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On the approximation by Chlodowsky type generalization of (p, q)-Bernstein operators

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Abstract

In the present article, we introduce Chlodowsky variant of (p, q)-Bernstein operators and compute the moments for these operators which are used in proving our main results. Further, we study some approximation properties of these new operators, which include the rate of convergence using usual modulus of continuity and also the rate of convergence when the function f belongs to the class $\mathrm{Lip}_M(\alpha)$. Moreover, we also discuss convergence and rate of approximation in weighted spaces and weighted statistical approximation properties of the sequence of positive linear operators defined by us.

Keywords: (p, q)-integers; Bernstein operators; positive linear operators; Korovkin type approximation theorem; statistical approximation.
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1. Introduction

In the field of approximation theory, the Bernstein polynomials discovered by S.N. Bernstein [3] in 1912, possess many remarkable properties, so new generalizations and applications are being discovered of it. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations etc.

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During the last two decades, the applications of q-calculus emerged as a new area in the field of approximation theory. The rapid development of q-calculus has led to the discovery of various generalizations of Bernstein polynomials involving q-integers. The first q-analogue of the well-known Bernstein polynomials was introduced by Lupaş [16] and another generalization of it was due to Phillips [24]. Since approximation by q-Bernstein polynomials is better than classical one under convenient choice of q, many authors introduced q-generalization of various operators and investigated several approximation properties, for more details we refer the readers to [13, 17, 19].

Recently, Mursaleen *et al* introduced (p, q)-calculus in approximation theory and constructed the (p, q)-analogue of Bernstein operators [20] and (p, q)-analogue of Bernstein-Stancu operators [21], (p, q)-analogue of Bleimann-Butzer-Hahn operators [22], Bernstein-Schurer operarors [23] and investigated their approximation properties. The (p, q)-analog of Szász-Mirakyan operators [1], Kantorovich type Bernstein-Stancu-Schurer operators [4] and Kantorovich variant of (p, q)-Szász-Mirakjan operators [18] have recently been studied too.

Motivated by their work, in this article, we introduce Chlodowsky variant of (p, q)-Bernstein operators. We have organised our paper as follows: In Section 2, we define (p, q)-Bernstein-Chlodowsky operators and estimate the moments for these operators which are used in proving main results. Section 3 is devoted to study some approximation properties of these new operators, which include the rate of convergence using usual modulus of continuity and also the rate of convergence when the function f belongs to the class $\mathrm{Lip}_M(\alpha)$. In section 4, we discuss convergence and rate of approximation in weighted spaces. Moreover, we study weighted statistical approximation properties of the operators in the last section.

Let us recall certain definitions and notations of (p, q)-calculus:

The (p, q)-integer was introduced in order to generalize or unify several forms of q-oscillator algebras well known in the earlier physics literature related to the representation theory of single parameter quantum algebras [5]. The (p, q)-integer $[n]_{p,q}$ is defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2, \ldots, \quad 0 < q < p \leq 1.$$

The (p, q)-factorial $[n]_{p,q}!$ and the (p, q)-binomial coefficients are defined as :

$$[n]_{p,q}! := \begin{cases} [n]_{p,q} [n-1]_{p,q} \ldots [1]_{p,q}, & n \in \mathbb{N} \\ 1, & n = 0 \end{cases}$$

and

$$\left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}! \, [n-k]_{p,q}!}, 0 \leq k \leq n.$$

Further, the (p, q)-binomial expansions are given as

$$(ax + by)^n_{p,q} = \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} a^{n-k} b^k x^{n-k} y^k,$$

and

$$(x - y)^n_{p,q} = (x - y)(px - qy)(p^2x - q^2y) \ldots (p^{n-1}x - q^{n-1}y).$$

Details on (p, q)-calculus can be found in [9, 11, 25, 26]. For $p = 1$, all the notions of (p, q)-calculus are reduced to q-calculus [12].

2. Construction of operators

For a function f defined on the interval $[0, b_n]$, the q-Bernstein-Chlodowsky operators $C_{n,q}(f),\ n \geq 1$ [14] are defined as:

$$(C_{n,q}f)(x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q} b_n\right) \left[\begin{array}{c} n \\ k \end{array}\right]_q \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n-k-1} \left(1 - q^s \frac{x}{b_n}\right)^{n-k},\ 0 \leq x \leq b_n,$$

where (b_n) is a positive increasing sequence of real numbers with $b_n \to \infty$ as $n \to \infty$. Some approximation properties of the above said operators including the rate of convergence studied in [14].

Now, by means of (p,q)-calculus, we introduce (p,q)-analogue of Bernstein-Chlodowsky operators as follows:

$$C_{n,p,q}(f;x) = \frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)_{p,q}^{n-k} f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}} b_n\right), \tag{2.1}$$

where

$$\left(1 - \frac{x}{b_n}\right)_{p,q}^{n-k} = \prod_{s=0}^{n-k-1} \left(p^s - q^s \frac{x}{b_n}\right).$$

For $p = 1$, the sequence of operators (2.1) turns out to be the classical q-Bernstein-Chlodowsky operators defined in [14].

Now, we need the following basic lemmas for studying our main results:

Lemma 2.1. (i) $C_{n,p,q}(e_0;x) = 1$,

(ii) $C_{n,p,q}(e_1;x) = x$,

(iii) $C_{n,p,q}(e_2;x) = \frac{p^{n-1}b_n}{[n]_{p,q}}x + \frac{q[n-1]_{p,q}}{[n]_{p,q}}x^2$,

(iv) $C_{n,p,q}(e_3;x) = \frac{b_n^2 x}{[n]_{p,q}^2}p^{2n-2} + \frac{(2p+q)q[n-1]_{p,q}x^2 b_n}{[n]_{p,q}^2}p^{n-1} + \frac{q^3[n-1]_{p,q}[n-2]_{p,q}x^3}{[n]_{p,q}^2}$,

(v)

$$\begin{aligned} C_{n,p,q}(e_4;x) = {} & \frac{b_n^3 x}{[n]_{p,q}^3}p^{3n-3} + \frac{q(3p^2+3qp+q^3)[n-1]_{p,q}b_n^2 x^2}{[n]_{p,q}^3}p^{2n-4} \\ & + \frac{q^3(3p^2+2pq+q^2)[n-1]_{p,q}[n-2]_{p,q}b_n x^3}{[n]_{p,q}^3}p^{n-3} \\ & + \frac{q^6[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}x^4}{[n]_{p,q}^3}, \end{aligned}$$

where $e_v(t) = t^v,\ v = 0,1,2,3,4$.

Proof . It is obvious that (i)

$$C_{n,p,q}(e_0;x) = \frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)_{p,q}^{n-k-1} = 1.$$

(ii)

$$
\begin{aligned}
&C_{n,p,q}\left(e_1;x\right)\\
&=\frac{1}{p^{n(n-1)/2}}\sum_{k=0}^{n}\left[\begin{array}{c} n\\ k\end{array}\right]_{p,q}p^{k(k-1)/2}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-1}\frac{[k]_{p,q}}{[n]_{p,q}\,p^{k-n}}b_n\\
&=\frac{b_n}{p^{n(n-3)/2}[n]_{p,q}}\sum_{k=1}^{n}\left[\begin{array}{c} n-1\\ k-1\end{array}\right]_{p,q}[n]_{p,q}p^{(k+1)(k-2)/2}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-1}\\
&=\frac{b_n x}{b_n p^{(n-1)(n-2)/2}}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1\\ k\end{array}\right]_{p,q}p^{k(k-1)/2}\left(\frac{x}{b_n}\right)^{k+1}\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-2}=x.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
&C_{n,p,q}\left(e_2;x\right)\\
&=\frac{1}{p^{n(n-5)/2}}\sum_{k=0}^{n}\left[\begin{array}{c} n\\ k\end{array}\right]_{p,q}p^{(k+1)(k-4)/2}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-1}\frac{[k]_{p,q}^2}{[n]_{p,q}^2}b_n^2\\
&=\frac{b_n^2}{p^{n(n-5)/2}[n]_{p,q}}\sum_{k=0}^{n}\left[\begin{array}{c} n-1\\ k\end{array}\right]_{p,q}[k+1]_{p,q}p^{(k+1)(k-4)/2}\left(\frac{x}{b_n}\right)^{k+1}\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-2}
\end{aligned}
$$

