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Some new Ostrowski type fractional integral inequalities for generalized $(r; g, s, m, \varphi)$-preinvex functions via Caputo k-fractional derivatives

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Abstract

In the present paper, the notion of generalized $(r; g, s, m, \varphi)$-preinvex function is applied to establish some new generalizations of Ostrowski type integral inequalities via Caputo k-fractional derivatives. At the end, some applications to special means are given.

Keywords: Ostrowski type inequality; Hölder's inequality; Minkowski's inequality; s-convex function in the second sense; m-invex.
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1. Introduction and preliminaries

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a n-dimensional vector space. The set of continuous differentiable functions of order n on the interval $[a, b]$ is denoted by $C^n[a, b]$.

The following result is known in the literature as the Ostrowski inequality (see [30]), which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a, b]$.

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Theorem 1.1. Let $f : I \longrightarrow \mathbb{R}$ be a mapping differentiable on I° and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq M(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right], \quad \forall x \in [a, b]. \tag{1.1}$$

For other recent results concerning Ostrowski type inequalities (please see [2]-[4], [12]-[16], [18], [21], [23], [28]-[31], [33]-[35], [38], [40], [42], [43], [45], [46]). Ostrowski inequality is playing a very important role in all the fields of mathematics, especially in the theory of approximations. Thus such inequalities were studied extensively by many researches and numerous generalizations, extensions and variants of them for various kind of functions like bounded variation, synchronous, Lipschitzian, monotonic, absolutely, continuous and n-times differentiable mappings etc. appeared in a number of papers (please see [12]-[16],[19]). In recent years, one more dimension has been added to this studies, by introducing a number of integral inequalities involving various fractional operators like Riemann-Liouville, Erdelyi-Kober, Katugampola, conformable fractional integral operators etc. by many authors (please see [1],[7],[25],[26],[37],[41]). Riemann-Liouville fractional integral operators are the most central between these fractional operators.

In numerical analysis many quadrature rules have been established to approximate the definite integrals. Ostrowski inequality provides the bounds of many numerical quadrature rules. In recent decades Ostrowski and Hermite-Hadamard inequality is studied in fractional calculus point of view by many mathematicians (please see [8]-[11],[17],[19],[20],[22],[24], [27],[32],[39]).

Now, let us evoke some definitions.

Definition 1.2. (see [20]) A function $f : [0, +\infty) \longrightarrow \mathbb{R}$ is said to be s-convex in the second sense, if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.2}$$

for all $x, y \geq 0,\ \lambda \in [0, 1]$ and $s \in (0, 1]$.

It is clear that a 1-convex function must be convex on $[0, +\infty)$ as usual. The s-convex functions in the second sense have been investigated in (see [20]).

Definition 1.3. (see [5]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true (please see [5],[44]).

Definition 1.4. (see [36]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0, 1]$, we have that

$$f\left(x + t\eta(y, x)\right) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

Definition 1.5. For $k \in \mathbb{R}^+$ and $x \in \mathbb{C}$, the k-gamma function is defined by

$$\Gamma_k(x) = \lim_{n \longrightarrow \infty} \frac{n! k^n n k^{\frac{x}{k}-1}}{(x)_{n,k}}. \tag{1.3}$$

Its integral representation is given by

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt. \tag{1.4}$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha).$$

For $k = 1$, (1.4) gives integral representation of gamma function.

Definition 1.6. For $k \in \mathbb{R}^+$ and $x, y \in \mathbb{C}$, the k-beta function with two parameters x and y is defined as

$$\beta_k(x, y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1} (1-t)^{\frac{y}{k}-1} dt. \tag{1.5}$$

For $k = 1$, (1.5) gives integral representation of beta function.

Theorem 1.7. Let $x, y > 0$, then for k-gamma and k-beta function the following equality holds:

$$\beta_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. \tag{1.6}$$

Definition 1.8. (see [27]) Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$, $f \in C^n[a, b]$ such that $f^{(n)}$ exists and are continuous on $[a, b]$. The Caputo fractional derivatives of order α are defined as follows:

$${}^c D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \ x > a \tag{1.7}$$

and

$${}^c D_{b-}^{\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, \ x < b. \tag{1.8}$$

If $\alpha = n \in \{1, 2, 3, \ldots\}$ and usual derivative of order n exists, then Caputo fractional derivative $\left({}^c D_{a+}^{\alpha} f\right)(x)$ coincides with $f^{(n)}(x)$. In particular we have

$$\left({}^c D_{a+}^{0} f\right)(x) = \left({}^c D_{b-}^{0} f\right)(x) = f(x) \tag{1.9}$$

where $n = 1$ and $\alpha = 0$.

In the following we recall Caputo k-fractional derivatives.

Definition 1.9. (see [19]) Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$, $f \in C^n[a, b]$. The Caputo k-fractional derivatives of order α are defined as follows:

$${}^c D_{a+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k\left(n - \frac{\alpha}{k}\right)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}} dt, \ x > a \tag{1.10}$$

and

$${}^c D_{b-}^{\alpha,k} f(x) = \frac{(-1)^n}{k\Gamma_k\left(n - \frac{\alpha}{k}\right)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}} dt, \ x < b. \tag{1.11}$$

The aim of this paper is to establish some generalizations of Ostrowski type inequalities using new integral identity given in Section 2 for generalized $(r; g, s, m, \varphi)$-preinvex functions via Caputo k-fractional derivatives. In Section 3, some applications to special means are establish. In Section 4, some conclusions and future research are given.

