*)}(e, f_A, \alpha, \beta) \sqcap \mathfrak{I}_{(\tau_E,\tau_E^*)}(e, g_B, \alpha, \beta) \sqsubseteq \mathfrak{I}_{(\tau_E,\tau_E^*)}(e, f_A \sqcap g_B, \alpha, \beta).$$

Hence

$$\mathfrak{I}_{(\tau_E,\tau_E^*)}(e, f_A \sqcap g_B, \alpha, \beta) \tilde{=} \mathfrak{I}_{(\tau_E,\tau_E^*)}(e, f_A, \alpha, \beta) \sqcap \mathfrak{I}_{(\tau_E,\tau_E^*)}(e, g_B, \alpha, \beta).$$

□

Theorem 3.6. *Let $\mathfrak{I} : E \times (I^X)^E \times I_1 \times I_0 \longrightarrow (I^X)^E$ be a soft double fuzzy interior operator on X and let $\tau_{\mathfrak{I}}$, $\tau_{\mathfrak{I}}^* : E \longrightarrow (I^X)^E$ be a pair of maps defined by*

$$(\tau_{\mathfrak{I}})_e(f_A) = \bigvee\{\alpha \in I : \mathfrak{I}(e, f_A, \alpha, \beta) \tilde{=} f_A\},$$

and

$$(\tau_{\mathfrak{I}})_e^*(f_A) = \bigwedge\{\beta \in I : \mathfrak{I}(e, f_A, \alpha, \beta) \tilde{=} f_A\},$$

for each $e \in E$ and $f_A \in (I^X)^E$. Then $((\tau_{\mathfrak{I}})_E, (\tau_{\mathfrak{I}})_E^*)$ is a soft double fuzzy topology on X.

Proof . The proof of (1) is obvious. We will prove (2) and (3) as follows:

(2) Suppose that

$$(\tau_{\mathfrak{I}})_e(f_A \sqcap g_B) \not\geq (\tau_{\mathfrak{I}})_e(f_A) \wedge (\tau_{\mathfrak{I}})_e(g_B)$$

or

$$(\tau_{\mathfrak{I}})_e^*(f_A \sqcap g_B) \not\leq (\tau_{\mathfrak{I}})_e^*(f_A) \wedge (\tau_{\mathfrak{I}})_e^*(g_B)$$

for each $e \in E$ and $f_A, g_B \in (I^X)^E$. Then there exist t_1, $t_2 \in I$ such that

$$(\tau_{\mathfrak{I}})_e(f_A \sqcap g_B) < t_1 < (\tau_{\mathfrak{I}})_e(f_A) \wedge (\tau_{\mathfrak{I}})_e(g_B)$$

or

$$(\tau_{\mathfrak{I}})_e^*(f_A \sqcap g_B) > t_2 > (\tau_{\mathfrak{I}})_e^*(f_A) \wedge (\tau_{\mathfrak{I}})_e^*(g_B).$$

Since $t_1 < (\tau_{\mathfrak{I}})_e(f_A)$, $t_1 < (\tau_{\mathfrak{I}})_e(g_B)$, $t_2 > (\tau_{\mathfrak{I}})_e^*(f_A)$ or $t_2 > (\tau_{\mathfrak{I}})_e^*(g_B)$, there exist α_1, $\alpha_2 \in I_1$ and β_1, $\beta_2 \in I_0$ such that $t_1 < \alpha_1 \leq (\tau_{\mathfrak{I}})_e(f_A)$, $t_1 < \alpha_2 \leq (\tau_{\mathfrak{I}})_e(g_B)$, or $t_2 > \beta_1 \geq (\tau_{\mathfrak{I}})_e^*(f_A)$, $t_2 > \beta_2 \geq (\tau_{\mathfrak{I}})_e^*(g_B)$. Let $\alpha = \alpha_1 \wedge \alpha_2$ and $\beta = \beta_1 \vee \beta_2$, then $\mathfrak{I}(e, f_A, \alpha, \beta) = f_A$ and $\mathfrak{I}(e, g_B, \alpha, \beta) = g_B$. By Definition 3.4(5), we have

$$\mathfrak{I}(e, f_A \sqcap g_B, \alpha, \beta) = \mathfrak{I}(e, f_A, \alpha, \beta) \sqcap \mathfrak{I}(e, g_B, \alpha, \beta) = f_A \sqcap g_B.$$

Thus

$$t_1 > (\tau_{\mathfrak{J}})_e(f_A \sqcap g_B) = \bigvee\{\alpha^* \in I : \mathfrak{J}(e, f_A \sqcap g_B, \alpha^*, \beta^*) = f_A \sqcap g_B\}$$
$$\geq \alpha = \alpha_1 \wedge \alpha_2 > t_1,$$

or

$$t_2 < (\tau_{\mathfrak{J}})^*_e(f_A \sqcap g_B) = \bigwedge\{\beta^* \in I : \mathfrak{J}(e, f_A \sqcap g_B, \alpha^*, \beta^*) = f_A \sqcap g_B\}$$
$$\leq \beta = \beta_1 \vee \beta_2 < t_2.$$

It is a contradiction. Hence

$$(\tau_{\mathfrak{J}})_e(f_A \sqcap g_B) \geq (\tau_{\mathfrak{J}})_e(f_A) \wedge (\tau_{\mathfrak{J}})_e(g_B)$$

and

$$(\tau_{\mathfrak{J}})^*_e(f_A \sqcap g_B) \leq (\tau_{\mathfrak{J}})^*_e(f_A) \vee (\tau_{\mathfrak{J}})^*_e(g_B).$$

(3) Suppose that

$$(\tau_{\mathfrak{J}})_e(\bigsqcup_{i\in\Gamma}(f_A)_i) \ngeq \bigwedge_{i\in\Gamma}(\tau_{\mathfrak{J}})_e((f_A)_i)$$

or

$$(\tau_{\mathfrak{J}})^*_e(\bigsqcup_{i\in\Gamma}(f_A)_i) \nleq \bigvee_{i\in\Gamma}(\tau_{\mathfrak{J}})^*_e((f_A)_i)$$

for each $e \in E$ and $\{(f_A)_i | i \in \Gamma\} \subset (I^X)^E$. Then there exist t_1, $t_2 \in I$ such that

$$(\tau_{\mathfrak{J}})_e(\bigsqcup_{i\in\Gamma}(f_A)_i) < t_1 < \bigwedge_{i\in\Gamma}(\tau_{\mathfrak{J}})_e((f_A)_i)$$

or

$$(\tau_{\mathfrak{J}})^*_e(\bigsqcup_{i\in\Gamma}(f_A)_i) > t_2 > \bigvee_{i\in\Gamma}(\tau_{\mathfrak{J}})^*_e((f_A)_i).$$

Since $t_1 < (\tau_{\mathfrak{J}})_e((f_A)_i)$ or $t_2 > (\tau_{\mathfrak{J}})^*_e((f_A)_i)$ for each $i \in \Gamma$, there exist $\alpha_i \in I_1$ and $\beta_i \in I_0$ such that

$$t_1 \leq \alpha_i \leq (\tau_{\mathfrak{J}})_e((f_A)_i)$$

or $t_2 \geq \beta_i \geq (\tau_{\mathfrak{J}})^*_e((f_A)_i)$. Let $\alpha = \bigwedge_{i\in\Gamma} \alpha_i$ and $\beta = \bigvee_{i\in\Gamma} \beta_i$. Then

$$\mathfrak{J}(e, \bigsqcup_{i\in\Gamma}(f_A)_i, \alpha, \beta) \tilde{=} \bigsqcup_{i\in\Gamma}(f_A)_i.$$

Thus

$$t_1 > (\tau_{\mathfrak{J}})_e(\bigsqcup_{i\in\Gamma}(f_A)_i) = \bigvee\{\alpha^* \in I : \mathfrak{J}(e, \bigsqcup_{i\in\Gamma}(f_A)_i, \alpha^*, \beta^*) \tilde{=} \bigsqcup_{i\in\Gamma}(f_A)_i\}$$
$$\geq \alpha = \bigwedge_{i\in\Gamma} \alpha_i \geq t_1,$$

or

$$t_2 > (\tau_{\mathfrak{J}})^*_e(\bigsqcup_{i\in\Gamma}(f_A)_i) = \bigwedge\{\beta^* \in I : \mathfrak{J}(e, \bigsqcup_{i\in\Gamma}(f_A)_i, \alpha^*, \beta^*) \tilde{=} \bigsqcup_{i\in\Gamma}(f_A)_i\}$$
$$\leq \beta = \bigvee_{i\in\Gamma} \beta_i \leq t_2.$$

It is a contradiction. Hence $(\tau_{\mathfrak{J}})_e(\bigsqcup_{i\in\Gamma}(f_A)_i) \geq \bigwedge_{i\in\Gamma}(\tau_{\mathfrak{J}})_e((f_A)_i)$ and $(\tau_{\mathfrak{J}})^*_e(\bigsqcup_{i\in\Gamma}(f_A)_i) \leq \bigvee_{i\in\Gamma}(\tau_{\mathfrak{J}})^*_e((f_A)_i)$.

