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Modified degenerate Carlitz's q-Bernoulli polynomials and numbers with weight (α, β)

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Abstract

The main goal of the present paper is to construct some families of the Carlitz's q-Bernoulli polynomials and numbers. We firstly introduce the modified Carlitz's q-Bernoulli polynomials and numbers with weight $(_p$. We then define the modified degenerate Carlitz's q-Bernoulli polynomials and numbers with weight (α, β) and obtain some recurrence relations and other identities. Moreover, we derive some correlations with the modified Carlitz's q-Bernoulli polynomials with weight (α, β), the modified degenerate Carlitz's q-Bernoulli polynomials with weight (α, β), the Stirling numbers of the first kind and second kind.

Keywords: Carlitz's q-Bernoulli polynomials; Stirling numbers of the first kind; Stirling numbers of the second kind; p-adic q-integral.
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1. Introduction

In the complex plane, the Bernoulli polynomials $B_n(x)$ are defined via the following Taylor series expansion about $t = 0$:

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t - 1}e^{xt} \quad (|t| < 2\pi),$$

from which one can obtain the Bernoulli numbers B_n as values $B_n(0) := B_n$ (see, [2]-[15]).

Carlitz [2] defined the q-analogue of the Bernoulli numbers $\beta_n = \beta_n(q)$ and polynomials $\beta_n(x : q)$ as follows

$$\beta_0 = 1,\ q(q\beta + 1)^n - \beta_n = \begin{cases} 1 \text{ if } n = 1, \\ 0 \text{ if } n > 1 \end{cases}$$

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and

$$\beta_n\left(x:q\right)=\sum_{s=0}^{n}\binom{n}{s}\beta_s\left(q\right)q^{sx}\left[x\right]_q^{n-s}.$$

Carlitz [3] also introduced the degenerate Bernoulli polynomials $\beta_n(\lambda, x)$ for $\lambda \neq 0$ by the following generating function to be

$$\frac{t}{(1+\lambda t)^{1/\lambda}-1}\left(1+\lambda t\right)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_n(\lambda,x)\frac{t^n}{n!}.$$

In recent years, q-Bernoulli polynomials and their generalizations have been studied and investigated extensively by many mathematicians. For instance, Kim et al. [7] considered a novel degenerate Bernoulli numbers and polynomials, different from Carlitz's degenerate Bernoulli numbers and polynomials, and provided some identities and formulas associated with these numbers and polynomials. Young [15] proved a general symmetric identity involving the degenerate Bernoulli polynomials and sums of generalized falling factorials, which unifies several known identities for Bernoulli and degenerate Bernoulli numbers and polynomials. Kim [5] obtained the Carlitz's q-Bernoulli polynomials arising from the p-adic q-integral on $\mathbb{Z}_p$ and obtained some recurrence relations. In [4], Duran et al. acquired some symmetric identities and properties for the Carlitz's twisted q-Bernoulli polynomials under the symmetric group of degree n. Kim et al. [6] introduced the q-Bernoulli polynomials and numbers with weight α and showed some interesting properties. In [10] and [12], Lee et al. defined and studied the modified degenerate Carlitz q-Bernoulli numbers and polynomials. Lee [11] considered the weighted degenerate Carlitz's q-Bernoulli polynomials and numbers and provided some interesting properties. Park [13] constructed new q-extension of Bernoulli polynomials with weight α and weak weight β.

Supposing that p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of rational numbers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}=\{1,2,3,\cdots\}$ and $\mathbb{N}^*=\mathbb{N}\cup\{0\}$. The normalized p-adic norm is given by $|p|_p=p^{-1}$. The notation q can be considered as an indeterminate, a complex number $q\in\mathbb{C}$ with $|q|<1$, or a p-adic number $q\in\mathbb{C}_p$ with $|q-1|_p<p^{-\frac{1}{p-1}}$ and $q^x=\exp(x\log q)$ for $|x|_p\leq 1$. The q-analogue of x is defined by $[x]_q=(1-q^x)/(1-q)$. It is clear that $\lim_{q\to 1}[x]_q=x$ for any x with $|x|_p\leq 1$ in the p-adic case, see [1, 4-14].

For

$$g\in UD\left(\mathbb{Z}_p\right)=\left\{g\,|g:\mathbb{Z}_p\to\mathbb{C}_p \text{ is uniformly differentiable function}\right\},$$

the bosonic p-adic q-integral on $\mathbb{Z}_p$ of a function $g\in UD(\mathbb{Z}_p)$ is introduced by Kim [5] as follows:

$$I_q(g)=\int_{\mathbb{Z}_p}g\left(x\right)d\mu_q\left(x\right)=\lim_{N\to\infty}\frac{1}{\left[p^N\right]_q}\sum_{j=0}^{p^N-1}g\left(j\right)q^j. \tag{1.1}$$

A more detailed statement of the above is found in each of the references [1], [4]-[14].