From $[k+1]_{p,q}=p^k+q[k]_{p,q}$, we get

$$
\begin{aligned}
&C_{n,p,q}\left(e_2;x\right)\\
&=\frac{b_n^2}{p^{n(n-5)/2}[n]_{p,q}}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1\\ k\end{array}\right]_{p,q}p^{(k+1)(k-4)/2}\left(\frac{px}{b_n}\right)^{k+1}\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-2}\left(p^k+q[k]_{p,q}\right)\\
&=\frac{b_n^2}{p^{n(n-5)/2}[n]_{p,q}}\frac{x}{b_n}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1\\ k\end{array}\right]_{p,q}p^{(k^2-k-4)/2}\left(\frac{px}{b_n}\right)^{k+1}\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-2}\\
&\quad+\frac{b_n^2q[n-1]_{p,q}}{p^{(n-2)(n-3)/2}[n]_{p,q}}\frac{x^2}{b_n^2}\sum_{k=0}^{n-2}\left[\begin{array}{c} n-2\\ k\end{array}\right]_{p,q}\left(\frac{x}{b_n}\right)^{k+2}\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-3}\\
&=\frac{p^{n-1}xb_n}{[n]_{p,q}}+\frac{q[n-1]_{p,q}x^2}{[n]_{p,q}}.
\end{aligned}
$$

(iv)

$$
\begin{aligned}
&C_{n,p,q}\left(e_3;x\right)\\
&=\frac{1}{p^{n(n-7)/2}}\sum_{k=0}^{n}\left[\begin{array}{c} n\\ k\end{array}\right]_{p,q}p^{k(k-7)/2}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-1}\frac{[k]_{p,q}^3}{[n]_{p,q}^3}b_n^3\\
&=\frac{b_n^3}{p^{n(n-7)/2}\,[n]_{p,q}^2}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1\\ k\end{array}\right]_{p,q}p^{(k+1)(k-6)/2}[k+1]_{p,q}^2\left(\frac{x}{b_n}\right)^{k+1}\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-2}.
\end{aligned}
$$

By $[k+1]_{p,q}^2 = p^{2k} + 2qp^k[k]_{p,q} + q^2[k]_{p,q}^2$, we have

$$\begin{aligned}
&C_{n,p,q}(e_3;x) \\
&= \frac{b_n^3}{p^{n(n-7)/2}[n]_{p,q}^2}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q} p^{(k^2-k-6)/2}\left(\frac{x}{b_n}\right)^{k+1}\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-2} \\
&\quad + 2q\frac{b_n^3}{p^{n(n-7)/2}[n]_{p,q}^2}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q}[k]_{p,q}p^{(k^2-3k-6)/2}\left(\frac{x}{b_n}\right)^{k+1}\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-2} \\
&\quad + \frac{q^2b_n^3}{p^{n(n-7)/2}[n]_{p,q}^2}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q}[k]_{p,q}^2p^{(k+1)(k-6)/2}\left(\frac{x}{b_n}\right)^{k+1}\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-2} \\
&= \frac{b_n^2x}{[n]_{p,q}^2}p^{2n-2} + \frac{2q[n-1]_{p,q}x^2b_n}{[n]_{p,q}^2}p^{n-1} + \frac{p^{n-2}q^2[n-1]_{p,q}b_nx^2}{[n]_{p,q}^2} + \frac{q^3[n-1]_{p,q}[n-2]_{p,q}x^3}{[n]_{p,q}^2} \\
&= \frac{b_n^2x}{[n]_{p,q}^2}p^{2n-2} + \frac{(2p+q)q[n-1]_{p,q}x^2b_n}{[n]_{p,q}^2}p^{n-2} + \frac{q^3[n-1]_{p,q}[n-2]_{p,q}x^3}{[n]_{p,q}^2}.
\end{aligned}$$

(v)

$$\begin{aligned}
&C_{n,p,q}(e_4;x) \\
&= \frac{1}{p^{n(n-9)/2}}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q}\left(\frac{x}{b_n}\right)^{k}\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-1} p^{k(k-1)/2}p^{-4k}\frac{[k]_{p,q}^4}{[n]_{p,q}^4}b_n^4 \\
&= \frac{b_n^4}{p^{n(n-9)/2}[n]_{p,q}^3}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q}p^{(k+1)(k-8)/2}[k+1]_{p,q}^3\left(\frac{x}{b_n}\right)^{k+1}\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-2}.
\end{aligned}$$

Using the fact $[k+1]_{p,q}^3 = p^{3k} + 3p^{2k}q[k]_{p,q} + 3p^kq^2[k]_{p,q}^2 + q^3[k]_{p,q}^3$, we obtain

$$\begin{aligned}
&C_{n,p,q}(e_4;x) \\
&= \frac{b_n^4p^{3n-3}}{p^{(n-1)(n-2)/2}[n]_{p,q}^3}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q}p^{k(k-1)/2}\left(\frac{x}{b_n}\right)^{k+1}\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-2} \\
&\quad + \frac{3qb_n^4}{p^{n(n-9)/2}[n]_{p,q}^3}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q}[k]_{p,q}p^{(k^2-3k-8)/2}\left(\frac{x}{b_n}\right)^{k+1}\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-2} \\
&\quad + \frac{3q^2b_n^4}{p^{n(n-9)/2}[n]_{p,q}^3}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q}[k]_{p,q}^2p^{(k^2-5k-8)/2}\left(\frac{x}{b_n}\right)^{k+1}\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-2} \\
&\quad + \frac{q^3b_n^4}{p^{n(n-9)/2}[n]_{p,q}^3}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q}[k]_{p,q}^3p^{(k+1)(k-8)/2}\left(\frac{x}{b_n}\right)^{k+1}\left(1-\frac{x}{b_n}\right)_{p,q}^{n-k-2} \\
&= \frac{b_n^3x}{[n]_{p,q}^3}p^{3n-3} + \frac{q(3p^2+3qp+q^2)[n-1]_{p,q}b_n^2x^2}{[n]_{p,q}^3}p^{2n-4} \\
&\quad + \frac{q^3(3p^2+2pq+q^2)[n-1]_{p,q}[n-2]_{p,q}b_nx^3}{[n]_{p,q}^3}p^{n-3} + \frac{q^6[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}x^4}{[n]_{p,q}^3}.
\end{aligned}$$

□

Form Lemma 2.1, we have the following equalities:

$$C_{n,p,q}\big((t-x);x\big)=0, \quad C_{n,p,q}\big((t-x)^2;x\big)=\frac{p^{n-1}x(b_n-x)}{[n]_{p,q}}. \tag{2.2}$$

Remark 2.2. For $q\in(0,1)$ and $p\in(q,1]$ it is obvious that (i) when $p=1$, $\lim\limits_{n\to\infty}[n]_{p,q}=\lim\limits_{n\to\infty}\frac{1-q^n}{1-q}=\frac{1}{1-q}$ and (ii) when $p<1$, $\lim\limits_{n\to\infty}[n]_{p,q}=\lim\limits_{n\to\infty}\frac{p^n-q^n}{p-q}=0$. This implies that $C_{n,p,q}(e_2;x)$ and $C_{n,p,q}\big((t-x)^2;x\big)$ do not converge to x^2 and 0, respectively, as $n\to\infty$, as in the case of original (p,q)-Bernstein-Chlodowsky operators. This situation arises due to two reasons. The first one belongs to (p,q)-integers and the second one belongs to the sequence (b_n). In order to reach to convergence results of the operators $C_{n,p,q}$ we take sequences $q_n\in(0,1)$ and $p_n\in(q_n,1]$ such that $\lim\limits_{n\to\infty}p_n=1$ $\lim\limits_{n\to\infty}q_n=1$. So we get $\lim\limits_{n\to\infty}[n]_{p_n,q_n}=\infty$ and $\lim\limits_{n\to\infty}\frac{b_n}{[n]_{p_n,q_n}}=0$.