□

4. Soft double fuzzy topogenous structure

In this section, we start with softification of the concept of double fuzzy semi-topogenous structure. Hence, we give its properties and the relationship with soft double fuzzy interior operator and soft double fuzzy topology.

Definition 4.1. A soft double fuzzy semi-topogenous order on a set X is a pair of functions $\eta, \eta^* : E \longrightarrow I^{(I^X)^E \times (I^X)^E}$ (where $(\eta_e, \eta_e^*) \equiv (\eta(e), \eta(e)^*) : (I^X)^E \times (I^X)^E \longrightarrow I$ are mappings for all $e \in E$) which satisfy the following axioms for any $e \in E$, f_A, $(f_A)_1$, $(f_A)_2$, (g_B), $(g_B)_1$, $(g_B)_2 \in (I^X)^E$:

(S1) $\eta_e(f_A, g_B) \leq \eta_e^*(f_A, g_B)'$.

(S2) If $\eta_e(f_A, g_B) \neq 0$ and $\eta_e^*(f_A, g_B) \neq 1$, then $f_A \sqsubseteq g_B$.

(S3) If $(f_A)_1 \sqsubseteq f_A \sqsubseteq g_B \sqsubseteq (g_B)_1$, then $\eta_e((f_A)_1, (g_B)_1) \geq \eta_e(f_A, g_B)$ and $\eta_e^*((f_A)_1, (g_B)_1) \leq \eta_e^*(f_A, g_B)$.

A soft double fuzzy semi-topogenous order (η_E, η_E^*) is said to be finer than (θ_E, θ_E^*) ((θ_E, θ_E^*) is coarser than (η_E, η_E^*)), denoted by $(\eta_E, \eta_E^*) \Subset (\theta_E, \theta_E^*)$, iff $\eta_e(f_A, g_B) \geq \theta_e(f_A, g_B)$ and $\eta_e^*(f_A, g_B) \leq \theta_e^*(f_A, g_B)$, for all f_A, $g_B \in (I^X)^E$ and $e \in E$. The composition $(\eta_E \circ \theta_E, \eta_E^* \circ \theta_E^*)$ of soft double fuzzy semi-topogenous orders (η_E, η_E^*) and (θ_E, θ_E^*) on X is defined by

$$\eta_e \circ \theta_e(f_A, g_B) = \bigvee_{h_C \in (I^X)^E} \{\theta_e(f_A, h_C) \wedge \eta_e(h_C, g_B)\}$$

and

$$\eta_e^* \circ \theta_e^*(f_A, g_B) = \bigwedge_{h_C \in (I^X)^E} \{\theta_e^*(f_A, h_C) \vee \eta_e^*(h_C, g_B)\},$$

for each $f_A, g_B \in (I^X)^E$ and $e \in E$. A soft double fuzzy semi-topogenous order (η_E, η_E^*) is called a soft double fuzzy topogenous order if it satisfies the following conditions:

(S4) $\eta_e((f_A)_1 \sqcup (f_A)_2, g_B) \geq \eta_e((f_A)_1, g_B) \wedge \eta_e((f_A)_2, g_B)$ and $\eta_e^*((f_A)_1 \sqcup (f_A)_2, g_B) \leq \eta_e^*((f_A)_1, g_B) \vee \eta_e^*((f_A)_2, g_B)$.

(S5) $\eta_e(f_A, (g_B)_1 \sqcap (g_B)_2) \geq \eta_e(f_A, (g_B)_1) \sqcap \eta_e(f_A, (g_B)_2)$ and $\eta_e^*(f_A, (g_B)_1 \sqcap (g_B)_2) \leq \eta_e^*(f_A, (g_B)_1) \vee \eta_e^*(f_A, (g_B)_2)$.

A soft double fuzzy topogenous order (η_E, η_E^*) on X is called a soft double fuzzy topogenous structure iff $(\eta_E \circ \eta_E, \eta_E^* \circ \eta_E^*) \Subset (\eta_E, \eta_E^*)$. The triplet (X, η_E, η_E^*) is called a soft double fuzzy topogenous space.

Example 4.2. Let $X = \{a, b\}$, $E = \{e_1, e_2, e_3\}$ and let the fuzzy soft sets h_C, $l_D \in (I^X)^E$ defined by

$$(h_C)_{e_i}(a) = (h_C)_{e_i}(b) = 0.5, \quad \text{and} \quad (l_D)_{e_i}(a) = 0.3, \quad (l_D)_{e_i}(b) = 0.7,$$

for each $i = 1, 2, 3$. Define the soft double fuzzy topogenous structure $\eta, \eta^* : E \longrightarrow I^{(I^X)^E \times (I^X)^E}$ by:

$$\eta_{e_1}(f, g) = \begin{cases} 1, & \text{if } f = \tilde{0} \text{ or } g = \tilde{1}; \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad \eta_{e_1}^*(f, g) = \begin{cases} 0, & \text{if } f = \tilde{0} \text{ or } g = \tilde{1}; \\ 1, & \text{otherwise.} \end{cases}$$

$$\eta_{e_2}(f, g) = \begin{cases} 1, & \text{if } f = \tilde{0} \text{ or } g = \tilde{1}; \\ \frac{1}{3}, & \text{if } \tilde{0} \neq f \sqsubseteq h_C \sqsubseteq g \neq \tilde{1}; \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad \eta_{e_2}^*(f, g) = \begin{cases} 0, & \text{if } f = \tilde{0} \text{ or } g = \tilde{1}; \\ \frac{1}{3}, & \text{if } \tilde{0} \neq f \sqsubseteq h_C \sqsubseteq g \neq \tilde{1}; \\ 1, & \text{otherwise.} \end{cases}$$

$$\eta_{e_3}(f,g)=\begin{cases}1, & \text{if } f=\tilde{0} \text{ or } g=\tilde{1};\\ 0.3, & \text{if } f\sqsubseteq h_C\sqsubseteq g,\ g\not\sqsupseteq l_D;\\ 0.5, & \text{if } f\sqsubseteq l_D\sqsubseteq g,\ f\not\sqsubseteq h_C;\\ 0.7, & \text{if } f\sqsubseteq h_C\sqcup l_D\sqsubseteq g;\\ 0.6, & \text{if } f\sqsubseteq h_C\sqcap l_D\sqsubseteq g;\\ 0, & \text{otherwise.}\end{cases} \quad \text{and} \quad \eta^*_{e_3}(f,g)=\begin{cases}0, & \text{if } f=\tilde{0} \text{ or } g=\tilde{1};\\ 0.6, & \text{if } f\sqsubseteq h_C\sqsubseteq g,\ g\not\sqsupseteq l_D;\\ 0.4, & \text{if } f\sqsubseteq l_D\sqsubseteq g,\ f\not\sqsubseteq h_C;\\ 0.3, & \text{if } f\sqsubseteq h_C\sqcup l_D\sqsubseteq g;\\ 0.2, & \text{if } f\sqsubseteq h_C\sqcap l_D\sqsubseteq g;\\ 1, & \text{otherwise.}\end{cases}$$

Definition 4.3. Let (X,η_E,η^*_E), (Y,θ_F,θ^*_F) and (Z,ξ_M,ξ^*_M) be soft double fuzzy topogenous spaces. A fuzzy soft function $\phi_\psi:(I^X)^E\longrightarrow(I^Y)^F$ is called a soft double fuzzy topogenous function if

$$\theta_{\psi(e)}(f_A,g_B)\leq\eta_e(\phi^{-1}_\psi(f_A),\phi^{-1}_\psi(g_B)) \qquad \text{and} \qquad \theta^*_{\psi(e)}(f_A,g_B)\geq\eta^*_e(\phi^{-1}_\psi(f_A),\phi^{-1}_\psi(g_B)),$$

for each f_A, $g_B\in(I^Y)^F$ and $e\in E$.