The Carlitz's q-Bernoulli polynomials with Witt's formula are defined by the following p-adic q-integral on $\mathbb{Z}_p$, with respect to μ_q (see [5]):

$$\int_{\mathbb{Z}_p}\left[x+y\right]_q^n d\mu_q\left(y\right)=\beta_{n,q}(x)\quad\left(n\geq 0\right).$$

Moreover, when $x = 0$, we have the Carlitz's q-Bernoulli numbers with Witt's formula given by $\int_{\mathbb{Z}_p} [y]_q^n \, d\mu_q(y) = \beta_{n,q}(0) := \beta_{n,q}$. By (1.1), $\beta_{n,q}$ holds the following relation

$$\begin{aligned}\beta_{n,q} &= \int_{\mathbb{Z}_p} [y]_q^n \, d\mu_q(y) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{j=0}^{p^N-1} [j]_q^n \, q^j = \lim_{N\to\infty} \frac{1-q}{1-q^{p^N}} \sum_{j=0}^{p^N-1} \left(\frac{1-q^j}{1-q}\right)^n q^j \\ &= \frac{1}{(1-q)^{n-1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \lim_{N\to\infty} \frac{1}{1-q^{p^N}} \frac{1-q^{p^N(k+1)}}{1-q^{k+1}} \\ &= \frac{1}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{k+1}{[k+1]_q}.\end{aligned}$$

The modified Carlitz's q-Bernoulli polynomials are defined as follows (see [12]):

$$\int_{\mathbb{Z}_p} q^{-y} [x+y]_q^n \, d\mu_q(y) = B_{n,q}(x).$$

The modified Carlitz's q-Bernoulli numbers can be obtained $B_{n,q}(0) := B_{n,q} = \int_{\mathbb{Z}_p} q^{-y} [y]_q^n \, d\mu_q(y)$ and satisfies the following identity

$$\begin{aligned}B_{n,q} &= \int_{\mathbb{Z}_p} q^{-y} [y]_q^n \, d\mu_q(y) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{j=0}^{p^N-1} [j]_q^n = \lim_{N\to\infty} \frac{1-q}{1-q^{p^N}} \sum_{j=0}^{p^N-1} \left(\frac{1-q^j}{1-q}\right)^n \\ &= \frac{1}{(1-q)^{n-1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \lim_{N\to\infty} \frac{1}{1-q^{p^N}} \frac{1-q^{p^N k}}{1-q^k} \\ &= \frac{1}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{k}{[k]_q}.\end{aligned}$$

Suppose that $\lambda, t \in \mathbb{C}_p$ with $0 < |\lambda|_p \leq 1$ and $|t|_p \leq p^{-\frac{1}{p-1}}$. In [12], Lee and Jang introduced the modified degenerate Carlitz's q-Bernoulli polynomials by the following generating function to be

$$\int_{\mathbb{Z}_p} q^{-y} (1+\lambda t)^{\frac{1}{\lambda}[x+y]_q} \, d\mu_q(y) = \sum_{n=0}^{\infty} B_{n,\lambda,q}(x) \frac{t^n}{n!}.$$

When $x = 0$, $B_{n,\lambda,q}(0) := B_{n,\lambda,q}$ are called the modified degenerate Carlitz's q-Bernoulli numbers.

This paper consists of the three sections. The first part is introduction which provides the required information, notations and definitions. The second part gives the definition of the modified Carlitz's q-Bernoulli polynomials and numbers with weight (α, β), and satisfies some identities and properties of them. Then, we introduce the modified degenerate Carlitz's q-Bernoulli polynomials and numbers with weight (α, β) and investigate their some identities and relations related to Stirling numbers of first kind and second kind in the last part.

2. The modified Carlitz's q-Bernoulli polynomials and numbers with weight (α, β)

We introduce the modified Carlitz's q-Bernoulli polynomials with weight (α, β) as follows:

$$\int_{\mathbb{Z}_p} q^{-\beta y} e^{t[x+y]_{q^\alpha}} \, d\mu_{q^\beta}(y) = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha,\beta)}(x) \frac{t^n}{n!} \tag{2.1}$$

or with another expression

$$\int_{\mathbb{Z}_p} q^{-\beta y} [x+y]_{q^\alpha}^n \, d\mu_{q^\beta}(y) = B_{n,q}^{(\alpha,\beta)}(x). \tag{2.2}$$

When we put $x=0$ in Eq. (2.2), we get $B_{n,q}^{(\alpha,\beta)}(0) := B_{n,q}^{(\alpha,\beta)} = \int_{\mathbb{Z}_p} q^{-\beta y} [y]_{q^\alpha}^n \, d\mu_{q^\beta}(y)$ called the modified Carlitz's q-Bernoulli numbers with weight (α,β).