To solve the difficulty about (b_n), one can study point-wise convergence, uniform convergence on any closed finite subinterval of $[0,\infty)$ and also on weighted spaces.

Lemma 2.3. *Let $q:=(q_n)$, $p:=(p_n)$, $0<q_n<p_n\le 1$, be sequences such that $p_n,q_n\longrightarrow 1$ and $p_n^n\longrightarrow a$, $q_n^n\longrightarrow b$ as $n\longrightarrow\infty$. Then, we have the following limits:*

(i) $\lim\limits_{n\to\infty}\frac{[n]_{q_n}}{b_n}C_{n,p_n,q_n}((t-x)^2;x)=ax,$

(ii) $\lim\limits_{n\to\infty}\frac{[n]_{q_n}^2}{b_n^2}C_{n,p_n,q_n}((t-x)^4;x)=3ax^2.$

Proof . (i) From (2.2), we have

$$C_{n,p_n,q_n}((t-x)^2;x)=\frac{-p_n^{n-1}x^2}{[n]_{q_n}}+\frac{xp_n^{n-1}b_n}{[n]_{q_n}}.$$

Then, we get

$$\frac{[n]_{p_n,q_n}}{b_n}C_{n,p_n,q_n}((t-x)^2;x)=\frac{-p_n^{n-1}x^2}{b_n}+xp_n^{n-1}.$$

Let us take the limit of both sides of the above equality as $n\to\infty$, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{[n]_{q_n}}{b_n}\left\{C_{n,p_n,q_n}((t-x)^2,x)\right\}&=\lim_{n\to\infty}\left\{\frac{-p_n^{n-1}x^2}{b_n}+xp_n^{n-1}\right\}\\&=ax.\end{aligned}$$

(ii) Again from Lemma 2.1 and by the linearity of the operators $C_{n,p_n,q_n}(f;x)$, we get

$$C_{n,p_n,q_n}((t-x)^4;x)=A_{1,n}x^4+A_{2,n}x^3+A_{3,n}x^2+A_{4,n}x$$

where

$$\begin{aligned}&A_{1,n}\\&=\frac{p_n^{n-3}[n]_{p_n,q_n}^2(-p_n^2+2p_nq_n-q_n^2)+p_n^{n-5}[n]_{p_n,q_n}(-p_n^3+3p_nq_n^2+q_n^3)-p_n^{3n-6}(p_n^2+p_n^3+2p_nq_n^2+q_n^3)}{[n]_{p_n,q_n}^3},\end{aligned}$$

$$A_{2,n} = \frac{p_n^{n-3}[n]_{p_n,q_n}^2(p_n^2-2p_nq_n+q_n^2)}{[n]_{p_n,q_n}^3}b_n + \frac{p_n^{2n-5}[n]_{p_n,q_n}(-q_n^3-4p_nq_n^2-3p_n^2q_n+2p_n^3)-p_n^{3n-6}(3p_n^3+3p_nq_n^2+5p_n^2q_n+q_n^3)}{[n]_{p_n,q_n}^3}b_n,$$

$$A_{3,n} = \frac{p_n^{2n-4}[n]_{p_n,q_n}(-p_n^2+3p_nq_n+q_n^2)-p_n^{3n-5}(3p_n^2+q_n^2+3p_nq_n)}{[n]_{q_n}^3}b_n^2,$$

$$A_{4,n} = \frac{p^{3n-3}b_n^3}{[n]_{q_n}^3}.$$

Taking the limit of both sides of $A_{1,n}$, we get

$$\begin{aligned}
&\lim_{n\to\infty}\frac{[n]_{q_n}^2}{b_n^2}\{A_{1,n}\}\\
&= \lim_{n\to\infty}\left\{\frac{-p_n^{n-3}[n]_{p_n,q_n}(p_n-q_n)^2}{b_n^2}+\frac{p_n^{n-5}(-p_n^3+3p_nq_n^2+q_n^3)}{b_n^2}-\frac{p_n^{3n-6}(p_n^2+p_n^3+2p_nq_n^2+q_n^3)}{[n]_{q_n}b_n^2}\right\}\\
&= \lim_{n\to\infty}\left\{\frac{-p_n^{n-3}(p_n^n-q_n^n)(p_n-q_n)}{b_n^2}+\frac{p^{n-5}(-p_n^3+3p_nq_n^2+q_n^3)}{b_n^2}-\frac{p_n^{3n-6}(p_n^2+p_n^3+2p_nq_n^2+q_n^3)}{[n]_{q_n}b_n^2}\right\}\\
&= 0. \qquad (2.3)
\end{aligned}$$

Similarly, we can compute

$$\lim_{n\to\infty}\frac{[n]_{q_n}^2}{b_n^2}\{A_{2,n}\}=0, \qquad (2.4)$$

$$\lim_{n\to\infty}\frac{[n]_{q_n}^2}{b_n^2}\{A_{3,n}\}=3ax^2 \qquad (2.5)$$

and

$$\lim_{n\to\infty}\frac{[n]_{q_n}^2}{b_n^2}\{A_{4,n}x\}=0. \qquad (2.6)$$

By combining (2.3)-(2.6), we attain our desired result. □

3. Local approximation properties of $C_{n,p,q}(f;x)$

In this section, we study the Korovkin's approximation property [15], order of convergence under usual modulus of continuity and Peetre's K-functional, and the rate of convergence when the function f belongs to the class $\mathrm{Lip}_M(\alpha)$, etc.

From Lemma 2.1, we can immediately give the following Bohman-Korovkin-type theorem:

Theorem 3.1. Let (p_n), (q_n) be sequences of real numbers such that $0 < q_n < p_n \leq 1$ and $A > 0$. Then for each $f \in C[0,\infty)$, the sequence of operators $C_{n,p_n,q_n}(f;x)$, $0 \leq x \leq b_n$ converges to f uniformly to $f(x)$ on any finite closed subinterval $[0,A]$ provided $\lim_{n\to\infty} p_n = 1$ and $\lim_{n\to\infty} q_n = 1$.

Now we will compute the rate of convergence in terms of modulus of continuity.

Theorem 3.2. *Let (p_n), (q_n) be sequences of real numbers such that $0 < q_n < p_n \leq 1$ and $\lim_{n\to\infty} p_n = 1$ and $\lim_{n\to\infty} q_n = 1$. If $f \in C[0,\infty)$, we have*

$$\left|C_{n,p_n,q_n}(f;x)-f(x)\right| \leq 2\omega\left(f,\sqrt{\frac{p^{n-1}x(b_n-x)}{[n]_{p_n,q_n}}}\right). \qquad (3.1)$$

Proof . Taking into account the sequence of positive linear operators $C_{n,p,q}$ for $p = p_n$ and $q = q_n$, we have

$$\begin{aligned}
&\left|C_{n,p_n,q_n}(f;x) - f(x)\right| \\
&= \left|\frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)_{p,q}^{n-k} f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}} b_n\right) - f(x)\right| \\
&\leq \frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)_{p,q}^{n-k} \left|f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}} b_n\right) - f(x)\right| \\
&\leq \frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)_{p,q}^{n-k} \left(1 + \frac{\left|\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}} b_n - x\right|}{\delta}\right) \omega(f,\delta) \\
&= \frac{\omega(f,\delta)}{p^{n(n-1)/2}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)_{p,q}^{n-k} \\
&+ \frac{\omega(f,\delta)}{\delta} \frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)_{p,q}^{n-k} \left|\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}} b_n - x\right| \\
&= \omega(f,\delta) + \frac{\omega(f,\delta)}{\delta} \left\{\frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)_{p,q}^{n-k} \left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}} b_n - x\right)^2\right\}^{\frac{1}{2}} \\
&= \omega(f,\delta) + \frac{\omega(f,\delta)}{\delta} \left\{\frac{p_n^{n-1} x(b_n - x)}{[n]_{p_n,q_n}}\right\}^{\frac{1}{2}}.
\end{aligned}$$

By choosing $\delta_n = \delta_n(x) = \frac{p_n^{n-1} x(b_n - x)}{[n]_{p_n,q_n}}$, we have

$$\left|C_{n,p_n,q_n}(f;x) - f(x)\right| \leq 2\omega\left(f, \sqrt{\frac{p_n^{n-1} x(b_n - x)}{[n]_{p_n,q_n}}}\right).$$

This completes the proof of the theorem. □

It is easy to see that, the right hand side of of relation (3.1) can diverge. Indeed, for $x = \frac{b_n}{2}$ we have $\delta = \frac{p_n^{n-1} b_n^2}{[n]_{p_n,q_n}}$. We can not guarantee $\delta \to 0$ as $n \to \infty$ in this case.