Proposition 4.4. *Let (X,η_E,η^*_E), (Y,θ_E,θ^*_E) and (Z,ξ_M,ξ^*_M) be soft double fuzzy topogenous spaces. If $\phi_\psi:(I^X)^E\longrightarrow(I^Y)^F$ and $\psi^\star{}_{\psi^\star}:(I^Y)^F\longrightarrow(I^Z)^M$ are soft double fuzzy topogenous functions, then $(\phi^\star\circ\phi)_{\psi^\star\circ\psi}:(I^X)^E\longrightarrow(I^Z)^M$ is soft double fuzzy topogenous function.*

Proof . For each f_A, $g_B\in(I^Z)^M$ and $e\in E$, we have

$$\begin{aligned}\eta_e\left((\phi^\star\circ\phi)^{-1}_{\psi^\star\circ\psi}(f_A),(\phi^\star\circ\phi)^{-1}_{\psi^\star\circ\psi}(g_B)\right)&=\eta_e\left(\phi^{\star}{}^{-1}_{\psi^\star}(\phi^{-1}_\psi(f_A)),\phi^{\star}{}^{-1}_{\psi^\star}(\phi^{-1}_\psi(g_B))\right)\\&\geq\theta_{\psi(e)}\left(\phi^{\star}{}^{-1}_{\psi^\star}(f_A),\phi^{\star}{}^{-1}_{\psi^\star}(g_B)\right)\\&\geq\xi_{\psi^\star\circ\psi(e)}(f_A,g_B),\end{aligned}$$

and

$$\begin{aligned}\eta^*_e\left((\phi^\star\circ\phi)^{-1}_{\psi^\star\circ\psi}(f_A),(\phi^\star\circ\phi)^{-1}_{\psi^\star\circ\psi}(g_B)\right)&=\eta^*_e\left(\phi^{\star}{}^{-1}_{\psi^\star}(\phi^{-1}_\psi(f_A)),\phi^{\star}{}^{-1}_{\psi^\star}(\phi^{-1}_\psi(g_B))\right)\\&\leq\theta^*_{\psi(e)}\left(\phi^{\star}{}^{-1}_{\psi^\star}(f_A),\phi^{\star}{}^{-1}_{\psi^\star}(g_B)\right)\\&\leq\xi^*_{\psi^\star\circ\psi(e)}(f_A,g_B).\end{aligned}$$

□

Theorem 4.5. *Let (η_E,η^*_E) be a soft double fuzzy semi-topogenous order on X. Define a function $\Im_{(\eta_E,\eta^*_E)}:E\times(I^X)^E\times I_1\times I_0\longrightarrow(I^X)^E$ by:*

$$\Im_{(\eta_E,\eta^*_E)}(e,f_A,\alpha,\beta)=\bigsqcup\{g_B\in(I^X)^E\,|\,\eta_e(g_B,f_A)\geq\alpha,\ \eta^*_e(g_B,f_A)\leq\beta\},$$

for each $e\in E$, $f_A\in(I^X)^E$, $\alpha\in I_1$ and $\beta\in I_0$. Then:

(i) $\Im_{\eta_E,\eta^*_E}$ is soft double fuzzy interior operator.
(ii) If (η_E,η^*_E) is a soft double fuzzy topogenous structure on X, then $\Im_{\eta_E,\eta^*_E}$ is topological soft double fuzzy interior operator.

Proof .

(i) (1) Obvious.
(2) Let $\eta_e(f_A,g_B)\geq\alpha$ and $\eta^*_e(f_A,g_B)\leq\beta$ for each $e\in E$, $f_A,g_B\in(I^X)^E$. By using Definition 4.1 (S2), we have $g_B\sqsubseteq f_A$. Then $\Im_{(\eta_E,\eta^*_E)}(e,f_A,\alpha,\beta)\sqsubseteq g_B$.

(3) If $f_A \sqsubseteq g_B$, $\eta_e(h_C, f_A) \geq \alpha$ and $\eta^*_e(h_C, f_A) \leq \beta$ for each $e \in E$. By Definition 4.1 (S3), $\eta_e(h_C, g_B) \geq \alpha$ and $\eta^*_e(h_C, g_B) \leq \beta$. Thus, $\mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta) \sqsubseteq \mathfrak{I}_{(\eta_E,\eta^*_E)}(e, g_B, \alpha, \beta)$.

(4) By using (3), we have

$$\mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A \sqcap g_B, \alpha, \beta) \sqsubseteq \mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta) \sqcap \mathfrak{I}_{(\eta_E,\eta^*_E)}(e, g_B, \alpha, \beta).$$

Now, suppose there exist f_A, $g_B \in (I^X)^E$ such that

$$\mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A \sqcap g_B, \alpha, \beta) \not\sqsupseteq \mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta) \sqcap \mathfrak{I}_{(\eta_E,\eta^*_E)}(e, g_B, \alpha, \beta).$$

Then, there exists $x \in X$ and $t \in I_1$ such that

$$\mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A \sqcap g_B, \alpha, \beta)(x) < t < \mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta)(x) \sqcap \mathfrak{I}_{(\eta_E,\eta^*_E)}(e, g_B, \alpha, \beta)(x).$$

Since $\mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta)(x) > t$ and $\mathfrak{I}_{(\eta_E,\eta^*_E)}(e, g_B, \alpha, \beta)(x) > t$, there exist h_C, $l_D \in (I^X)^E$ with $\eta_e(h_C, f_A) \geq \alpha$, $\eta^*_e(h_C, f_A) \leq \beta$ and $\eta_e(l_D, g_B) \geq \alpha$, $\eta^*_e(l_D, g_B) \leq \beta$ such that

$$\mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta)(x) \geq h_C(x) > t, \quad \mathfrak{I}_{(\eta_E,\eta^*_E)}(e, g_B, \alpha, \beta)(x) \geq l_D(x) > t.$$

On the other hand, since

$$\eta_e(h_C, f_A) \geq \alpha,\ \eta_e(l_D, g_B) \geq \alpha \Rightarrow \eta_e(h_C \sqcap l_D, f_A) \geq \alpha, \quad \eta_e(h_C \sqcap l_D, g_B) \geq \alpha,$$
$$\Rightarrow \eta_e(h_C \sqcap l_D, f_A \sqcap g_B) \geq \alpha.$$

$$\eta^*_e(h_C, f_A) \leq \beta,\ \eta^*_e(l_D, g_B) \leq \beta \Rightarrow \eta^*_e(h_C \sqcap l_D, f_A) \leq \beta, \eta^*_e(h_C \sqcap l_D, g_B) \leq \beta$$
$$\Rightarrow \eta^*_e(h_C \sqcap l_D, f_A \sqcap g_B) \leq \beta.$$

Then we have

$$\mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A \sqcap g_B, \alpha, \beta)(x) \geq (h_C \sqcap l_D)(x) > t.$$

It is contradiction. Thus,

$$\mathfrak{I}_{(\eta_E,\eta^*_E)}(f_A \sqcap g_B, \alpha, \beta) \geq \mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta) \sqcap \mathfrak{I}_{(\eta_E,\eta^*_E)}(e, g_B, \alpha, \beta).$$

(ii) Since $\mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta) \sqsubseteq f_A$. Then, we have

$$\mathfrak{I}_{(\eta_E,\eta^*_E)}(e, \mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta), \alpha, \beta) \sqsubseteq \mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta).$$

Suppose there exists $f_A \in (I^X)^E$ such that

$$\mathfrak{I}_{(\eta_E,\eta^*_E)}(e, \mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta), \alpha, \beta) \not\sqsupseteq \mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta).$$

There exist $x \in X$ and $t \in I_1$ such that

$$\mathfrak{I}_{(\eta_E,\eta^*_E)}(e, \mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta), \alpha, \beta)(x) < t < \mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta)(x).$$

Since $\mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta)(x) > t$, then there exists $g_B \in (I^X)^E$ with $\eta_e(g_B, f_A) \geq \alpha$ and $\eta^*_e(g_B, f_A) \leq \beta$ such that $\mathfrak{I}_{\eta_E,\eta^*_E}(f_A, \alpha, \beta) \geq g_B(x) > t$. Also, since $(\eta_e \circ \eta_e, \eta^*_e \circ \eta^*_e)$ is finer than (η_e, η^*_e), then $\eta_e \circ \eta_e(g_B, f_A) \geq \eta_e(g_B, f_A) \geq \alpha$ and $\eta^*_e \circ \eta^*_e(g_B, f_A) \leq \eta^*_e(g_B, f_A) \leq \beta$, and therefore, there exists $h_C \in (I^X)^E$ such that $\eta_e(g_B, h_C) \geq \alpha$, $\eta^*_e(g_B, h_C) \leq \beta$, $\eta_e(h_C, f_A) \geq \alpha$ and $\eta^*_e(h_C, f_A) \leq \beta$. Hence $\eta_e(g_B, \mathfrak{I}_{\eta_E,\eta^*_E}(f_A, \alpha, \beta)) \geq \alpha$ and $\eta^*_e(g_B, \mathfrak{I}_{\eta_E,\eta^*_E}(f_A, \alpha, \beta)) \leq \beta$. Thus

$$\mathfrak{I}_{\eta_E,\eta^*_E}(e, \mathfrak{I}_{\eta_E,\eta^*_E}(f_A, \alpha, \beta), \alpha, \beta)(x) \geq g_B(x) > t.$$