2. Main results

Definition 2.1. (see [17]) A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.

We next give new definition, to be referred as generalized $(r; g, s, m, \varphi)$-preinvex function.

Definition 2.3. Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$, $g : [0,1] \longrightarrow [0,1]$ be a differentiable function and $\varphi : I \longrightarrow K$ is a continuous function. The function $f : K \longrightarrow (0, +\infty)$ is said to be generalized $(r; g, s, m, \varphi)$-preinvex with respect to η, if

$$f\left(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)\right) \leq M_r(f(\varphi(x)), f(\varphi(y)), m, s; g(t)) \tag{2.1}$$

holds for any fixed $s, m \in (0,1]$ and for all $x, y \in I, t \in [0,1]$, where

$$M_r(f(\varphi(x)), f(\varphi(y)), m, s; g(t))$$
$$= \begin{cases} \Big[m(1-g(t))^s f^r(\varphi(x)) + g^s(t) f^r(\varphi(y))\Big]^{\frac{1}{r}}, & r \neq 0; \\ \big[f(\varphi(x))\big]^{m(1-g(t))^s} \big[f(\varphi(y))\big]^{g^s(t)}, & r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers $f(\varphi(x))$ and $f(\varphi(y))$.

Remark 2.4. In Definition 2.3, it is worthwhile to note that the class of generalized $(r; g, s, m, \varphi)$-preinvex function is a generalization of the class of s-convex in the second sense function given in Definition 1.2. For $g(t) = t$, we get the notion of generalized $(r; s, m, \varphi)$-preinvex function (see [22]). For $r = 1$ and $g(t) = t$, we get the notion of generalized (s, m, φ)-preinvex function (see [21]). Also, for $r = 1$, $g(t) = t$ and $\varphi(x) = x$, $\forall x \in I$, we get the notion of generalized (s, m)-preinvex function (see [17]).

In this section, in order to present some new Ostrowski type integral inequalities for generalized $(r; g, s, m, \varphi)$-preinvex functions via Caputo k-fractional derivatives, we need the following new interesting lemma to obtain our results.

Lemma 2.5. *Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $m \in (0,1]$. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ a differentiable function. Assume that $f : K \longrightarrow \mathbb{R}$ is* *a function on K° such that $f \in C^{n+1}[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, where $\eta(\varphi(b),\varphi(a),m)>0$. Then we have the following equality for Caputo k-fractional derivatives*

$$\begin{aligned}
&\frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}\\
&\quad\times\left[g^{n-\frac{\alpha}{k}}(1)f^{(n)}(m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m))-g^{n-\frac{\alpha}{k}}(0)f^{(n)}(m\varphi(a)+g(0)\eta(\varphi(x),\varphi(a),m))\right]\\
&\quad-\frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}\\
&\quad\times\left[g^{n-\frac{\alpha}{k}}(1)f^{(n)}(m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m))-g^{n-\frac{\alpha}{k}}(0)f^{(n)}(m\varphi(b)+g(0)\eta(\varphi(x),\varphi(b),m))\right]\\
&\quad-\frac{n-\frac{\alpha}{k}}{\eta(\varphi(b),\varphi(a),m)}\\
&\quad\times\left[\int_{m\varphi(a)+g(0)\eta(\varphi(x),\varphi(a),m)}^{m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m)}(t-m\varphi(a))^{n-\frac{\alpha}{k}-1}f^{(n)}(t)dt\right.\\
&\quad\left.-\int_{m\varphi(b)+g(0)\eta(\varphi(x),\varphi(b),m)}^{m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m)}(t-m\varphi(b))^{n-\frac{\alpha}{k}-1}f^{(n)}(t)dt\right]\\
&=\frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 g^{n-\frac{\alpha}{k}}(t)f^{(n+1)}(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))d[g(t)]\\
&\quad-\frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 g^{n-\frac{\alpha}{k}}(t)f^{(n+1)}(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))d[g(t)].
\end{aligned}$$

Proof . Throughout this paper we denote

$$\begin{aligned}
&I_{f,g,\eta,\varphi}(x;\alpha,k,n,m,a,b)\\
&\quad=\frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 g^{n-\frac{\alpha}{k}}(t)f^{(n+1)}(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))d[g(t)]\\
&\quad-\frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 g^{n-\frac{\alpha}{k}}(t)f^{(n+1)}(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))d[g(t)].
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
I_{f,g,\eta,\varphi}(x;\alpha,k,n,m,a,b)&=\frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}\\
&\times\left[\left.\frac{t^{n-\frac{\alpha}{k}}f^{(n)}(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(x),\varphi(a),m)}\right|_{g(0)}^{g(1)}\right.\\
&\left.-\frac{n-\frac{\alpha}{k}}{\eta(\varphi(x),\varphi(a),m)}\int_{g(0)}^{g(1)}t^{n-\frac{\alpha}{k}-1}f^{(n)}(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))dt\right]\\
&-\frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}\\
&\times\left[\left.\frac{t^{n-\frac{\alpha}{k}}f^{(n)}(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(x),\varphi(b),m)}\right|_{g(0)}^{g(1)}\right.
\end{aligned}$$