In view of the previous theorem, we can give the following results on the degree of pointwise convergence and uniform convergence as follows:

Theorem 3.3. *Let* (p_n), (q_n) *be sequences of real numbers such that* $0 < q_n < p_n \leq 1$ *and* $\lim_{n\to\infty} p_n = 1$ *and* $\lim_{n\to\infty} q_n = 1$. *If* $f \in C[0, b_n]$, *then*

$$\left|C_{n,p_n,q_n}(f;x_0) - f(x_0)\right| \leq 2\omega\left(f, \sqrt{\frac{p_n^{n-1} x_0 b_n}{[n]_{p_n,q_n}}}\right),$$

where x_0 *is any fixed point.*

Proof . Note that

$$\frac{p_n^{n-1} x(b_n - x)}{[n]_{p_n,q_n}} \leq \frac{p_n^{n-1} x_0 b_n}{[n]_{p_n,q_n}} \tag{3.2}$$

for any fixed point x_0. In view of the monotonicity properties of the modulus of continuity and (3.2), the remaining part of the proof of this theorem is analogous to the proof of the Theorem 3.2, therefore, we skip the details. □

We can also have a theorem similar to the proof of Theorem 3.2 as follows:

Theorem 3.4. Let (p_n), (q_n) be sequences of real numbers such that $0 < q_n < p_n \leq 1$ and $\lim_{n\to\infty} p_n = 1$ and $\lim_{n\to\infty} q_n = 1$. If $f \in C[0,\infty)$, we have for sufficiently large n

$$\left\|C_{n,p_n,q_n}(f) - f\right\| \leq 2\omega\left(f, \sqrt{\frac{p_n^{n-1}Ab_n}{[n]_{p_n,q_n}}}\right),$$

where $A > 0$ is a constant being appeared in Theorem 3.1.

Definition 3.5. We denote by $C^2[a,b]$, the space of functions f such that f, f', f'' belong to $C[a,b]$. The norm on the space $C^2[a,b]$ can be defined as

$$\|f\|_{C^2[a,b]} = \sum_{j=0}^{2} \|f^{(j)}\|_{C[a,b]}. \tag{3.3}$$

Definition 3.6. For $f \in C[a,b]$ and $t > 0$, the Peetre's K-functional is defined as

$$K(f,\delta) := \inf_{g\in C^2[a,b]} \left\{\|f-g\|_{C[a,b]} + t\|g\|_{C^2[a,b]}\right\}.$$

Theorem 3.7. *If $g \in C^2[0,b_n]$, then*

$$\left|C_{n,p_n,q_n}(g;x) - g(x)\right| \leq \frac{p^{n-1}x(b_n - x)}{2[n]_{p,q}}\|g\|_{C^2[0,b_n]},$$

where $0 < q < p \leq 1$.

Proof . Using Taylor's formula with integral remainder term, we can write

$$g(t) = g(x) + (t-x)\frac{dg}{dx} + \int_0^{t-x}(t-x-u)\frac{d^2g}{dx^2}du. \tag{3.4}$$

Applying the operators $C_{n,p,q}$ to (3.4), we have

$$\begin{aligned}
&|C_{n,p,q}(g;x) - g(x)| \\
&= \left|C_{n,p,q}\left((t-x)\frac{dg}{dx} + \int_0^{t-x}(t-x-u)\frac{d^2g}{dx^2}du; x\right)\right| \\
&\leq \left\|\frac{dg}{dx}\right\|_{C[0,b_n]} |C_{n,p,q}((t-x);x)| + \left\|\frac{d^2g}{dx^2}\right\|_{C[0,b_n]} \left|C_{n,p,q}\left(\int_0^{t-x}(t-x-u)\frac{d^2g}{dx^2}du; x\right)\right|.
\end{aligned}$$

Since $\int_0^{t-x}(t-x-u)du = \frac{1}{2}(t-x)^2$, we obtain from (2.2)

$$|C_{n,p,q}(g;x) - g(x)| \leq \frac{p^{n-1}x(b_n - x)}{2[n]_{p,q}}\left\|\frac{d^2g}{dx^2}\right\|_{C[0,b_n]}.$$

By the relation (3.3), finally we have

$$\left|C_{n,p,q}(g;x)-g(x)\right| \leq \frac{p^{n-1}x(b_n-x)}{2[n]_{p,q}}\|g\|_{C^2[0,b_n]}.$$

This ends the proof of the theorem. □

Now, we can prove the following theorem:

Theorem 3.8. *Let (p_n), (q_n) be sequences of real numbers such that $0<q_n<p_n\leq 1$ and $\lim_{n\to\infty} p_n=1$ and $\lim_{n\to\infty} q_n=1$. If $f\in C[0,\infty)$ and $A>0$ a constant, then*

$$\|C_{n,p_n,q_n}(f;x)-f(x)\|_{C[0,b_n]} \leq 2K\left(f,\frac{p_n^{n-1}Ab_n}{2[n]_{p_n,q_n}}\right).$$

Proof . By the linearity property of C_{n,p_n,q_n}, we have

$$\begin{aligned}
&\left|C_{n,p_n,q_n}(f;x)-f(x)\right| \\
&\leq |C_{n,p_n,q_n}(f;x)-C_{n,p_n,q_n}(g;x)|+|C_{n,p_n,q_n}(g;x)-g(x)|+|g(x)-f(x)| \\
&\leq \|f-g\|_{C[0,b_n]}\left|C_{n,p_n,q_n}(1;x)\right|+\|f-g\|_{C[0,b_n]}+\left|C_{n,p_n,q_n}(g;x)-g(x)\right|.
\end{aligned}$$

From Theorem 3.7, we have

$$\left|C_{n,p_n,q_n}(f;x)-f(x)\right| \leq 2\|f-g\|_{C[0,b_n]}+\frac{p_n^{n-1}x(b_n-x)}{2[n]_{p_n,q_n}}\|g\|_{C^2[0,b_n]},$$

and hence

$$\|C_{n,p_n,q_n}(f)-f\|_{C[0,b_n]} \leq 2\|f-g\|_{C[0,b_n]}+\frac{p_n^{n-1}Ab_n}{2[n]_{p_n,q_n}}\|g\|_{C^2[0,b_n]}. \tag{3.5}$$

Taking the infimum on the right hand side of (3.5) over all $g\in C^2[0,b_n]$, then

$$\|C_{n,p_n,q_n}(f;x)-f(x)\|_{C[0,b_n]} \leq 2K\left(f,\frac{p_n^{n-1}Ab_n}{2[n]_{p_n,q_n}}\right).$$

This completes the proof. □

Now we give the rate of convergence of the operators $C_{n,p,q}$ in terms of the elements of the usual Lipschitz class $\mathrm{Lip}_M(\alpha)$.