It is a contradiction. So,

$$\mathfrak{I}_{\eta_E,\eta^*_E}(e, \mathfrak{I}_{\eta_E,\eta^*_E}(f_A, \alpha, \beta), \alpha, \beta) \geq \mathfrak{I}_{\eta_E,\eta^*_E}(f_A, \alpha, \beta).$$

□

Definition 4.6. A soft double fuzzy semi-topogenous order (η_E, η^*_E) is called perfect if

$$\eta_e(\bigsqcup_{i\in\Gamma}(f_A)_i, g_B) \geq \bigwedge_{i\in\Gamma}\eta_e((f_A)_i, g_B)$$

and

$$\eta^*_e(\bigsqcup_{i\in\Gamma}(f_A)_i, g_B) \leq \bigvee_{i\in\Gamma}\eta^*_e((f_A)_i, g_B),$$

for each $\{(f_A)_i | i \in \Gamma\} \subset (I^X)^E$ and $e \in E$.

Theorem 4.7. *Let (η_E, η^*_E) be a soft double fuzzy topogenous order on X. Define the maps $\tau_{\Im_{(\eta_E,\eta^*_E)}}$, ${\tau^*}_{\Im_{(\eta_E,\eta^*_E)}} : E \longrightarrow I^{(I^X)^E}$ by:*

$$\tau_{\Im(\eta_E,\eta^*_E)}(f_A) = \bigvee\{\alpha \in I |\ \Im_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta) = f_A\},$$

$$\tau^*_{\Im(\eta_E,\eta^*_E)}(f_A) = \bigwedge\{\beta \in I |\ \Im_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta) = f_A\},$$

for each $e \in E$ and $f_A \in (I^X)^E$. Then:

(i) $(\tau_{\Im_{(\eta_E,\eta^*_E)}}, \tau^*_{\Im(\eta_E,\eta^*_E)})$ *is a soft double fuzzy topology on X.*

(ii) If (η_E, η^*_E) is perfect, then

$$\tau_{\Im_{(\eta_E,\eta^*_E)}}(f_A) = \eta_e(f_A, f_A) \quad \text{and} \quad \tau^*_{\Im_{(\eta_E,\eta^*_E)}}(f_A) = \eta^*_e(f_A, f_A).$$

Proof .

(i) Similar to the proof of Theorem 4.7.

(ii) Let $\tau_{\Im_{(\eta_E,\eta^*_E)}}(f_A) \geq \alpha$ and $\tau^*_{\Im_{(\eta_E,\eta^*_E)}}(f_A) \leq \beta$, then

$$f_A = \Im_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta) = \bigvee\{g_B \in (I^X)^E | \eta_e(g_B, f_A) \geq \alpha, \eta^*_e(g_B, f_A) \leq \beta\}.$$

Since (η_E, η^*_E) is perfect, then $\eta_e(f_A, f_A) \geq \alpha$ and $\eta^*_e(f_A, f_A) \leq \beta$. Conversely, let $\eta_e(f_A, f_A) \geq \alpha$ and $\eta^*_e(f_A, f_A) \leq \beta$, then $\Im_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta) \sqsupseteq f_A$. So, $\tau_{\Im_{(\eta_E,\eta^*_E)}}(f_A) \geq \alpha$, $\tau^*_{\Im_{(\eta_E,\eta^*_E)}}(f_A) \leq \beta$ and the result follows.

□

Theorem 4.8. *Let $(X, \Im_E)$ be a soft double fuzzy interior space. Define the functions $\eta_\Im, \eta^*_\Im : E \longrightarrow I^{(I^X)^E \times (I^X)^E}$ by:*

$$(\eta_\Im)_e(f_A, g_B) = \bigvee\{\alpha \in I | f_A \sqsubseteq \Im(e, g_B, \alpha, \beta)\},$$

$$(\eta_\Im)^*_e(f_A, g_B) = \bigwedge\{\beta \in I | f_A \sqsubseteq \Im(e, g_B, \alpha, \beta)\}.$$

Then:

(a) $(\eta_\Im, \eta^*_\Im)$ is perfect soft double fuzzy semi-topogenous order on X such that

$$\Im_{((\eta_\Im)_E,(\eta_\Im)^*_E)}(e, f_A, \alpha, \beta) = \Im(e, f_A, \alpha, \beta),$$

for each $e \in E$, $f_A \in (I^X)^E$, $\alpha \in I_1$ and $\beta \in I_0$.

(b) $(\eta_{\Im}, \eta^*_{\Im})$ *is a soft double fuzzy topogenous order on* X*.*

(c) If $\Im$ *is topological soft double fuzzy interior operator, then* $(\eta_{\Im} \circ \eta_{\Im}, \eta^*_{\Im} \circ \eta^*_{\Im}) \Subset (\eta_{\Im}, \eta^*_{\Im})$*.*

(d) If (η, η^*) *is a soft double fuzzy semi-topogenous order, then* $(\eta_{\Im_{(\eta,\eta^*)}}, \eta^*_{\Im_{(\eta,\eta^*)}}) \Subset (\eta, \eta^*)$*.*

(e) If (η, η^*) *is perfect soft double fuzzy semi-topogenous order, then*

$$(\eta, \eta^*) = (\eta_{\Im_{(\eta,\eta^*)}}, \eta^*_{\Im_{(\eta,\eta^*)}}).$$

Proof . (a) Firstly, we will show that $(\eta_{e\Im}, \eta^*_{e\Im})$ is perfect double fuzzy semi-topogenous order:

(1) Trivial.

(2) If $\eta_{e\Im}(f_A, g_B) \neq 0$ and $\eta^*_{e\Im}(f_A, g_B) \neq 1$, then $f_A \sqsubseteq \Im(e, g_B, \alpha, \beta) \sqsubseteq g_B$.

(3) Let $h_C \sqsubseteq F_A \sqsubseteq g_B \sqsubseteq l_D$, $\eta_{e\Im}(f_A, g_B) \geq \alpha$ and $\eta^*_{e\Im}(f_A, g_B) \leq \beta$. Since $f_A \sqsubseteq \Im(e, g_B, \alpha, \beta)$, then $h_C \sqsubseteq \Im(e, l_D, \alpha, \beta)$. Hence $\eta_{e\Im}(h_C, l_D) \geq \alpha$ and $\eta^*_{e\Im}(h_C, l_D) \leq \beta$. Thus $\eta_{e\Im}(h_C, l_D) \geq \eta_{e\Im}(f_A, g_B)$ and $\eta^*_{e\Im}(h_C, l_D) \leq \eta^*_{e\Im}(f_A, g_B)$.

(Pf) Let

$$\eta_{e\Im}(\bigsqcup_{i\in\Gamma} f_{A_i}, g_B) \geq \alpha$$

and

$$\eta^*_{e\Im}(\bigsqcup_{i\in\Gamma} f_{A_i}, g_B) \leq \beta.$$

Since

$$f_{A_i} \sqsubseteq \bigsqcup_{i\in\Gamma} f_{A_i},$$

then by (3) $\eta_{e\Im}(f_{A_i}, g_B) \geq \alpha$ and $\eta^*_{e\Im}(f_{A_i}, g_B) \leq \beta$, for each $i \in \Gamma$. Thus

$$\bigsqcup_{i\in\Gamma} \eta_{e\Im}(f_{A_i}, g_B) \geq \alpha$$

and

$$\bigsqcup_{i\in\Gamma} \eta^*_{e\Im}(f_{A_i}, g_B) \leq \beta$$

and therefore

$$\eta_{e\Im}(\bigsqcup_{i\in\Gamma} f_{A_i}, g_B) \leq \bigwedge_{i\in\Gamma} \eta_{e\Im}(f_{A_i}, g_B) \quad \text{and} \quad \eta^*_{e\Im}(\bigsqcup_{i\in\Gamma} f_{A_i}, g_B) \geq \bigvee_{i\in\Gamma} \eta^*_{e\Im}(f_{A_i}, g_B).$$

Now, let

$$\bigwedge_{i\in\Gamma} \eta_{e\Im}(f_{A_i}, g_B) \geq \alpha$$

and

$$\bigvee_{i\in\Gamma} \eta^*_{e\Im}(f_{A_i}, g_B) \leq \beta.$$

Then

$$f_{A_i} \sqsubseteq \Im(e, g_B, \alpha, \beta),$$

for each $i \in \Gamma$. Thus

$$\bigsqcup_{i\in\Gamma} f_{A_i} \sqsubseteq \Im(e, g_B, \alpha, \beta).$$

It implies

$$\eta_{e\Im}(\bigsqcup_{i\in\Gamma} f_{A_i}, g_B) \geq \alpha$$

and

$$\eta^*_{e\Im}(\bigvee_{i\in\Gamma} f_{A_i}, g_B) \leq \beta.$$

Thus, $(\eta_{e\Im}, \eta^*_{e\Im})$ is perfect soft double fuzzy semi-topogenous order.