$$
\begin{aligned}
&- \frac{n-\frac{\alpha}{k}}{\eta(\varphi(x),\varphi(b),m)} \int_{g(0)}^{g(1)} t^{n-\frac{\alpha}{k}-1} f^{(n)}(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))dt\Bigg] \\
&= \frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad \times \left[g^{n-\frac{\alpha}{k}}(1) f^{(n)}(m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m)) - g^{n-\frac{\alpha}{k}}(0) f^{(n)}(m\varphi(a)+g(0)\eta(\varphi(x),\varphi(a),m))\right] \\
&\quad - \frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad \times \left[g^{n-\frac{\alpha}{k}}(1) f^{(n)}(m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m)) - g^{n-\frac{\alpha}{k}}(0) f^{(n)}(m\varphi(b)+g(0)\eta(\varphi(x),\varphi(b),m))\right] \\
&\quad - \frac{n-\frac{\alpha}{k}}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad \times \Bigg[\int_{m\varphi(a)+g(0)\eta(\varphi(x),\varphi(a),m)}^{m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m)} (t-m\varphi(a))^{n-\frac{\alpha}{k}-1} f^{(n)}(t)dt \\
&\quad - \int_{m\varphi(b)+g(0)\eta(\varphi(x),\varphi(b),m)}^{m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m)} (t-m\varphi(b))^{n-\frac{\alpha}{k}-1} f^{(n)}(t)dt \Bigg].
\end{aligned}
$$

□

Remark 2.6. Under the same conditions as in Lemma 2.5 for $g(t)=t$, we get

$$
\begin{aligned}
&I_{f,\eta,\varphi}(x;\alpha,k,n,m,a,b) \\
&\quad = \frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x),\varphi(a),m) f^{(n)}(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad\quad - \frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x),\varphi(b),m) f^{(n)}(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad\quad + (-1)^{n+1} \frac{(nk-\alpha)\Gamma_k\left(n-\frac{\alpha}{k}\right)}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad\quad \times \left[{}^{c}D^{\alpha,k}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-} f(m\varphi(a)) - {}^{c}D^{\alpha,k}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-} f(m\varphi(b)) \right] \\
&\quad = \frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)} \int_0^1 t^{n-\frac{\alpha}{k}} f^{(n+1)}(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))dt \\
&\quad\quad - \frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)} \int_0^1 t^{n-\frac{\alpha}{k}} f^{(n+1)}(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))dt.
\end{aligned}
$$

By using Lemma 2.5, one can extend to the following results.

Theorem 2.7. *Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1,2,3,\ldots\}$, $n = [\alpha]+1$. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s,m \in (0,1]$. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ a differentiable function. Assume that $f : K \longrightarrow (0,+\infty)$ is a function on K° such that $f \in C^{n+1}[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, where $\eta(\varphi(b),\varphi(a),m) > 0$. If $0 < r \leq 1$ and $f^{(n+1)}$ is a generalized $(r;g,s,m,\varphi)$-preinvex function on $[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, then the following inequality for Caputo k-fractional derivatives* *holds:*

$$\begin{aligned}
|I_{f,g,\eta,\varphi}(x;\alpha,k,n,m,a,b)| &\le \frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)} \\
&\times \Bigg\{ m\left(f^{(n+1)}(\varphi(a))\right)^r B^r_{g(1)}\left(n-\frac{\alpha}{k}+1,\frac{s}{r}+1\right) \\
&+\left(f^{(n+1)}(\varphi(x))\right)^r \left(\frac{g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(1)-g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(0)}{n+\frac{s}{r}-\frac{\alpha}{k}+1}\right)^r \Bigg\}^{\frac{1}{r}} \\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)} \times \Bigg\{ m\left(f^{(n+1)}(\varphi(b))\right)^r B^r_{g(1)}\left(n-\frac{\alpha}{k}+1,\frac{s}{r}+1\right) \\
&+\left(f^{(n+1)}(\varphi(x))\right)^r \left(\frac{g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(1)-g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(0)}{n+\frac{s}{r}-\frac{\alpha}{k}+1}\right)^r \Bigg\}^{\frac{1}{r}},
\end{aligned}$$

where

$$B_{g(x)}(a,b)=\int_{g(0)}^{g(x)} t^{a-1}(1-t)^{b-1}dt.$$

Proof . Let $0<r\le 1$. From Lemma 2.5, generalized $(r;g,s,m,\varphi)$-preinvexity of $f^{(n+1)}$, Minkowski inequality and properties of the modulus, we have