Theorem 3.9. *Let (p_n), (q_n) be sequences of real numbers such that $0<q_n<p_n\leq 1$ and $\lim_{n\to\infty} p_n=1$ and $\lim_{n\to\infty} q_n=1$. If $f\in Lip_M[0,b_n]$ and $x\in[0,A]$, $A>0$ a constant, then*

$$\|C_{n,p_n,q_n}(f;x)-f(x)\|_{C[0,b_n]} \leq M\left\{\frac{p_n^{n-1}Ab_n}{[n]_{p_n,q_n}}\right\}^{\frac{\alpha}{2}}.$$

Proof .

$$\begin{aligned}
&\left|C_{n,p_n,q_n}(f;x)-f(x)\right| \\
&\leq \frac{1}{p_n^{n(n-1)/2}}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{p_n,q_n} p_n^{k(k-1)/2}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{p_n,q_n}^{n-k}\left|f\left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n\right)-f(x)\right| \\
&\leq \frac{M}{p_n^{n(n-1)/2}}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{p_n,q_n} p_n^{k(k-1)/2}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{p_n,q_n}^{n-k}\left|\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right|^{\alpha}.
\end{aligned}$$

Now applying the Hölder's inequality for the sum $p_1 = \frac{2}{\alpha}$ and $p_1 = \frac{2}{2-\alpha}$, we have

$$\begin{aligned}
&\left|C_{n,p_n,q_n}(f;x) - f(x)\right| \\
&\leq \frac{M}{p_n^{n(n-1)/2}} \sum_{k=0}^{n} \left\{ \left[\begin{array}{c} n \\ k \end{array} \right]_{p_n,q_n} p_n^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)_{p_n,q_n}^{n-k} \left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}\,[n]_{p_n,q_n}} b_n - x \right)^2 \right\}^{\frac{\alpha}{2}} \\
&\times \left\{ \left[\begin{array}{c} n \\ k \end{array} \right]_{p_n,q_n} p_n^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)_{p_n,q_n}^{n-k} \right\}^{\frac{2-\alpha}{2}}.
\end{aligned}$$

Form (2.2), we have

$$\left|C_{n,p_n,q_n}(f;x) - f(x)\right| \leq M \left\{ \frac{p_n^{n-1} x(b_n - x)}{[n]_{p_n,q_n}} \right\}^{\frac{\alpha}{2}}.$$

This implies that for $x \in [0, A]$,

$$\left\|C_{n,p_n,q_n}(f) - f\right\|_{C[0,b_n]} \leq M \left\{ \frac{p_n^{n-1} A b_n}{[n]_{p_n,q_n}} \right\}^{\frac{\alpha}{2}},$$

which tends to zero as $n \to \infty$. This completes the proof of the theorem. □

Theorem 3.10. *Let (p_n), (q_n) be sequences of real numbers such that $0 < q_n < p_n \leq 1$ and $\lim_{n\to\infty} p_n = 1$ and $\lim_{n\to\infty} q_n = 1$. If $f(x)$ has a continuous derivative $f'(x)$ and $\omega_1(\delta)$ is the modulus of continuity of $f'(x)$ in $[0, A]$. Then*

$$|C_{n,p_n,q_n}(f;x) - f(x)| \leq N \sqrt{\frac{p_n^{n-1} b_n}{[n]_{p_n,q_n}}}\ \omega_1 \left(\sqrt{\frac{p_n^{n-1} b_n}{[n]_{p_n,q_n}}} \right),$$

where N is the constant independent of n.

Proof . Using the mean value theorem, we can write

$$\begin{aligned}
f\left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}\,[n]_{p_n,q_n}} b_n \right) - f(x) &= \left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}\,[n]_{p_n,q_n}} b_n - x \right) f'(\xi) \\
&= \left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}\,[n]_{p_n,q_n}} b_n - x \right) f'(x) + \left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}\,[n]_{p_n,q_n}} b_n - x \right) (f'(\xi) - f'(x)),
\end{aligned}$$

where ξ is some point between x and $\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}} b_n$. From this equality we have

$$\begin{aligned}
&C_{n,p_n,q_n}(f;x) - f(x) \\
&= \frac{1}{p_n^{n(n-1)/2}} \left\{ f'(x) \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p_n,q_n} p_n^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)_{p_n,q_n}^{n-k} \left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}\,[n]_{p_n,q_n}} b_n - x \right) \right. \\
&\left. + \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p_n,q_n} p_n^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)_{p_n,q_n}^{n-k} \left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}\,[n]_{p_n,q_n}} b_n - x \right) (f'(\xi) - f'(x)) \right\}.
\end{aligned}$$

Since

$$|\xi - x| \leq \left| \frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}} b_n - x \right|,$$

we obtain the following inequality:

$$\begin{aligned}
&C_{n,p_n,q_n}(f;x) - f(x) \\
&= \frac{\omega_1(\delta)}{p_n^{n(n-1)/2}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p_n,q_n} p_n^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)_{p_n,q_n}^{n-k} \left| \frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}} b_n - x \right| \\
&+ \frac{\omega_1(\delta)}{\delta . p_n^{n(n-1)/2}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p_n,q_n} p_n^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)_{p_n,q_n}^{n-k} \left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}} b_n - x \right)^2.
\end{aligned}$$

Applying the Cauchy-Schwartz inequality for the first term in the above expression, we get

$$\begin{aligned}
&\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p_n,q_n} p_n^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)_{p_n,q_n}^{n-k} \left| \frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}} b_n - x \right| \\
&\leq \left(\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p_n,q_n} p_n^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)_{p_n,q_n}^{n-k} \left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}} b_n - x \right)^2 \right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{p_n^{n-1} x(b_n - x)}}{\sqrt{[n]_{p_n,q_n}}} \leq \frac{\sqrt{p_n^{n-1} A b_n}}{\sqrt{[n]_{p_n,q_n}}}.
\end{aligned}$$

Since $x \in [0, A]$, we have

$$\frac{p_n^{n-1} x(b_n - x)}{[n]_{p_n,q_n}} \leq \frac{p_n^{n-1} b_n A}{[n]_{p_n,q_n}},$$

and hence we have for the second term

$$\begin{aligned}
&\frac{\omega_1(\delta)}{\delta . p_n^{n(n-1)/2}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p_n,q_n} p_n^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)_{p_n,q_n}^{n-k} \left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}} b_n - x \right)^2 \\
&\leq \frac{\omega_1(\delta)}{\delta} \frac{p_n^{n-1} b_n A}{[n]_{p_n,q_n}}.
\end{aligned}$$

Consequently

$$|C_{n,p_n,q_n}(f;x) - f(x)| \leq \omega_1(\delta) \left\{ \frac{\sqrt{p_n^{n-1} A b_n}}{\sqrt{[n]_{p_n,q_n}}} + \frac{1}{\delta} \frac{p_n^{n-1} A b_n}{[n]_{p_n,q_n}} \right\}.$$

Choosing $\delta = \delta_n = \sqrt{\frac{p_n^{n-1} b_n}{[n]_{p_n,q_n}}}$, we get the following inequality:

$$|C_{n,p_n,q_n}(f;x) - f(x)| \leq \omega_1 \left(\sqrt{\frac{p_n^{n-1} b_n}{[n]_{p_n,q_n}}} \right) \left\{ \sqrt{A} \sqrt{\frac{p_n^{n-1} b_n}{[n]_{p_n,q_n}}} + A \sqrt{\frac{p_n^{n-1} b_n}{[n]_{p_n,q_n}}} \right\}.$$

Following this, we get our desired result. □

4. Approximation properties in weighted spaces

Now we give approximation properties of the operators $C_{n,p,q}$ of the weighted spaces of continuous functions with exponential growth on $\mathbb{R}_0^+ = [0, \infty)$ with the help of the weighted Korovkin type theorem proved by Gadjiev in [7, 8]. For this purpose, we consider the following weighted spaces of functions which are defined on the $\mathbb{R}_0^+ = [0, \infty)$.

Let $\rho(x)$ be the weighted function and M_f be a positive constant. Then we define

$$\begin{aligned} B_\rho(\mathbb{R}_0^+) &= \{f \in E(\mathbb{R}_0^+) : |f(x)| \le M_f \rho(x)\}, \\ C_\rho(\mathbb{R}_0^+) &= \{f \in B_\rho(\mathbb{R}_0^+) : f \text{ is continuous}\}, \\ C_\rho^k(\mathbb{R}_0^+) &= \left\{ f \in C_\rho(\mathbb{R}_0^+) : \lim_{n\to\infty} \frac{f(x)}{\rho(x)} = K_f < \infty \right\}. \end{aligned}$$

It is obvious that $C_\rho^k(\mathbb{R}_0^+) \subset C_\rho(\mathbb{R}_0^+) \subset B_\rho(\mathbb{R}_0^+)$. The space $B_\rho(\mathbb{R}_0^+)$ is a normed linear space with the following norm:

$$\|f\|_\rho = \sup_{x\in\mathbb{R}_0^+} \frac{|f(x)|}{\rho(x)}.$$

The following results on the sequence of positive linear operators in these spaces are given in [7, 8].