Secondly, we will show that

$$\Im_{\eta_{e\Im},\eta^*_{e\Im}}(e, f_A, \alpha, \beta) = \Im(e, f_A, \alpha, \beta),$$

for each $f_A \in (I^X)^E$, $\alpha \in I_1$ and $\beta \in I_0$:

Let $\eta_{e\Im}(g_B, \lambda) \geq \alpha$ and $\eta^*_{e\Im}(g_B, \lambda) \leq \beta$. Then $g_B \sqsubseteq \Im(e, f_A, \alpha, \beta)$. By the definition of $\Im_{\eta_{e\Im},\eta^*_{e\Im}}$, we have

$$\Im_{\eta_{e\Im},\eta^*_{e\Im}}(e, f_A, \alpha, \beta) \sqsubseteq \Im(e, f_A, \alpha, \beta).$$

Since

$$\Im(e, f_A, \alpha, \beta) \sqsubseteq \Im(e, f_A, \alpha, \beta),$$

then

$$\eta_{e\Im}(\Im(e, f_A, \alpha, \beta), f_A) \geq \alpha$$

and

$$\eta^*_{e\Im}(\Im(e, f_A, \alpha, \beta), f_A) \leq \beta.$$

Thus,

$$\Im_{\eta_{e\Im},\eta^*_{e\Im}}(e, f_A, \alpha, \beta) \sqsubseteq \Im(e, f_A, \alpha, \beta)$$

and the result follows.

(b) From (1), we will only show that $(\eta_{e\Im}, \eta^*_{e\Im})$ satisfies (T):

Let

$$\eta_{e\Im}(f_A, g_{B1} \sqcap g_{B2}) \geq \alpha$$

and

$$\eta^*_{e\Im}(f_A, g_{B1} \sqcap g_{B2}) \leq \beta.$$

Since

$$g_{B1} \sqcap g_{B2} \sqsubseteq g_{Bi}$$

for each $i \in \{1, 2\}$. Then, by Definition 4.1-(T), $\eta(f_A, g_{Bi}) \geq \alpha$ and $\eta^*(f_A, g_{Bi}) \leq \beta$, for each $i = \{1, 2\}$. Thus

$$\eta_{e\Im}(f_A, g_{B1} \sqcap g_{B2}) \leq \eta_{e\Im}(f_A, g_{B1}) \wedge \eta_{e\Im}(f_A, g_{B2})$$

and

$$\eta^*_{e\Im}(f_A, g_{B1} \sqcap g_{B2}) \geq \eta^*_{e\Im}(f_A, g_{B1}) \vee \eta^*_{e\Im}(f_A, g_{B2}).$$

Conversely, let $\eta_{e\Im}(f_A, g_{Bi}) \geq \alpha$ and $\eta^*_{e\Im}(f_A, g_{Bi}) \leq \beta$, for each $i \in \{1, 2\}$. Then $f_A \sqsubseteq \Im(e, g_{Bi}, \alpha, \beta)$, but $\Im$ is a soft double fuzzy interior operator, then

$$f_A \sqsubseteq \Im(e, g_{B1}, \alpha, \beta) \sqcap \Im(e, g_{B2}, \alpha, \beta) = \Im(e, g_{B1} \sqcap g_{B2}, \alpha, \beta).$$

Hence

$$\eta(f_A, g_{B1} \sqcap g_{B2}) \geq \alpha,$$

$$\eta^*(f_A, g_{B_1} \sqcap g_{B_2}) \leq \beta$$

and the result follows.

(c) Let $\eta_{e\mathfrak{I}}(f_A, g_B) \geq \alpha$ and $\eta^*_{e\mathfrak{I}}(f_A, g_B) \leq \beta$. Then $f_A \sqsubseteq \mathfrak{I}(e, g_B, \alpha, \beta)$. Since

$$f_A \sqsubseteq \mathfrak{I}(e, g_B, \alpha, \beta) = \mathfrak{I}(e, \mathfrak{I}(e, g_B, \alpha, \beta), \alpha, \beta)$$

and

$$\mathfrak{I}(e, g_B, \alpha, \beta) \sqsubseteq \mathfrak{I}(e, g_B, \alpha, \beta),$$

then

$$\eta_{e\mathfrak{I}}(f_A, \mathfrak{I}(e, g_B, \alpha, \beta)) \geq \alpha,$$
$$\eta^*_{e\mathfrak{I}}(f_A, \mathfrak{I}(e, g_B, \alpha, \beta)) \leq \beta$$

and

$$\eta_{e\mathfrak{I}}(\mathfrak{I}(e, g_B, \alpha, \beta), g_B) \geq \alpha,$$
$$\eta^*_{e\mathfrak{I}}(\mathfrak{I}(e, g_B, \alpha, \beta), g_B) \leq \beta.$$

Thus,

$$\eta_{e\mathfrak{I}} \circ \eta_{e\mathfrak{I}}(f_A, g_B) \geq \alpha$$

and

$$\eta^*_{e\mathfrak{I}} \circ \eta^*_{e\mathfrak{I}}(f_A, g_B) \leq \beta.$$

(d) Let $\eta_e(f_A, g_B) \geq \alpha$ and $\eta^*_e(f_A, g_B) \leq \beta$. Then

$$f_A \sqsubseteq \mathfrak{I}_{\eta,\eta^*}(e, g_B, \alpha, \beta).$$

Thus

$$\eta_{e\mathfrak{I}_{\eta,\eta^*}}(f_A, g_B) \geq \alpha$$

and

$$\eta^*_{e\mathfrak{I}_{\eta,\eta^*}}(f_A, g_B) \leq \beta.$$

Thus,

$$(\eta_{\mathfrak{I}_{\eta,\eta^*}}, \eta^*_{\mathfrak{I}_{\eta,\eta^*}})$$

is finer than (η, η^*).

(e) Let

$$\eta_{\mathfrak{I}_{\eta,\eta^*}}(f_A, g_B) \geq \alpha$$

and

$$\eta^*_{\mathfrak{I}_{\eta,\eta^*}}(f_A, g_B) \leq \beta.$$

Then

$$f_A \sqsubseteq \mathfrak{I}_{\eta,\eta^*}(e, g_B, \alpha, \beta) = \bigsqcup\{h_C \in (I^X)^E \,|\, \eta(h_C, g_B) \geq \alpha, \eta^*(h_C, g_B) \leq \beta\}.$$

Since (η, η^*) is perfect, then

$$\eta(\mathfrak{I}_{\eta,\eta^*}(e, g_B, \alpha, \beta), g_B) \geq \alpha$$

and

$$\eta^*(\mathfrak{I}_{\eta,\eta^*}(e, g_B, \alpha, \beta), g_B) \leq \beta.$$

From (T),

$$f_A \sqsubseteq \mathfrak{I}_{\eta,\eta^*}(e, g_B, \alpha, \beta).$$

It implies that $\eta_e(f_A, g_B) \geq r$ and $\eta^*_e(f_A, g_B) \leq \beta$. □

Theorem 4.9. *Let (τ_E, τ_E^*) be a soft double fuzzy topology on X. Define the maps $\eta_{\tau_E}, \eta_{\tau_E^*} : E \longrightarrow I^{(I^X)^E \times (I^X)^E}$ by:*

$$\eta_{\tau_e}(f_A, g_B) = \begin{cases} \bigvee \tau_e(h_C), & if\ h_C \in \Phi(f_A, g_B); \\ 0, & if\ \Phi(f_A, g_B) = \emptyset. \end{cases},$$

$$\eta_{\tau_e^*}(f_A, g_B) = \begin{cases} \bigwedge \tau_e^*(h_C), & if\ h_C \in \Phi(f_A, g_B); \\ 1, & if\ \Phi(f_A, g_B) = \emptyset. \end{cases}$$

where $\Phi(f_A, g_B) = \{h_C | f_A \sqsubseteq h_C \sqsubseteq g_B\}$. Then:

(i) $(\eta_{\tau_E}, \eta_{\tau_E^*})$ is a perfect soft double fuzzy topogenous order on X.