$$\begin{aligned}
&|I_{f,g,\eta,\varphi}(x;\alpha,k,n,m,a,b)| \\
&\le \frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |g^{n-\frac{\alpha}{k}}(t)||f^{(n+1)}(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))|d[g(t)] \\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |g^{n-\frac{\alpha}{k}}(t)||f^{(n+1)}(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))|d[g(t)] \\
&\le \frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)} \\
&\times\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)\Big[m(1-g(t))^s\left(f^{(n+1)}(\varphi(a))\right)^r+g^s(t)\left(f^{(n+1)}(\varphi(x))\right)^r\Big]^{\frac{1}{r}}d[g(t)]\right) \\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)} \\
&\times\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)\Big[m(1-g(t))^s\left(f^{(n+1)}(\varphi(b))\right)^r+g^s(t)\left(f^{(n+1)}(\varphi(x))\right)^r\Big]^{\frac{1}{r}}d[g(t)]\right) \\
&\le \frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)}\times\Bigg\{\left(\int_0^1 m^{\frac{1}{r}}g^{n-\frac{\alpha}{k}}(t)(1-g(t))^{\frac{s}{r}}f^{(n+1)}(\varphi(a))d[g(t)]\right)^r \\
&+\left(\int_0^1 g^{n+\frac{s}{r}-\frac{\alpha}{k}}(t)f^{(n+1)}(\varphi(x))d[g(t)]\right)^r\Bigg\}^{\frac{1}{r}}+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)} \\
&\times\Bigg\{\left(\int_0^1 m^{\frac{1}{r}}g^{n-\frac{\alpha}{k}}(t)(1-g(t))^{\frac{s}{r}}f^{(n+1)}(\varphi(b))d[g(t)]\right)^r
\end{aligned}$$

$$+\left(\int_0^1 g^{n+\frac{s}{r}-\frac{\alpha}{k}}(t)f^{(n+1)}(\varphi(x))d[g(t)]\right)^r\Bigg\}^{\frac{1}{r}}$$

$$=\frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)}\times\Bigg\{m\left(f^{(n+1)}(\varphi(a))\right)^r B^r_{g(1)}\left(n-\frac{\alpha}{k}+1,\frac{s}{r}+1\right)$$

$$+\left(f^{(n+1)}(\varphi(x))\right)^r\left(\frac{g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(1)-g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(0)}{n+\frac{s}{r}-\frac{\alpha}{k}+1}\right)^r\Bigg\}^{\frac{1}{r}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)}\times\Bigg\{m\left(f^{(n+1)}(\varphi(b))\right)^r B^r_{g(1)}\left(n-\frac{\alpha}{k}+1,\frac{s}{r}+1\right)$$

$$+\left(f^{(n+1)}(\varphi(x))\right)^r\left(\frac{g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(1)-g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(0)}{n+\frac{s}{r}-\frac{\alpha}{k}+1}\right)^r\Bigg\}^{\frac{1}{r}}.$$

□

Corollary 2.8. Under the same conditions as in Theorem 2.7, if we choose $m=k=r=1$, $\eta(\varphi(y),\varphi(x),m)=\varphi(y)-m\varphi(x)$, $\varphi(x)=x$, $\forall x\in I$, $g(t)=t$ and $f^{(n+1)}\le K$, we get the following inequality for Caputo fractional derivatives:

$$\left|\left[\frac{(x-a)^{n-\alpha}-(x-b)^{n-\alpha}}{b-a}\right]f^{(n)}(x)+(-1)^{n+1}\frac{\Gamma(n-\alpha+1)}{b-a}\left[{}^cD^{\alpha}_{x-}f(a)-{}^cD^{\alpha}_{x-}f(b)\right]\right|$$

$$\le K\left(\beta(n-\alpha+1,s+1)+\frac{1}{n+s-\alpha+1}\right)\left[\frac{(x-a)^{n-\alpha+1}+(b-x)^{n-\alpha+1}}{b-a}\right].\tag{2.2}$$

Theorem 2.9. *Let $\alpha>0$, $k\ge 1$ and $\alpha\notin\{1,2,3,\dots\}$, $n=[\alpha]+1$. Suppose $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$ for any fixed $s,m\in(0,1]$. Let $\varphi: I\longrightarrow K$ be a continuous function and $g:[0,1]\longrightarrow[0,1]$ a differentiable function. Assume that $f: K\longrightarrow(0,+\infty)$ is a function on K° such that $f\in C^{n+1}[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, where $\eta(\varphi(b),\varphi(a),m)>0$. If $0<r\le 1$ and $\left(f^{(n+1)}\right)^q$ is a generalized $(r;g,s,m,\varphi)$-preinvex function on $[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, $q>1$, $p^{-1}+q^{-1}=1$, then the following inequality for Caputo k-fractional derivatives holds:*

$$|I_{f,g,\eta,\varphi}(x;\alpha,k,n,m,a,b)|\le\left(\frac{r}{s+r}\right)^{\frac{1}{q}}\left(\frac{g^{(n-\frac{\alpha}{k})p+1}(1)-g^{(n-\frac{\alpha}{k})p+1}(0)}{(n-\frac{\alpha}{k})p+1}\right)^{\frac{1}{p}}\frac{1}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}\Big[m\left(f^{(n+1)}(\varphi(a))\right)^{rq}\left((1-g(0))^{\frac{s}{r}+1}-(1-g(1))^{\frac{s}{r}+1}\right)^r$$

$$+\left(f^{(n+1)}(\varphi(x))\right)^{rq}\left(g^{\frac{s}{r}+1}(1)-g^{\frac{s}{r}+1}(0)\right)^r\Big]^{\frac{1}{rq}}\tag{2.3}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}\Big[m\left(f^{(n+1)}(\varphi(b))\right)^{rq}\left((1-g(0))^{\frac{s}{r}+1}-(1-g(1))^{\frac{s}{r}+1}\right)^r$$

$$+\left(f^{(n+1)}(\varphi(x))\right)^{rq}\left(g^{\frac{s}{r}+1}(1)-g^{\frac{s}{r}+1}(0)\right)^r\Big]^{\frac{1}{rq}}\Bigg\}.$$