For special value $n=0$, we have

$$\begin{aligned} B_{0,q}^{(\alpha,\beta)} &= \int_{\mathbb{Z}_p} q^{-\beta y} d\mu_{q^\beta}(y) = \lim_{N\to\infty} \frac{1}{[p^N]_{q^\beta}} \sum_{j=0}^{p^N-1} q^{-\beta j} q^{\beta j} \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_{q^\beta}} \sum_{j=0}^{p^N-1} 1 = \lim_{N\to\infty} \frac{(1-q)^\beta}{1-q^{\beta p^N}} p^N = -\frac{(1-q)^\beta}{\beta \log q}. \end{aligned}$$

By using the definition of the q-number and using (2.2), it is observed that

$$\begin{aligned} B_{n,q}^{(\alpha,\beta)}(x) &= \int_{\mathbb{Z}_p} q^{-\beta y} [x+y]_{q^\alpha}^n \, d\mu_{q^\beta}(y) = \int_{\mathbb{Z}_p} q^{-\beta y} \left([x]_{q^\alpha} + q^{\alpha x} [y]_{q^\alpha} \right)^n d\mu_{q^\beta}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} q^{l\alpha x} [x]_{q^\alpha}^{n-l} \int_{\mathbb{Z}_p} q^{-\beta y} [y]_{q^\alpha}^l \, d\mu_{q^\beta}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} q^{l\alpha x} [x]_{q^\alpha}^{n-l} B_{l,q}^{(\alpha,\beta)}. \end{aligned}$$

So, we deduce

$$B_{n,q}^{(\alpha,\beta)}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{l\alpha x} [x]_{q^\alpha}^{n-l} B_{l,q}^{(\alpha,\beta)} = \left(q^{\alpha x} B_q^{(\alpha,\beta)} + [x]_{q^\alpha} \right)^n, \tag{2.3}$$

with the usual convention about replacing $\left(B_q^{(\alpha,\beta)}\right)^n$ by $B_{n,q}^{(\alpha,\beta)}$. Here we have used the binomial formula $(x+y)^n = \sum_{l=0}^{n} \binom{n}{l} x^l y^{n-l}$.

Let $g_1(x) = g(x+1)$. By using (1.1), we easily derive

$$\begin{aligned} q^\beta I_{q^\beta}(g_1) &= \lim_{N\to\infty} \frac{1}{[p^N]_{q^\beta}} \sum_{j=0}^{p^N-1} g(j+1)\, q^{\beta(j+1)} \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_{q^\beta}} \left(\sum_{j=0}^{p^N-1} g(j)\, q^{\beta j} - g(0) + g\left(p^N\right) q^{\beta p^N} \right) \\ &= I_{q^\beta}(g) + \lim_{N\to\infty} \frac{g'\left(p^N\right) + g\left(p^N\right) \beta \log q}{\beta \log q} \\ &= I_{q^\beta}(g) + \left(q^\beta - 1\right) g(0) + \frac{q^\beta - 1}{\beta \log q} g'(0). \end{aligned}$$

Hence, we develop

$$q^\beta I_{q^\beta}(g_1) - I_{q^\beta}(g) = \left(q^\beta - 1\right) g(0) + \frac{q^\beta - 1}{\beta \log q} g'(0). \tag{2.4}$$

Taking $g(x) = q^{-\beta x}[x]_{q^\alpha}^n$ in (2.4) yields to the following identity

$$\left(q^\beta - 1\right) g(0) + \frac{q^\beta - 1}{\beta \log q} g'(0) = q^\beta \int_{\mathbb{Z}_p} q^{-\beta(x+1)} [x+1]_{q^\alpha}^n \, d\mu_{q^\beta}(x) - \int_{\mathbb{Z}_p} q^{-\beta x} [x]_{q^\alpha}^n \, d\mu_{q^\beta}(x)$$

which gives the following recurrence relation:

$$B_{0,q}^{(\alpha,\beta)} = -\frac{(1-q)^\beta}{\beta \log q} \text{ and } \left(q^\alpha B_q^{(\alpha,\beta)} + 1\right)^n - B_{n,q}^{(\alpha,\beta)} = \left\{ \begin{array}{ll} \frac{\alpha[\beta]_q}{\beta[\alpha]_q} & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{array} \right| \tag{2.5}$$

in conjunction with the replacing method $\left(B_q^{(\alpha,\beta)}\right)^n$ by $B_{n,q}^{(\alpha,\beta)}$.