(ii) $(\eta_{\tau_E}, \eta_{\tau_E^*}) = (\eta_{\mathfrak{I}_{(\tau_E,\tau_E^*)}}, \eta^*_{\mathfrak{I}_{(\tau_E,\tau_E^*)}})$.

Proof .(i) (1) Trivial.

(2) If $\eta_{\tau_e}(f_A, g_B) \neq 0$ and $\eta_{\tau_e^*}(f_A, g_B) \neq 1$. Then there exists $\Phi(f_A, g_B) \neq \phi$. Thus $f_A \sqsubseteq g_B$.

(3) It follows from the definition of $(\eta_{\tau_e}, \eta_{\tau_e^*})$.

(T) Let $\eta_{\tau_e}(f_A, (g_B)_i) \geq \alpha$ and $\eta_{\tau_e^*}(f_A, (g_B)_i) \leq \beta$, for each $i \in \{1, 2\}$. Then there exist $(h_C)_1 \in \Phi(f_A, (g_B)_1)$ and $(h_C)_2 \in \Phi(f_A, (g_B)_2)$ such that $\tau_e((h_C)_1) \geq \alpha$, $\tau_e^*((h_C)_1) \leq \beta$ and $\tau_e((h_C)_2) \geq \alpha$, $\tau_e^*((h_C)_2) \leq \beta$. Since $\tau_e((h_C)_1 \sqcap (h_C)_2) \geq \alpha$, $\tau_e^*((h_C)_1 \sqcap (h_C)_2) \leq \beta$ and $(h_C)_1 \sqcap (h_C)_2 \in \Phi(f_A, (g_B)_1 \sqcap (g_B)_2)$, we have $\eta_{\tau_e}(f_A, (g_B)_1 \sqcap (g_B)_2) \geq \alpha$, $\eta_{\tau_e^*}(f_A, (g_B)_1 \sqcap (g_B)_2) \leq \beta$.

Now, to prove that $(\eta_{\tau_E}, \eta_{\tau_E^*})$ is perfect, let $\eta_{\tau_e}((f_A)_i, g_B) \geq \alpha$ and $\eta_{\tau_e^*}((f_A)_i, g_B) \leq \beta$, for each $i \in \Gamma$. For each $i \in \Gamma$, there exist $(h_C)_i \in \Phi((f_A)_i, g_B)$ such that $\tau_e((h_C)_i) \geq \alpha$ and $\tau_e^*((h_C)_i) \leq \beta$. It implies that

$$\bigsqcup_{i\in\Gamma}(f_A)_i \sqsubseteq \bigsqcup_{i\in\Gamma}(h_C)_i \sqsubseteq g_B$$

and $\tau_e(\bigsqcup_{i\in\Gamma}(h_C)_i) \geq \alpha$, $\tau_e^*(\bigsqcup_{i\in\Gamma}(h_C)_i) \leq \beta$. Hence $\eta_{\tau_e}(\bigsqcup_{i\in\Gamma}(f_A)_i, g_B) \geq \alpha$ and $\eta_{\tau_e^*}(\bigsqcup_{i\in\Gamma}(f_A)_i, g_B) \leq \beta$.

(ii) Let $\eta_{\mathfrak{I}_{(\tau_E,\tau_E^*)}}(f_A, g_B) \geq \alpha$ and $\eta^*_{\mathfrak{I}_{(\tau_E,\tau_E^*)}}(f_A, g_B) \leq \beta$. Since

$$f_A \sqsubseteq \mathfrak{I}_{\tau_E,\tau_E^*}(e, g_B, \alpha, \beta) \sqsubseteq g_B$$

and

$$\tau_e(\mathfrak{I}_{(\tau_E,\tau_E^*)}(e, g_B, \alpha, \beta)) \geq \alpha,$$
$$\tau_e^*(\mathfrak{I}_{(\tau_E,\tau_E^*)}(e, g_B, \alpha, \beta)) \leq \beta,$$

then $\eta_{\tau_e}(f_A, g_B) \geq \alpha$, $\eta_{\tau_e^*}(f_A, g_B) \leq \beta$.

Conversely, let $\eta_{\tau_e}(f_A, g_B) \geq \alpha$ and $\eta_{\tau_e^*}(f_A, g_B) \leq \beta$. Then, there exists $h_C \in \Phi(f_A, g_B)$ such that $\tau_e(h_C) \geq \alpha$ and $\tau_e^*(h_C) \leq \beta$. Since $\mathfrak{I}_{(\tau_E,\tau_E^*)}(e, h_C, \alpha, \beta) \tilde{=} h_C$, $\eta_{\mathfrak{I}_{(\tau_E,\tau_E^*)}}(f_A, h_C) \geq \alpha$, $\eta^*_{\mathfrak{I}_{\tau_E,\tau_E^*}}(f_A, h_C) \leq \beta$ implies $\eta_{\mathfrak{I}_{(\tau_E,\tau_E^*)}}(f_A, g_B) \geq \alpha$ and $\eta^*_{\mathfrak{I}_{(\tau_E,\tau_E^*)}}(f_A, g_B) \leq \beta$. □

Theorem 4.10. *Let (η_E, η_E^*) be a perfect soft double fuzzy semi-topogenous order on X. Define the maps $\eta_{\eta_E}, \tau_{\eta_E^*} : E \longrightarrow (I^X)^E$ by*

$$\tau_{\eta_e}(f_A) = \eta_e(f_A, f_A), \quad \tau_{\eta_e^*}(f_A) = \eta_e^*(f_A, f_A)$$

for each $f_A \in (I^X)^E$. Then:

(i) $(\tau_{\eta_E}, \tau_{\eta^*_E}) = (\tau_{\Im_{(\eta_E,\eta^*_E)}}, \tau_{\Im_{(\eta_E,\eta^*_E)}})$.

(ii) If (η_E, η^*_E) *is perfect soft double fuzzy topogenous order on* X*, then* $(\tau_{\eta_E}, \tau_{\eta^*_E})$ *is a soft double fuzzy topology on* X*.*

(iii) $(\eta_E, \eta^*_E) \Subset (\eta_{\tau_{\eta_E}}, \eta_{\tau_{\eta^*_E}})$.

(iv) If $(\eta_E \circ \eta_E, \eta^*_E \circ \eta^*_E) \Subset (\eta_E, \eta^*_E)$*, then* $(\eta_E, \eta^*_E) = (\eta_{\tau_{\eta_E}}, \eta_{\tau_{\eta^*_E}})$.

Proof .(i) From Theorem 4.5(2), $\tau_{\eta_e}(f_A) \geq \alpha$, $\tau_{\eta^*_e}(f_A) \leq \beta$ iff $\eta_e(f_A, f_A) \geq \alpha$, $\eta^*_e(f_A, f_A) \leq \beta$ iff

$$f_A = \Im_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta),$$

$$f_A = \Im_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta)$$

iff $\tau_{\Im_{(\eta_E,\eta^*_E)}}(f_A) \geq \alpha$, $\tau^*_{\Im_{(\eta_E,\eta^*_E)}}(f_A) \leq \beta$.

(ii) (1) Trivial.

(2) Let $\tau_{\eta_e}((f_A)_1) \geq \alpha$, $\tau_{\eta^*_e}((f_A)_1) \leq \beta$ and $\tau_{\eta_e}((f_A)_2) \geq \beta$, $\tau_{\eta^*_e}((f_A)_2) \leq \beta$. Then $\eta_e((f_A)_1, (f_A)_1) \geq \alpha$, $\eta^*_e((f_A)_1, (f_A)_1) \leq \beta$, $\eta_e((f_A)_2, (f_A)_2) \geq \alpha$ and $\eta^*_e((f_A)_2, (f_A)_2) \leq \beta$. Thus, $\eta_e((f_A)_1 \sqcap (f_A)_2, (f_A)_1) \geq \alpha$, $\eta^*_e((f_A)_1 \sqcap (f_A)_2, (f_A)_1) \leq \beta$, $\eta_e((f_A)_1 \sqcap (f_A)_2, (f_A)_2) \geq \alpha$ and $\eta^*_e((f_A)_1 \sqcap (f_A)_2, (f_A)_2) \leq \beta$, which tends to

$$\eta_e((f_A)_1 \sqcap (f_A)_2, (f_A)_1 \sqcap (f_A)_2) \geq \alpha, \eta^*_e((f_A)_1 \sqcap (f_A)_2, (f_A)_1 \sqcap (f_A)_2) \leq \beta.$$

Thus $\tau_{\eta_e}((f_A)_1 \sqcap (f_A)_2) \geq \alpha$, $\tau_{\eta^*_e}((f_A)_1 \sqcap (f_A)_2) \leq \beta$.