Proof . Suppose that $q>1$ and $0<r\le 1$. From Lemma 2.5, generalized $(r;g,s,m,\varphi)$-preinvexity of $\left(f^{(n+1)}\right)^q$, Hölder inequality, Minkowski inequality and properties of the modulus, we have

$$
\begin{aligned}
&|I_{f,g,\eta,\varphi}(x;\alpha,k,n,m,a,b)| \\
&\quad\le \frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |g^{n-\frac{\alpha}{k}}(t)||f^{(n+1)}(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))|d[g(t)] \\
&\quad\quad+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |g^{n-\frac{\alpha}{k}}(t)||f^{(n+1)}(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))|d[g(t)] \\
&\quad\le \frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 g^{\left(n-\frac{\alpha}{k}\right)p}(t)d[g(t)]\right)^{\frac{1}{p}} \\
&\quad\quad\times\left(\int_0^1 \left(f^{(n+1)}(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))\right)^q d[g(t)]\right)^{\frac{1}{q}} \\
&\quad\quad+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 g^{\left(n-\frac{\alpha}{k}\right)p}(t)d[g(t)]\right)^{\frac{1}{p}} \\
&\quad\quad\times\left(\int_0^1 \left(f^{(n+1)}(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))\right)^q d[g(t)]\right)^{\frac{1}{q}} \\
&\quad\le \frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 g^{\left(n-\frac{\alpha}{k}\right)p}(t)d[g(t)]\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1 \left[m(1-g(t))^s\left(f^{(n+1)}(\varphi(a))\right)^{rq}+g^s(t)\left(f^{(n+1)}(\varphi(x))\right)^{rq}\right]^{\frac{1}{r}}d[g(t)]\right)^{\frac{1}{q}} \\
&\quad\quad+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 g^{\left(n-\frac{\alpha}{k}\right)p}(t)d[g(t)]\right)^{\frac{1}{p}} \\
&\quad\quad\times\left(\int_0^1 \left[m(1-g(t))^s\left(f^{(n+1)}(\varphi(b))\right)^{rq}+g^s(t)\left(f^{(n+1)}(\varphi(x))\right)^{rq}\right]^{\frac{1}{r}}d[g(t)]\right)^{\frac{1}{q}} \\
&\quad\le \frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 g^{\left(n-\frac{\alpha}{k}\right)p}(t)d[g(t)]\right)^{\frac{1}{p}} \\
&\quad\quad\times\left\{\left(\int_0^1 m^{\frac{1}{r}}(1-g(t))^{\frac{s}{r}}\left(f^{(n+1)}(\varphi(a))\right)^q d[g(t)]\right)^r\right. \\
&\quad\quad\left.+\left(\int_0^1 g^{\frac{s}{r}}(t)\left(f^{(n+1)}(\varphi(x))\right)^q d[g(t)]\right)^r\right\}^{\frac{1}{rq}} \\
&\quad\quad+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 g^{\left(n-\frac{\alpha}{k}\right)p}(t)d[g(t)]\right)^{\frac{1}{p}} \\
&\quad\quad\times\left\{\left(\int_0^1 m^{\frac{1}{r}}(1-g(t))^{\frac{s}{r}}\left(f^{(n+1)}(\varphi(b))\right)^q d[g(t)]\right)^r\right. \\
&\quad\quad\left.+\left(\int_0^1 g^{\frac{s}{r}}(t)\left(f^{(n+1)}(\varphi(x))\right)^q d[g(t)]\right)^r\right\}^{\frac{1}{rq}}
\end{aligned}
$$

$$= \left(\frac{r}{s+r}\right)^{\frac{1}{q}} \left(\frac{g^{(n-\frac{\alpha}{k})p+1}(1) - g^{(n-\frac{\alpha}{k})p+1}(0)}{(n-\frac{\alpha}{k})p+1}\right)^{\frac{1}{p}} \frac{1}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times \Bigg\{ |\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1} \Big[m\left(f^{(n+1)}(\varphi(a))\right)^{rq} \left((1-g(0))^{\frac{s}{r}+1} - (1-g(1))^{\frac{s}{r}+1}\right)^r$$
$$+ \left(f^{(n+1)}(\varphi(x))\right)^{rq} \left(g^{\frac{s}{r}+1}(1) - g^{\frac{s}{r}+1}(0)\right)^r \Big]^{\frac{1}{rq}}$$
$$+ |\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1} \Big[m\left(f^{(n+1)}(\varphi(b))\right)^{rq} \left((1-g(0))^{\frac{s}{r}+1} - (1-g(1))^{\frac{s}{r}+1}\right)^r$$
$$+ \left(f^{(n+1)}(\varphi(x))\right)^{rq} \left(g^{\frac{s}{r}+1}(1) - g^{\frac{s}{r}+1}(0)\right)^r \Big]^{\frac{1}{rq}} \Bigg\}.$$