Lemma 4.1. ([7, 8]) The sequence of positive linear operators $(L_n)_{n\ge1}$ which act from $C_\rho(\mathbb{R}_0^+)$ to $B_\rho(\mathbb{R}_0^+)$ if and only if there exists a positive constant k such that

$$L_n(\rho; x) \le k\rho(x), \quad \text{i.e.}$$

$$\|L_n(\rho; x)\|_\rho \le k.$$

Theorem 4.2. ([7, 8]) Let $(L_n)_{n\ge1}$ be the sequence of positive linear operators which act from $C_\rho(\mathbb{R}_0^+)$ to $B_\rho(\mathbb{R}_0^+)$ such that

$$\lim_{n\to\infty} \|L_n(t^i; x) - x^i\|_\rho = 0, \quad i \in \{0, 1, 2\}.$$

Then for any function $f \in C_\rho^k(\mathbb{R}_0^+)$,

$$\lim_{n\to\infty} \|L_n f - f\|_\rho = 0.$$

Lemma 4.3. *Let (p_n) and (q_n) be the sequences such that $0 < q_n < p_n \le 1$ and $\rho(x) = 1 + x^2$ a weight function. If $f \in C_\rho(\mathbb{R}_0^+)$, then*

$$\|C_{n,p_n,q_n}(\rho; x)\|_\rho \le 1 + M$$

provided $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$.

Proof . Using Lemma 2.1 (i) and (iii), one has

$$\begin{aligned} C_{n,p_n,q_n}(\rho; x) &= 1 + \frac{p_n^{n-1} b_n}{[n]_{p_n,q_n}} x + \frac{q_n [n-1]_{p_n,q_n}}{[n]_{p_n,q_n}} x^2 \|C_{n,p_n,q_n}(\rho; x)\|_\rho \\ &= \sup_{x\ge0} \left\{ \frac{1}{1+x^2} \left(1 + \frac{p_n^{n-1} b_n}{[n]_{p_n,q_n}} x + \frac{q_n [n-1]_{p_n,q_n}}{[n]_{p_n,q_n}} x^2 \right) \right\} \\ &\le 1 + \frac{p_n^{n-1} b_n}{[n]_{p_n,q_n}} + \frac{q_n [n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}. \end{aligned}$$

Since $\lim_{n\to\infty} \frac{b_n}{[n]_{p_n,q_n}} = 0$, there exists a positive M such that

$$\|C_{n,p_n,q_n}(\rho; x)\|_\rho \leq 1 + M.$$

This completes the proof. □

By using Lemma 4.3, we can easily see that the operators C_{n,p_n,q_n} act from $C_\rho(\mathbb{R}_0^+)$ to $B_\rho(\mathbb{R}_0^+)$.

Theorem 4.4. *Let (p_n), (q_n) be the sequences such that $0 < q_n < p_n \leq 1$ and $p_n \to 1$, $q_n \to 1$ as $n \to \infty$. Let $\rho(x) = 1 + x^2$, then for each $f \in C_\rho^k(\mathbb{R}_0^+)$*

$$\lim_{n\to\infty} \|C_{n,p_n,q_n}(f; x) - f(x)\|_\rho = 0.$$

Proof . It is enough to prove that the conditions of the weighted Korovkin type theorem given by Theorem 4.2 are satisfied. From Lemma 2.1 (i)-(ii), it is immediate that

$$\lim_{n\to\infty} \|C_{n,p_n,q_n}(e_0; x) - e_0(x)\|_\rho = 0, \tag{4.1}$$

$$\lim_{n\to\infty} \|C_{n,p_n,q_n}(e_1; x) - e_1(x)\|_\rho = 0. \tag{4.2}$$

By means of Lemma 2.1 (iii), we get

$$\begin{aligned} &\|C_{n,p_n,q_n}(e_2; x) - e_2(x)\| \\ &= \sup_{x\in\mathbb{R}_0^+} \left| \frac{p_n^{n-1} b_n}{[n]_{p_n,q_n}} \frac{x}{1+x^2} + \left(\frac{q_n[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}} - 1 \right) \frac{x^2}{1+x^2} \right| \leq \frac{p_n^{n-1} b_n}{[n]_{p_n,q_n}} - \frac{p_n^{n-1}}{[n]_{p_n,q_n}}. \end{aligned}$$

Using the conditions $\lim_{n\to\infty} \frac{b_n}{[n]_{p_n,q_n}} = 0$, it follows that

$$\lim_{n\to\infty} \|C_{n,p_n,q_n}(e_2; x) - e_2(x)\|_\rho = 0. \tag{4.3}$$

From (4.1), (4.2) and (4.3), for $i \in \{0, 1, 2\}$, we have

$$\lim_{n\to\infty} \|C_{n,p_n,q_n}(t^i; x) - x^i\|_\rho = 0.$$

Applying Theorem 4.2, we obtain the desired result. □

Definition 4.5. ([2, 10]) For $f \in C_\rho^k[0, \infty)$, $\delta > 0$, we define the weighted modulus of continuity $\Omega(f; \delta)$ as follows:

$$\Omega(f, \delta) = \sup_{t\in[0,\infty), |h|\leq\delta} \frac{|f(t+h) - f(t)|}{\rho(t)\rho(h)}.$$

$\Omega(f, \delta)$ has the following properties:

(i) monotonically increasing function of δ.

(ii) $\lim_{\delta\to 0} \Omega(f, \delta) = 0$.

(iii) For any $\lambda > 0$, $\Omega(f, \lambda\delta) \leq 2(1 + \lambda)(1 + \delta^2)\Omega(f, \delta)$.

By property (iii), we have

$$|f(t)-f(x)| \leq 2\big(1+\frac{|t-x|}{\delta}\big)(1+\delta^2)\rho(x)(1+(t-x)^2)\Omega(f,\delta). \tag{4.4}$$

Theorem 4.6. *If $f \in C_\rho^k$, then the inequality*

$$\sup_{x\geq 0}\frac{|C_{n,p_n,q_n}(f;x)-f(x)|}{\rho^3(x)} \leq K\Omega\left(f,\sqrt{\frac{p_n^{n-1}b_n}{[n]_{p_n,q_n}}}\right)$$

holds, where K is a constant independent of b_n, and (p_n), (q_n) be the sequences such that $0<q_n<p_n\leq 1$ and $p_n,\ q_n \to 1$ as $n\to\infty$.