(3) Let $\tau_{\eta_e}((f_A)_i) \geq \alpha$ and $\tau_{\eta^*_e}((f_A)_i) \leq \beta$, for each $i \in \Gamma$. Then $\eta_e((f_A)_i, (f_A)_i) \geq \alpha$ and $\eta^*_e((f_A)_i, (f_A)_i) \leq \beta$, for each $i \in \Gamma$. By Definition 4.1-(3), we have $\eta_e((f_A)_i, \bigsqcup_{i\in\Gamma}) \geq \alpha$ and $\eta^*_e((f_A)_i, \bigsqcup_{i\in\Gamma}) \leq \beta$. Then $\eta_e(\bigsqcup_{i\in\Gamma}(f_A)_i, \bigsqcup_{i\in\Gamma}(f_A)_i) \geq \alpha$ and $\eta^*_e(\bigsqcup_{i\in\Gamma}(f_A)_i, \bigsqcup_{i\in\Gamma}(f_A)_i) \leq \beta$ (Since (η_E, η^*_E) is perfect). Therefore, $\tau_{\eta_e}(\bigsqcup_{i\in\Gamma}(f_A)_i) \geq \alpha$, $\tau_{\eta^*_e}(\bigsqcup_{i\in\Gamma}(f_A)_i) \leq \beta$ and the result follows.

(iii) Let $\eta_{\tau_{\eta_e}}(f_A, g_B) \geq \alpha$ and $\eta_{\tau_{\eta^*_e}}(f_A, g_B) \leq \beta$. Then, there exists $h_C \in \Phi(f_A, g_B)$ such that $\tau_{\eta_e}(h_C) \geq \alpha$ and $\tau_{\eta^*_e}(h_C) \leq \beta$. Since $\tau_{\eta_e}(h_C) \geq \alpha$, $\tau_{\eta^*_e}(h_C) \leq \beta$ iff $\eta_e(h_C, h_C) \geq \alpha$, $\eta^*_e(h_C, h_C) \leq \beta$ and $f_A \sqsubseteq h_C \sqsubseteq g_B$. By Definition 4.1-(3), $\eta_e(f_A, g_B) \geq \alpha$ and $\eta^*_e(f_A, g_B) \leq \beta$. Then, $(\eta_E, \eta^*_E) \Subset (\eta_{\tau_{\eta_E}}, \eta_{\tau_{\eta^*_E}})$.

(iv) We need only to show that $(\eta_{\tau_{\eta_E}}, \eta_{\tau_{\eta^*_E}}) \Subset (\eta_E, \eta^*_E)$. Let $\eta_e(f_A, g_B) \geq \alpha$ and $\eta^*_e(f_A, g_B) \leq \beta$. Then, $f_A \sqsubseteq \Im_{(\eta_E,\eta^*_E)}(e, g_B, \alpha, \beta) \sqsubseteq g_B$. Since

$$\Im_{(\eta_E,\eta^*_E)}(e, \Im_{(\eta_E,\eta^*_E)}(e, g_B, \alpha, \beta), \alpha, \beta) = \Im_{(\eta_E,\eta^*_E)}(e, g_B, \alpha, \beta),$$

we have $\tau_{\Im_{(\eta_E,\eta^*_E)}}(\Im_{(\eta_E,\eta^*_E)}(e, g_B, \alpha, \beta)) \geq \alpha$ and $\tau^*_{\Im_{(\eta_E,\eta^*_E)}}(\Im_{(\eta_E,\eta^*_E)}(e, g_B, \alpha, \beta)) \leq \alpha$. Since $\tau_{\Im_{(\eta_E,\eta^*_E)}} = \tau_{\eta_e}$ and $\tau^*_{\Im_{(\eta_E,\eta^*_E)}} = \tau_{\eta^*_e}$, then we have $\tau_{\eta_e}(\Im_{(\eta_E,\eta^*_E)}(e, g_B, \alpha, \beta)) \geq \alpha$ and $\tau_{\eta^*_e}(\Im_{(\eta_E,\eta^*_E)}(e, g_B, \alpha, \beta)) \leq \beta$ and therefore, $\eta_{\tau_{\eta_e}}(f_A, g_B) \geq \alpha$ and $\eta_{\tau_{\eta^*_e}}(f_A, g_B) \leq \beta$. $\square$

Theorem 4.11. *Let* (τ_E, τ^*_E) *be a soft double fuzzy topology on* X*. Then*

$$(\tau_{\eta_{\tau_E}}, \tau_{\eta_{\tau^*_E}}) = (\tau_{\eta_{\Im_{(\tau_E,\tau^*_E)}}}, \tau_{\eta_{\Im_{(\tau_E,\tau^*_E)}}}) = (\tau_E, \tau^*_E).$$

Proof . By Theorem 4.9, $(\eta_{\tau_E}, \eta_{\tau^*_E})$ is a perfect soft double fuzzy semi-topogenous order on X and $(\eta_{\tau_E}, \eta_{\tau^*_E}) = (\eta_{\Im_{(\tau_E,\tau^*_E)}}, \eta^*_{\Im_{(\tau_E,\tau^*_E)}})$. Then, by Theorem 4.10,

$$(\tau_{\eta_{\tau_E}}, \tau_{\eta_{\tau^*_E}}) = (\tau_{\eta_{\Im_{(\tau_E,\tau^*_E)}}}, \tau_{\eta^*_{\Im_{(\tau_E,\tau^*_E)}}}).$$

Let

$$\tau_{\eta_{\Im_{(\tau_E,\tau_E^*)}}}(f_A) \geq \alpha$$

and

$$\tau_{\eta^*_{\Im_{(\tau_E,\tau_E^*)}}}(f_A) \leq \beta.$$

Then, by definition of

$$(\tau_{\eta_{\Im_{(\tau_E,\tau_E^*)}}}, \tau_{\eta^*_{\Im_{(\tau_E,\tau_E^*)}}}),$$

$\eta_{\Im_{(\tau_E,\tau_E^*)}}(f_A, f_A) \geq \alpha$ and $\eta^*_{\Im_{(\tau_E,\tau_E^*)}}(f_A, f_A) \leq \beta$. Thus, $\eta_{\tau_e}(f_A, f_A) \geq \alpha$ and $\eta_{\tau_e^*}(f_A, f_A) \leq \beta$. By definition of $(\eta_{\tau_E}, \eta_{\tau_E^*})$, $\tau_e(f_A) \geq \alpha$ and $\tau_e^*(f_A) \leq \beta$. So,

$$\tau_{\eta_{\Im_{(\tau_E,\tau_E^*)}}} \leq \tau$$

and

$$\tau_{\eta^*_{\Im_{(\tau_E,\tau_E^*)}}} \geq \tau^*.$$

Let $\tau_e(f_A) \geq \alpha$ and $\tau_e^*(f_A) \leq \beta$. Then, $\Im_{(\tau_E,\tau_E^*)}(e, f_A, \alpha, \beta) = f_A$. By Theorem 4.8, it implies that $\eta_{\Im_{(\tau_E,\tau_E^*)}}(f_A, f_A) \geq \alpha$ and $\eta^*_{\Im_{(\tau_E,\tau_E^*)}}(f_A, f_A) \leq \beta$ iff $\tau_{\eta_{\Im_{(\tau_E,\tau_E^*)}}}(f_A) \geq \alpha$ and $\tau_{\eta^*_{\Im_{(\tau_E,\tau_E^*)}}}(f_A) \leq \beta$. Thus $\tau \leq \tau_{\eta_{\Im_{(\tau_E,\tau_E^*)}}}$ and $\tau^* \geq \tau_{\eta^*_{\Im_{(\tau_E,\tau_E^*)}}}$. Hence $(\tau_E, \tau_E^*) = (\tau_{\eta_{\Im_{(\tau_E,\tau_E^*)}}}, \tau_{\eta^*_{\Im_{(\tau_E,\tau_E^*)}}})$. □

Theorem 4.12. *Let $(X, \Im_E)$ be a soft double fuzzy interior space. Define the maps $\tau_{\Im_E}, \tau^*_{\Im_E}$ as in Theorem 4.11. Then:*

- *(a)* $(\tau_{\Im_E}, \tau^*_{\Im_E}) = (\tau_{\eta_{\Im_E}}, \tau_{\eta^*_{\Im_W}})$.
- *(b)* $(\eta_{\Im_E}, \eta^*_{\Im_E}) \Subset (\eta_{\tau_{\Im_E}}, \eta_{\tau^*_{\Im_E}})$.
- *(c) If* $(X, \Im_E)$ *is topological, then* $(\eta_{\Im_E}, \eta^*_{\Im_E}) = (\eta_{\tau_{\Im_E}}, \eta_{\tau^*_{\Im_E}})$.