□

Corollary 2.10. Under the same conditions as in Theorem 2.9, if we choose $m = k = r = 1$, $\eta(\varphi(y),\varphi(x),m) = \varphi(y) - m\varphi(x)$, $\varphi(x) = x$, $\forall x \in I$, $g(t) = t$ and $f^{(n+1)} \leq K$, we get the following inequality for Caputo fractional derivatives:

$$\left|\left[\frac{(x-a)^{n-\alpha} - (x-b)^{n-\alpha}}{b-a}\right] f^{(n)}(x) + (-1)^{n+1} \frac{\Gamma(n-\alpha+1)}{b-a} \left[{}^cD^{\alpha}_{x-}f(a) - {}^cD^{\alpha}_{x-}f(b)\right]\right|$$
$$\leq \frac{K}{\left((n-\alpha)p+1\right)^{\frac{1}{p}}} \left(\frac{2}{s+1}\right)^{\frac{1}{q}} \left[\frac{(x-a)^{n-\alpha+1} + (b-x)^{n-\alpha+1}}{b-a}\right]. \tag{2.4}$$

Theorem 2.11. *Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1,2,3,\ldots\}$, $n = [\alpha]+1$. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s, m \in (0,1]$. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ a differentiable function. Assume that $f : K \longrightarrow (0,+\infty)$ is a function on K° such that $f \in C^{n+1}[m\varphi(a), m\varphi(a) + \eta(\varphi(b),\varphi(a),m)]$, where $\eta(\varphi(b),\varphi(a),m) > 0$. If $0 < r \leq 1$ and $\left(f^{(n+1)}\right)^q$ is a generalized $(r; g, s, m, \varphi)$-preinvex function on $[m\varphi(a), m\varphi(a) + \eta(\varphi(b),\varphi(a),m)]$, $q \geq 1$, then the following inequality for Caputo k-fractional derivatives holds:*

$$|I_{f,g,\eta,\varphi}(x;\alpha,k,n,m,a,b)| \leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)} \left(\frac{g^{n-\frac{\alpha}{k}+1}(1) - g^{n-\frac{\alpha}{k}+1}(0)}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}$$
$$\times \Bigg\{ m\left(f^{(n+1)}(\varphi(a))\right)^{rq} B^r_{g(1)}\left(n-\frac{\alpha}{k}+1, \frac{s}{r}+1\right)$$
$$+ \left(f^{(n+1)}(\varphi(x))\right)^{rq} \left(\frac{g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(1) - g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(0)}{n+\frac{s}{r}-\frac{\alpha}{k}+1}\right)^r \Bigg\}^{\frac{1}{rq}} \tag{2.5}$$
$$+ \frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)} \left(\frac{g^{n-\frac{\alpha}{k}+1}(1) - g^{n-\frac{\alpha}{k}+1}(0)}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}$$
$$\times \Bigg\{ m\left(f^{(n+1)}(\varphi(b))\right)^{rq} B^r_{g(1)}\left(n-\frac{\alpha}{k}+1, \frac{s}{r}+1\right)$$

$$+\left(f^{(n+1)}(\varphi(x))\right)^{rq}\left(\frac{g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(1)-g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(0)}{n+\frac{s}{r}-\frac{\alpha}{k}+1}\right)^{r}\Bigg\}^{\frac{1}{rq}}.$$

Proof . Suppose that $q\geq 1$ and $0<r\leq 1$. From Lemma 2.5, generalized $(r;g,s,m,\varphi)$-preinvexity of $\left(f^{(n+1)}\right)^{q}$, the well-known power mean inequality, Minkowski inequality and properties of the modulus, we have

$$\begin{aligned}
&|I_{f,g,\eta,\varphi}(x;\alpha,k,n,m,a,b)|\\
&\leq\frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1|g^{n-\frac{\alpha}{k}}(t)||f^{(n+1)}(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))|d[g(t)]\\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1|g^{n-\frac{\alpha}{k}}(t)||f^{(n+1)}(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))|d[g(t)]\\
&\leq\frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)d[g(t)]\right)^{1-\frac{1}{q}}\\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)d[g(t)]\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)\left(f^{(n+1)}(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))\right)^{q}d[g(t)]\right)^{\frac{1}{q}}\\
&\times\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)\left(f^{(n+1)}(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))\right)^{q}d[g(t)]\right)^{\frac{1}{q}}\\
&\leq\frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)d[g(t)]\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)\left[m(1-g(t))^{s}\left(f^{(n+1)}(\varphi(a))\right)^{rq}+g^{s}(t)\left(f^{(n+1)}(\varphi(x))\right)^{rq}\right]^{\frac{1}{r}}d[g(t)]\right)^{\frac{1}{q}}\\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)d[g(t)]\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)\left[m(1-g(t))^{s}\left(f^{(n+1)}(\varphi(b))\right)^{rq}+g^{s}(t)\left(f^{(n+1)}(\varphi(x))\right)^{rq}\right]^{\frac{1}{r}}d[g(t)]\right)^{\frac{1}{q}}\\
&\leq\frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)d[g(t)]\right)^{1-\frac{1}{q}}\\
&\times\Bigg\{\left(\int_0^1 m^{\frac{1}{r}}g^{n-\frac{\alpha}{k}}(t)(1-g(t))^{\frac{s}{r}}\left(f^{(n+1)}(\varphi(a))\right)^{q}d[g(t)]\right)^{r}\\
&+\left(\int_0^1 g^{n+\frac{s}{r}-\frac{\alpha}{k}}(t)\left(f^{(n+1)}(\varphi(x))\right)^{q}d[g(t)]\right)^{r}\Bigg\}^{\frac{1}{rq}}\\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)d[g(t)]\right)^{1-\frac{1}{q}}
\end{aligned}$$