Proof . From (4.4), we have

$$|f(t)-f(x)| \leq 2\big(1+\frac{|t-x|}{\delta}\big)(1+\delta^2)\rho(x)(1+(t-x)^2)\Omega(f,\delta).$$

Letting $t=\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n$, we obtain

$$\begin{aligned}&\left|f\left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n\right)-f(x)\right|\\ &\leq 2\left(1+\frac{\left|\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right|}{\delta_n}\right)(1+\delta_n^2)\rho(x)\left(1+\left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right)^2\right)\Omega(f,\delta_n).\end{aligned}$$

Hence,

$$\begin{aligned}&|C_{n,p_n,q_n}(f,x)-f(x)|\\ &\leq \sum_{k=0}^{n}\left[2\left(1+\frac{\left|\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right|}{\delta_n}\right)(1+\delta_n^2)\rho(x)\left(1+\left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right)^2\right)\Omega(f,\delta_n)\right]P_{n,k}(x)\\ &\leq 4\rho(x)\Omega(f,\delta_n)\sum_{k=0}^{n}\left(1+\frac{\left|\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right|}{\delta_n}\right)\left(1+\left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right)^2\right)P_{n,k}(x)\\ &\leq 4\rho(x)\Omega(f,\delta_n)\Bigg[1+\frac{1}{\delta_n}\sum_{k=0}^{n}\left|\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right|P_{n,k}(x)+\sum_{k=0}^{n}\left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right)^2P_{n,k}(x)\\ &+\frac{1}{\delta_n}\sum_{k=0}^{n}\left|\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right|\left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right)^2P_{n,k}(x)\Bigg],\end{aligned}$$

where

$$P_{n,k}(x)=\frac{1}{p_n^{n(n-1)/2}}\left[\begin{array}{c}n\\k\end{array}\right]_{p_n,q_n}p_n^{k(k-1)/2}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{p_n,q_n}^{n-k}.$$

Applying Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} &|C_{n,p_n,q_n}(f,x)-f(x)| \\ &\le 4\rho(x)\Omega(f,\delta_n)\Bigg[1+\frac{1}{\delta_n}\sqrt{\sum_{k=0}^{n}\left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right)^2 P_{n,k}(x)} \\ &\quad+\sum_{k=0}^{n}\left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right)^2 P_{n,k}(x) \\ &+\frac{1}{\delta_n}\sqrt{\sum_{k=0}^{n}\left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right)^4 P_{n,k}(x)\sum_{k=0}^{n}\left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right)^2 P_{n,k}(x)}\;\Bigg]. \end{aligned} \tag{4.5}$$

By simple calculation, we get

$$\sum_{k=0}^{n}\left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right)^2 P_{n,k}(x)=\frac{p_n^{n-1}b_n}{[n]_{p_n,q_n}}x-\frac{p_n^{n-1}}{[n]_{p_n,q_n}}x^2\le\frac{p_n^{n-1}b_n}{[n]_{p_n,q_n}}x, \tag{4.6}$$

$$\begin{aligned} &\sum_{k=0}^{n}\left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right)^4 P_{n,k}(x) \\ &=\left\{\frac{q_n^6[n-1]_{p_n,q_n}[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}}{[n]^3_{p_n,q_n}}-\frac{4q_n^3[n-1]_{p_n,q_n}[n-2]_{p_n,q_n}}{[n]^2_{p_n,q_n}}+\frac{6q_n[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}}-3\right\}x^4 \\ &\quad+\left\{\frac{p_n^{n-3}q_n^3(3p_n^2+2p_nq_n+q_n^2)[n-1]_{p_n,q_n}[n-2]_{p_n,q_n}b_n}{[n]^3_{p_n,q_n}}\right. \\ &\quad\left.-\frac{4p_n^{n-1}q_n(2p_n+q_n)[n-1]_{p_n,q_n}b_n}{[n]^2_{p_n,q_n}}+\frac{6p_n^{n-1}b_n}{[n]_{p_n,q_n}}\right\}x^3 \\ &\quad+\left\{\frac{p_n^{2n-4}q_n(3p_n^2+3p_nq_n+q_n^3)[n-1]_{p_n,q_n}b_n^2}{[n]^3_{p_n,q_n}}-\frac{4p_n^{2n-2}b_n^2}{[n]^2_{p_n,q_n}}\right\}x^2+\frac{p_n^{3n-3}b_n^3}{[n]^3_{p_n,q_n}}x. \end{aligned}$$

Using the relation $[n]_{p,q}=q^k[n-k]_{p,q}+\sum_{j=0}^{k-1}p^{n-j-1}q^j$, the above equality can be written as

$$\begin{aligned} &\sum_{k=0}^{n}\left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right)^4 P_{n,k}(x) \\ &=\frac{p_n^{n-1}}{[n]_{p_n,q_n}}\left\{\left(-1-\frac{4q_n}{p_n}-\frac{q_n^2}{p_n^2}\right)+\frac{p_n^{n-1}}{[n]_{p_n,q_n}}\left(2-\frac{2q_n}{p_n}+\frac{q_n^2}{p_n^2}-\frac{q_n^3}{p_n^3}\right)+\frac{p_n^{2n-3}q_n}{[n]^2_{p_n,q_n}}\left(1+\frac{q_n}{p_n}+\frac{q_n^2}{p_n^2}\right)\right\}x^4 \\ &\quad+\frac{p_n^{n-1}b_n}{[n]_{p_n,q_n}}\left\{\left(3+\frac{2q_n}{p_n}+\frac{q_n^2}{p_n^2}\right)\frac{([n]_{p_n,q_n}-p_n^{n-1})([n]_{p_n,q_n}-p_n^{n-1}-p_n^{n-2}q_n)}{[n]^2_{p_n,q_n}}\right. \\ &\quad\left.-\frac{4(2p_n+q_n)([n]_{p_n,q_n}-p_n^{n-1})}{[n]_{p_n,q_n}}\right\}x^3 \\ &\quad+\frac{p_n^{n-1}b_n^2}{[n]_{p_n,q_n}}\left\{\frac{p_n^{n-3}(3p_n^2+3p_nq_n+q_n^3)([n]_{p_n,q_n}-p_n^{n-1})}{[n]^2_{p_n,q_n}}-\frac{4p_n^{n-1}}{[n]_{p_n,q_n}}\right\}x^2+\frac{p_n^{3n-3}b_n^3}{[n]^3_{p_n,q_n}}x \end{aligned}$$

and so

$$\begin{aligned}
&\sum_{k=0}^{n}\left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right)^4 P_{n,k}(x)\\
&\leq \frac{p_n^{n-1}}{[n]_{p_n,q_n}}\left\{\frac{3p_n^{n-1}}{[n]_{p_n,q_n}}+\frac{3p_n^{2n-3}q_n}{[n]_{p_n,q_n}^2}\right\}x^4+\frac{p_n^{n-1}b_n}{[n]_{p_n,q_n}}\left\{6\left(1+\frac{p_n^{2n-2}}{[n]_{p_n,q_n}^2}+\frac{p_n^{2n-3}q_n}{[n]_{p_n,q_n}^2}\right)+\frac{12p_n^{n-1}}{[n]_{p_n,q_n}}\right\}x^3\\
&\quad+\frac{p_n^{n-1}b_n^2}{[n]_{p_n,q_n}}\left\{\frac{p_n^{n-3}(3p_n^2+3p_nq_n+q_n^3)}{[n]_{p_n,q_n}}\right\}x^2+\frac{p_n^{3n-3}b_n^3}{[n]_{p_n,q_n}^3}x\\
&\leq \frac{6p_n^{n-1}}{[n]_{p_n,q_n}}x^4+\frac{30p_n^{n-1}b_n}{[n]_{p_n,q_n}}x^3+\frac{7p_n^{2n-4}b_n^2}{[n]_{p_n,q_n}^2}x^2+\frac{p_n^{3n-3}b_n^3}{[n]_{p_n,q_n}^3}x\\
&\leq \frac{p_n^{n-1}b_n}{[n]_{p_n,q_n}}\left\{6x^4+30x^3+\frac{7p_n^{n-3}b_n}{[n]_{p_n,q_n}}x^2+\frac{p_n^{2n-2}b_n^2}{[n]_{p_n,q_n}^2}x\right\}.
\end{aligned}$$

Thus, if we consider $\frac{b_n}{[n]_{p_n,q_n}}\leq 1$ for sufficiently large n, since $\lim\limits_{n\to\infty}\frac{b_n}{[n]_{p_n,q_n}}=0$, we get

$$\begin{aligned}
\sum_{k=0}^{n}\left(\frac{[k]_{p_n,q_n}}{p_n^{k-n}[n]_{p_n,q_n}}b_n-x\right)^4 P_{n,k}(x) &\leq \frac{p_n^{n-1}b_n}{[n]_{p_n,q_n}}(6x^4+30x^3+x^2+x)\\
&\leq \frac{30p_n^{n-1}b_n}{[n]_{p_n,q_n}}(x^4+x^3+x^2+x).
\end{aligned}$$