Proof .(a) Let $(\tau_\Im)_e(f_A) \geq \alpha$ and $(\tau_\Im)^*_e(f_A) \leq \beta$ with $\Im(e, f_A, \alpha, \beta) = f_A$. It implies that $(\eta_\Im)_e(f_A, f_A) \geq \alpha$ and $(\eta^*_\Im)_e(f_A, f_A) \leq \beta$ iff $\tau_{\eta_\Im}(f_A) \geq \alpha$ and $\tau_{\eta^*_\Im}(f_A) \leq \beta$. Thus, $\tau_\Im \leq \tau_{\eta_\Im}$ and $\tau^*_\Im \geq \tau_{\eta^*_\Im}$. Let $\tau_{\eta_\Im}(f_A) \geq \alpha$ and $\tau_{\eta^*_\Im}(f_A) \leq \beta$. Then $\eta_\Im(f_A, f_A) \geq \alpha$ and $\eta^*_\Im(f_A, f_A) \leq \beta$. It implies that $f_A \sqsubseteq \Im(e, f_A, \alpha, \beta)$. Thus, $\tau_\Im(f_A) \geq \alpha$ and $\tau^*_\Im(f_A) \leq \beta$. So, $\tau_{\eta_\Im} \leq \tau_\Im$ and $\tau_{\eta^*_\Im} \geq \tau^*_\Im$.

(b) Let $\eta_{\tau_\Im}(f_A, g_B) \geq \alpha$ and $\eta_{\tau^*_\Im}(f_A, g_B) \leq \beta$. Then, there exists $h_C \in \Phi(f_A, g_B)$ such that $\tau_\Im(h_C) \geq \alpha$ and $\tau^*_\Im(h_C) \leq \beta$. Since $f_A \sqsubseteq h_C = \Im(e, h_C, \alpha, \beta)$, we have $\eta_\Im(f_A, h_C) \geq \alpha$ and $\eta^*_\Im(f_A, h_C) \leq \beta$, which implies $\eta_\Im(f_A, g_B) \geq \alpha$ and $\eta^*_\Im(f_A, g_B) \leq \beta$. Hence $(\eta_\Im, \eta^*_\Im)$ is finer than $(\eta_{\tau_\Im}, \eta_{\tau^*_\Im})$.

(c) Let $(\eta_\Im)_e(f_A, g_B) \geq \alpha$ and $(\eta_\Im)^*_e(f_A, g_B) \leq \beta$. Then $f_A \sqsubseteq \Im(e, g_B, \alpha, \beta) \sqsubseteq g_B$. Since $(X, \Im_E)$ is topological, $\tau_\Im(\Im(e, g_B, \alpha, \beta)) \geq \alpha$ and $\tau^*_\Im(\Im(e, g_B, \alpha, \beta)) \leq \beta$. Thus, $\eta_{\tau_\Im}(f_A, g_B) \geq \alpha$ and $\eta_{\tau^*_\Im}(f_A, g_B) \leq \beta$. Hence $(\eta_{\tau_\Im}, \eta_{\tau^*_\Im})$ is finer than $(\eta_\Im, \eta^*_\Im)$ and this complete the proof. □

Theorem 4.13. *Let (X, η_E, η^*_E) and $(Y, \theta_F, \theta^*_F)$ be two soft double fuzzy topogenous spaces. If $\phi_\psi : (I^X)^E \longrightarrow (I^Y)^F$ is soft double fuzzy topogenous function, then it satisfies the following statements:*

1. *(1)* $\phi_\psi\left(\Im_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta)\right) \sqsubseteq \Im_{(\theta_F,\theta^*_F)}(\psi(e), \phi_\psi(f_A), \alpha, \beta)$ *for each* $f_A \in (I^X)^E$, $\alpha \in I_1$, $\beta \in I_0$ *and* $e \in E$.
2. *(2)* $\Im_{(\eta_E,\eta^*_E)}\left(\psi^{-1}(e), \phi_\psi^{-1}(g_B), \alpha, \beta\right) \leq \phi_\psi^{-1}\left(\Im_{(\theta_F,\theta^*_F)}(e, g_B, \alpha, \beta)\right)$, *for each* $g_B \in (I^Y)^F$, $\alpha \in I_1$, $\beta \in I_0$ *and* $e \in F$.

(3) $\phi_\psi : (X, \tau_{(\eta_E,\eta^*_E)}, \tau^*_{(\eta_E,\eta^*_E)}) \longrightarrow (Y, \tau_{(\theta_F,\theta^*_F)}, \tau^*_{(\theta_F,\theta^*_F)})$ *is soft double fuzzy continuous function.*

Proof .

(1) Suppose there exist $f_A \in (I^X)^E$, $\alpha \in I_1$, $\beta \in I_0$ and $e \in E$ such that

$$\phi_\psi\left(\mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta)\right) \not\leq \mathfrak{I}_{(\theta_F,\theta^*_F)}(\psi(e), \phi_\psi(f_A), \alpha, \beta).$$

By using the definition of $\mathfrak{I}_{(\theta_F,\theta^*_F)}$, there exists $g_B \in (I^Y)^F$ with $\theta_{\psi(e)}(g_B, \phi_\psi(f_A)) \geq \alpha$ and $\theta^*_{\psi(e)}(g_B, \phi_\psi(f_A)) \leq \beta$ such that $\phi_\psi\left(\mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta)\right) \not\sqsubseteq g_B$. Since ϕ_ψ is soft double fuzzy topogenous function, then

$$\eta_e\left(\phi^{-1}_\psi(g_B), \phi^{-1}_\psi(\phi_\psi(f_A))\right) \geq \theta_e(g_B, \phi_\psi(f_A)) \geq \alpha,$$

and

$$\eta^*_e\left(\phi^{-1}_\psi(g_B), \phi^{-1}_\psi(\phi_\psi(f_A))\right) \leq \theta^*_e(g_B, \phi_\psi(f_A)) \leq \beta.$$

Since

$$\eta_e(\phi^{-1}_\psi(g_B), f_A) \geq \eta_e(\phi^{-1}_\psi(g_B), \phi^{-1}_\psi(\phi_\psi(f_A)))$$

and

$$\eta^*_e(\phi^{-1}_\psi(g_B), f_A) \leq \eta^*_e(\phi^{-1}_\psi(g_B), \phi^{-1}_\psi(\phi_\psi(f_A))),$$

we have

$$\mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta) = \phi^{-1}_\psi(g_B).$$

Then

$$\phi_\psi\left(\mathfrak{I}_{(\eta_E,\eta^*_E)}(e, f_A, \alpha, \beta)\right) \sqsubseteq \phi_\psi(\phi^{-1}_\psi(g_B)) \sqsubseteq g_B.$$

It is a contradiction.

(2) For each $g_B \in (I^Y)^F$, $\alpha \in I_1$, $\beta \in I_0$ and $f \in F$, put $f_A \tilde{=} \phi^{-1}_\psi(g_B)$ in (1), we get

$$\begin{aligned}\phi_\psi\left(\mathfrak{I}_{(\eta_E,\eta^*_E)}(\psi^{-1}(f), \phi^{-1}_\psi(g_B), \alpha, \beta)\right) &\leq \mathfrak{I}_{(\theta_F,\theta^*_F)}(f, \phi_\psi(\phi^{-1}_\psi(g_B), \alpha, \beta) \\ &\leq \mathfrak{I}_{(\theta_F,\theta^*_F)}(f, g_B, \alpha, \beta).\end{aligned}$$

Then

$$\begin{aligned}\mathfrak{I}_{(\eta_E,\eta^*_E)}(\psi^{-1}(f), \phi^{-1}_\psi(g_B), \alpha, \beta) &\leq \phi^{-1}_\psi\left(\phi_\psi\left(\mathfrak{I}_{(\eta_E,\eta^*_E)}(\psi^{-1}(f), \phi^{-1}_\psi(g_B), \alpha, \beta)\right)\right) \\ &\leq \phi^{-1}_\psi\left(\mathfrak{I}_{(\theta_F,\theta^*_F)}(f, g_B, \alpha, \beta)\right).\end{aligned}$$

(3) It easily proved from (2) and Theorem 4.7.

□

5. Conclusion

In this paper, we have introduced the notions of soft double fuzzy topology, soft double fuzzy interior operator, and soft double fuzzy semi-topogenous structure. In this way, we constructed soft double fuzzy interior operator and soft double fuzzy semi-topogenous structure from soft double fuzzy topology and vice versa. Finally, we presented many properties of the new structures. The results of this paper are considered as a generalization of the same concepts that have been studied before.

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