$$\times\left\{\left(\int_0^1 m^{\frac{1}{r}} g^{n-\frac{\alpha}{k}}(t)(1-g(t))^{\frac{s}{r}}\left(f^{(n+1)}(\varphi(b))\right)^q d[g(t)]\right)^r\right.$$
$$\left.+\left(\int_0^1 g^{n+\frac{s}{r}-\frac{\alpha}{k}}(t)\left(f^{(n+1)}(\varphi(x))\right)^q d[g(t)]\right)^r\right\}^{\frac{1}{rq}}$$
$$=\frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\frac{g^{n-\frac{\alpha}{k}+1}(1)-g^{n-\frac{\alpha}{k}+1}(0)}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}$$
$$\times\left\{m\left(f^{(n+1)}(\varphi(a))\right)^{rq} B^r_{g(1)}\left(n-\frac{\alpha}{k}+1,\frac{s}{r}+1\right)\right.$$
$$\left.+\left(f^{(n+1)}(\varphi(x))\right)^{rq}\left(\frac{g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(1)-g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(0)}{n+\frac{s}{r}-\frac{\alpha}{k}+1}\right)^r\right\}^{\frac{1}{rq}}$$
$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),\varphi(a),m)}\left(\frac{g^{n-\frac{\alpha}{k}+1}(1)-g^{n-\frac{\alpha}{k}+1}(0)}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}$$
$$\times\left\{m\left(f^{(n+1)}(\varphi(b))\right)^{rq} B^r_{g(1)}\left(n-\frac{\alpha}{k}+1,\frac{s}{r}+1\right)\right.$$
$$\left.+\left(f^{(n+1)}(\varphi(x))\right)^{rq}\left(\frac{g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(1)-g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(0)}{n+\frac{s}{r}-\frac{\alpha}{k}+1}\right)^r\right\}^{\frac{1}{rq}}.$$

□

Corollary 2.12. Under the same conditions as in Theorem 2.11, if we choose $m=k=r=1$, $\eta(\varphi(y),\varphi(x),m)=\varphi(y)-m\varphi(x)$, $\varphi(x)=x$, $\forall x\in I$, $g(t)=t$ and $f^{(n+1)}\leq K$, we get the following inequality for Caputo fractional derivatives:

$$\left|\left[\frac{(x-a)^{n-\alpha}-(x-b)^{n-\alpha}}{b-a}\right]f^{(n)}(x)+(-1)^{n+1}\frac{\Gamma(n-\alpha+1)}{b-a}\left[{}^cD^\alpha_{x-}f(a)-{}^cD^\alpha_{x-}f(b)\right]\right|$$
$$\leq K\left(\frac{1}{n-\alpha+1}\right)^{1-\frac{1}{q}}\left(\beta(n-\alpha+1,s+1)+\frac{1}{n+s-\alpha+1}\right)^{\frac{1}{q}}$$
$$\times\left[\frac{(x-a)^{n-\alpha+1}+(b-x)^{n-\alpha+1}}{b-a}\right].$$

Corollary 2.13. Under the same conditions as in Theorem 2.11 for $q=1$, we get Theorem 2.7.

3. Applications to special means

Definition 3.1. (see [6]) A function $M:\mathbb{R}^2_+\longrightarrow\mathbb{R}_+$, is called a Mean function if it has the internality property:

$$\min\{x,y\}\leq M(x,y)\leq\max\{x,y\},\ \forall x,y\in\mathbb{R}_+.$$

It follows that a mean $M(x, y)$ must have the property $M(x, x) = x$, $\forall x \in \mathbb{R}_+$. Now, let us consider some means for arbitrary positive real numbers α, β $(\alpha \neq \beta)$.