Substituting the inequalities (4.6)-(4.7) in (4.5), we have

$$\begin{aligned}
&|C_{n,p_n,q_n}(f,x)-f(x)|\\
&\leq 4\rho(x)\Omega(f,\delta_n)\left[1+\frac{1}{\delta_n}\sqrt{\frac{p_n^{n-1}b_n}{[n]_{p_n,q_n}}x}+\frac{p_n^{n-1}b_n}{[n]_{p_n,q_n}}x+\sqrt{30}\frac{1}{\delta_n}\frac{p_n^{n-1}b_n}{[n]_{p_n,q_n}}\sqrt{x^5+x^4+x^3+x^2}\right].
\end{aligned}$$

Choosing $\delta_n=\frac{p_n^{n-1}b_n}{[n]_{p_n,q_n}}$, for sufficiently large n, we have

$$\sup_{x\geq 0}\frac{|C_{n,p_n,q_n}(f;x)-f(x)|}{\rho^3(x)}\leq K\Omega\left(f,\sqrt{\frac{p_n^{n-1}b_n}{[n]_{p_n,q_n}}}\right)$$

holds, where K is a constant independent of b_n. □

5. Weighted statistical approximation properties

In this section, we give Korovkin type weighted statistical approximation properties of the our operator. At this moment, we give some basic notations and some known results related to the statistical convergence which will be used in this section.

The density of a subset K of $\mathbb{N}$ is given

$$\delta(K)=\lim_{n}\frac{1}{n}\sum_{k=1}^{n}\chi_K(k),$$

whenever the limit exists, where χ_K is the characteristic function of K. A sequence $x=(x_k)$ is called statistically convergent to the number $\ell\in\mathbb{R}$, if for any $\epsilon>0$, $\delta\{k\in\mathbb{N}:|x_k-\ell|\geq\epsilon\}=0$, for each $\epsilon>0$ and is denoted by $st-\lim x=\ell$, (see [6, 27]).

Theorem 5.1. *Let $q := (q_n), p := (p_n)$, $0 < q_n < p_n \leq 1$ be sequences such that satisfying following condition:*

$$st - \lim_n \frac{b_n}{[n]_{p_n,q_n}} = 0. \tag{5.1}$$

Then we have the following:

$$st - \lim_n \| C_{n,p_n,q_n}(f;.) - f \|_\rho = 0$$

for each $f \in C_\rho^k[0,\infty)$.

Proof . It is sufficient to prove that

$$st - \lim_n \| C_{n,p_n,q_n}(e_v;.) - e_v \|_\rho = 0$$

where $e_v(t) = t^v$, $v = 0, 1, 2$.

From Lemma 2.1 (i)-(ii), it is easy to obtain

$$st - \lim_n \| C_{n,p_n,q_n}(e_0;x) - e_0 \|_\rho = 0,$$

and

$$st - \lim_n \| C_{n,p_n,q_n}(e_1;x) - e_1 \|_\rho = 0.$$

By Lemma 2.1 (iii), one can see that

$$\| C_{n,p_n,q_n}(e_2;x) - e_2 \|_\rho \leq \frac{p_n^{n-1}}{[n]_{p_n,q_n}} \| e_2 \|_\rho + \frac{p^{n-1}b_n^2}{[n]_{p_n,q_n}} \| e_1 \|_\rho .$$

Let us define the following sets for $\epsilon > 0$

$$\begin{aligned} \mathcal{C} &:= \{k : \|C_{n,p_n,q_n}(e_2;x) - e_2\|_\rho \geq \epsilon\}, \\ \mathcal{C}_1 &:= \left\{k : \frac{p_k^{k-1}}{[k]_{p_k,q_k}} \geq \frac{\epsilon}{2}\right\}, \\ \mathcal{C}_2 &:= \left\{k : \frac{p_k^{k-1}b_k^2}{[k]_{p_k,q_k}} \geq \frac{\epsilon}{2}\right\}, \end{aligned}$$

such that $\mathcal{C} \subseteq \mathcal{C}_1 \cup \mathcal{C}_2$.

Hence, we get

$$\begin{aligned} &\delta \{k \leq n : \| C_{n,p_n,q_n}(e_2;x) - e_2 \|_\rho \geq \epsilon\} \\ &\leq \delta \left\{k \leq n : \frac{p_k^{k-1}}{[k]_{p_k,q_k}} \geq \frac{\epsilon}{2}\right\} + \delta \left\{k \leq n : \frac{p_k^{k-1}b_k^2}{[k]_{p_k,q_k}} \geq \frac{\epsilon}{2}\right\}. \end{aligned}$$

By (5.1) and (5.2), we have

$$st - \lim_n \| C_{n,p_n,q_n}(e_2;x) - e_2 \|_\rho = 0.$$

□

Now, we give a Voronovskaja type theorem for $C_{n,p_n,q_n}(f;x)$.

Theorem 5.2. *Let $f \in C_\rho^k[0,\infty)$ such that $f', f'' \in C_\rho^k[0,\infty)$. Then, we have*

$$\lim_{n\to\infty} \frac{[n]_{p_n,q_n}}{b_n} \left(C_{n,p_n,q_n}(f;x) - f(x)\right) = axf''(x).$$

Proof . We write Taylor's expansion of f as follows:

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \varepsilon(t,x)(t-x)^2, \tag{5.2}$$

where $\varepsilon(t,x) \to 0$ as $t \to x$.

Applying the operators $C_{n,p_n,q_n}(f;.)$ on (5.2), we get

$$\begin{aligned} &C_{n,p_n,q_n}(f,x) - f(x) \\ &= f'(x)C_{n,p_n,q_n}((t-x);x) + \frac{1}{2}f''(x)C_{n,p_n,q_n}((t-x)^2;x) + C_{n,p_n,q_n}\left(\varepsilon(t,x)(t-x)^2,x\right). \end{aligned}$$

Let us take the limit of both sides of the above equality as $n \to \infty$, we have

$$\begin{aligned} &\lim_{n\to\infty} \frac{[n]_{q_n}}{b_n} C_{n,p_n,q_n}((f;x) - f(x)) \\ &= \lim_{n\to\infty} \frac{[n]_{q_n}}{b_n} \left\{ \frac{1}{2}f''(x) \left(\frac{-p_n^{n-1}x^2}{[n]_{q_n}} + \frac{xp_n^{n-1}b_n}{[n]_{q_n}} \right) + C_{n,p_n,q_n}\left(\varepsilon(t,x)(t-x)^2;x\right) \right\}. \end{aligned}$$

For the last term on the right hand side, using Cauchy-Schwartz inequality, we get

$$\begin{aligned} &\lim_{n\to\infty} \frac{[n]_{q_n}}{b_n} C_{n,p_n,q_n}\left(\varepsilon(t,x)(t-x)^2;x\right) \\ &\leq \sqrt{\lim_{n\to\infty} C_{n,p_n,q_n}\left(\varepsilon^2(t,x);x\right)} \sqrt{\lim_{n\to\infty} \frac{[n]_{p_n,q_n}^2}{b_n^2} C_{n,p_n,q_n}\left((t-x)^4;x\right)}. \end{aligned}$$

Because of $\lim_{n\to\infty} C_{n,p_n,q_n}\left(\varepsilon^2(t,x);x\right) = 0$ and using Lemma 2.3 (ii),

$$\lim_{n\to\infty} \frac{[n]_{p_n,q_n}^2}{b_n^2} C_{n,p_n,q_n}\left((t-x)^4;x\right)$$

is finite, then we obtain

$$\lim_{n\to\infty} \frac{[n]_{p_n,q_n}}{b_n} C_{n,p_n,q_n}\left(\varepsilon(t,x)(t-x)^2;x\right) = 0.$$

Hence, one can see that

$$\begin{aligned} &\lim_{n\to\infty} \frac{[n]_{p_n,q_n}}{b_n} \left(C_{n,p_n,q_n}(f;x) - f(x)\right) \\ &= \frac{1}{2}f''(x) \lim_{n\to\infty} \left(\frac{-p_n^{n-1}x^2}{b_n} + xp_n^{n-1} \right) = axf''(x). \end{aligned}$$

This step completes the proof. □

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