1. The arithmetic mean:
$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2};$$
2. The geometric mean:
$$G := G(\alpha, \beta) = \sqrt{\alpha\beta};$$
3. The harmonic mean:
$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}};$$
4. The power mean:
$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1;$$
5. The identric mean:
$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta; \end{cases}$$
6. The logarithmic mean:
$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln(\beta) - \ln(\alpha)};$$
7. The generalized log-mean:
$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\};$$
8. The weighted p-power mean:
$$M_p\begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & , \alpha_n \\ u_1, & u_2, & \cdots & , u_n \end{pmatrix} = \left(\sum_{i=1}^n \alpha_i u_i^p\right)^{\frac{1}{p}}$$
where $0 \leq \alpha_i \leq 1$, $u_i > 0\,(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^n \alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(\varphi(x), \varphi(y)) : [\varphi(x), \varphi(x) + \eta(\varphi(y), \varphi(x))] \times [\varphi(x), \varphi(x) + \eta(\varphi(y), \varphi(x))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means and $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ a differentiable function. Therefore one can obtain various inequalities using the results of Section 2 for these means as follows: Replace $\eta(\varphi(y), \varphi(x), m)$ with $\eta(\varphi(y), \varphi(x))$ and setting $\eta(\varphi(y), \varphi(x)) = M(\varphi(x), \varphi(y))$ for value $m = 1$ and $\forall x, y \in I$ in (2.3), one can obtain the following interesting inequalities involving means:

$$\begin{aligned} &|I_{f,g,M(\cdot,\cdot),\varphi}(x; \alpha, k, n, 1, a, b)| \\ &\leq \left(\frac{r}{s+r}\right)^{\frac{1}{q}} \left(\frac{g^{(n-\frac{\alpha}{k})p+1}(1) - g^{(n-\frac{\alpha}{k})p+1}(0)}{(n-\frac{\alpha}{k})p+1}\right)^{\frac{1}{p}} \frac{1}{M(\varphi(a), \varphi(b))} \end{aligned}$$

$$
\begin{aligned}
&\times \Bigg\{ M^{n-\frac{\alpha}{k}+1}(\varphi(a),\varphi(x)) \Big[\left(f^{(n+1)}(\varphi(a))\right)^{rq} \left((1-g(0))^{\frac{s}{r}+1} - (1-g(1))^{\frac{s}{r}+1}\right)^{r} \\
&+ \left(f^{(n+1)}(\varphi(x))\right)^{rq} \left(g^{\frac{s}{r}+1}(1) - g^{\frac{s}{r}+1}(0)\right)^{r} \Big]^{\frac{1}{rq}} \\
&+ M^{n-\frac{\alpha}{k}+1}(\varphi(b),\varphi(x)) \Big[\left(f^{(n+1)}(\varphi(b))\right)^{rq} \left((1-g(0))^{\frac{s}{r}+1} - (1-g(1))^{\frac{s}{r}+1}\right)^{r} \\
&+ \left(f^{(n+1)}(\varphi(x))\right)^{rq} \left(g^{\frac{s}{r}+1}(1) - g^{\frac{s}{r}+1}(0)\right)^{r} \Big]^{\frac{1}{rq}} \Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
&|I_{f,g,M(\cdot,\cdot),\varphi}(x;\alpha,k,n,1,a,b)| \le \frac{M^{n-\frac{\alpha}{k}+1}(\varphi(a),\varphi(x))}{M(\varphi(a),\varphi(b))} \left(\frac{g^{n-\frac{\alpha}{k}+1}(1) - g^{n-\frac{\alpha}{k}+1}(0)}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \\
&\times \Bigg\{ \left(f^{(n+1)}(\varphi(a))\right)^{rq} B^{r}_{g(1)}\left(n-\frac{\alpha}{k}+1, \frac{s}{r}+1\right) \\
&+ \left(f^{(n+1)}(\varphi(x))\right)^{rq} \left(\frac{g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(1) - g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(0)}{n+\frac{s}{r}-\frac{\alpha}{k}+1}\right)^{r} \Bigg\}^{\frac{1}{rq}} \\
&+ \frac{M^{n-\frac{\alpha}{k}+1}(\varphi(b),\varphi(x))}{M(\varphi(a),\varphi(b))} \left(\frac{g^{n-\frac{\alpha}{k}+1}(1) - g^{n-\frac{\alpha}{k}+1}(0)}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \\
&\times \Bigg\{ \left(f^{(n+1)}(\varphi(b))\right)^{rq} B^{r}_{g(1)}\left(n-\frac{\alpha}{k}+1, \frac{s}{r}+1\right) \\
&+ \left(f^{(n+1)}(\varphi(x))\right)^{rq} \left(\frac{g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(1) - g^{n+\frac{s}{r}-\frac{\alpha}{k}+1}(0)}{n+\frac{s}{r}-\frac{\alpha}{k}+1}\right)^{r} \Bigg\}^{\frac{1}{rq}}.
\end{aligned}
\tag{3.1}
$$

Letting $M(\varphi(x),\varphi(y)) = A, G, H, P_r, I, L, L_p, M_p$, $\forall x, y \in I$, in (3.1), we get the inequalities involving means for a particular choice of a generalized $(r; g, s, 1, \varphi)$-preinvex functions $\left(f^{(n+1)}\right)^q$. The details are left to the interested reader.

4. Conclusions

In the present paper, the notion of generalized $(r; g, s, m, \varphi)$-preinvex function was applied for established some new generalizations of Ostrowski type inequalities via Caputo k-fractional derivatives. Motivated by this new interesting class of generalized $(r; g, s, m, \varphi)$-preinvex functions we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Ostrowski, Hermite-Hadamard and Simpson type integral inequalities for various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, k-fractional integrals, local fractional integrals, fractional integral operators, Caputo k-fractional derivatives, q-calculus, (p, q)-calculus, time scale calculus and conformable fractional integrals.

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