For $n = 1$, we observe that

$$\left(q^\alpha B_q^{(\alpha,\beta)} + 1\right)^1 - B_{1,q}^{(\alpha,\beta)} = \frac{\alpha[\beta]_q}{\beta[\alpha]_q} \Rightarrow \sum_{k=0}^{1} \binom{1}{k} q^{k\alpha} B_{k,q}^{(\alpha,\beta)} = \frac{\alpha[\beta]_q}{\beta[\alpha]_q}$$

$$\Rightarrow B_{0,q}^{(\alpha,\beta)} + (q^\alpha - 1) B_{1,q}^{(\alpha,\beta)} = \frac{\alpha[\beta]_q}{\beta[\alpha]_q} \Rightarrow B_{1,q}^{(\alpha,\beta)} = \frac{1}{q^\alpha - 1} \left(\frac{\alpha[\beta]_q}{\beta[\alpha]_q} + \frac{(1-q)^\beta}{\beta \log q} \right).$$

For $n = 2$, we readily obtain

$$B_{2,q}^{(\alpha,\beta)} = \frac{1}{1 - q^{2\alpha}} \left(2q^\alpha \frac{1}{q^\alpha - 1} \left(\frac{\alpha[\beta]_q}{\beta[\alpha]_q} + \frac{(1-q)^\beta}{\beta \log q} \right) - \frac{(1-q)^\beta}{\beta \log q} \right).$$

Using the relation (2.5), one can subsequently derive the all modified Carlitz's q-Bernoulli numbers with weight (α, β).

By the virtue of (1.1), we see that

$$\begin{aligned} B_{n,q}^{(\alpha,\beta)} &= \int_{\mathbb{Z}_p} q^{-\beta x} [x]_{q^\alpha}^n \, d\mu_{q^\beta}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{q^\beta}} \sum_{j=0}^{p^N - 1} [j]_{q^\alpha}^n = \lim_{N \to \infty} \frac{1}{[p^N]_{q^\beta}} \sum_{j=0}^{p^N - 1} \left(\frac{1 - q^{\alpha j}}{1 - q^\alpha} \right)^n \\ &= \frac{1}{(1 - q^\alpha)^n} \lim_{N \to \infty} \frac{1 - q^{\beta p^N}}{1 - q^\beta} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \sum_{j=0}^{p^N - 1} \left(q^{\alpha l}\right)^j \\ &= \frac{\alpha}{\beta} \frac{1 - q^\beta}{(1 - q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{l}{1 - q^{\alpha l}} = \frac{\alpha/\beta}{(1 - q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{l}{[\alpha l/\beta]_{q^\beta}}. \end{aligned}$$

Then we get

$$B_{n,q}^{(\alpha,\beta)} = \frac{\alpha/\beta}{(1 - q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{l}{[\alpha l/\beta]_{q^\beta}}. \tag{2.6}$$

From Eqs. (2.3) and (2.6), we get the explicit formulas for $B_{n,q}^{(\alpha,\beta)}(x)$:

$$B_{n,q}^{(\alpha,\beta)}(x) = \frac{\alpha/\beta}{(1 - q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha x l} \frac{l}{[\alpha l/\beta]_{q^\beta}} \tag{2.7}$$

and

$$B_{n,q}^{(\alpha,\beta)}(x) = \frac{\alpha}{\beta} \sum_{l=0}^{n} \binom{n}{l} [x]_{q^\alpha}^{n-l} \frac{q^{l\alpha x}}{(1 - q^\alpha)^l} \sum_{k=0}^{l} \binom{l}{k} (-1)^k \frac{k}{[\alpha k/\beta]_{q^\beta}}.$$

For $r \in \mathbb{N}$, we define the modified Carlitz's q-Bernoulli polynomials of order r with weight (α, β):

$$\sum_{n=0}^{\infty} B_{n,q}^{(\alpha,\beta:r)}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-\beta(x_1+\cdots+x_r)} e^{t[x_1+\cdots+x_r+x]_{q^\alpha}} d\mu_{q^\beta}(x_1) \cdots d\mu_{q^\beta}(x_r). \tag{2.8}$$

The modified Carlitz's q-Bernoulli polynomials of order r with weight (α, β) satisfy the following recurrence relation

$$B_{n,q}^{(\alpha,\beta:r)}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{l\alpha x} [x]_{q^\alpha}^{n-l} B_{l,q}^{(\alpha,\beta:r)},$$

where $B_{n,q}^{(\alpha,\beta:r)}$ $(n \in \mathbb{N}^*)$ are called the modified Carlitz's q-Bernoulli numbers of order r with weight (α, β) and defined by

$$B_{n,q}^{(\alpha,\beta:r)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-\beta(x_1+\cdots+x_r)} [x_1 + \cdots + x_r]_{q^\alpha}^n d\mu_{q^\beta}(x_1) \cdots d\mu_{q^\beta}(x_r).$$

3. The modified degenerate Carlitz's q-Bernoulli polynomials and numbers with weight (α, β)

In this part, we provide the main results of this paper including some correlations and identities among the modified Carlitz's q-Bernoulli polynomials $B_{n,q}^{(\alpha,\beta)}(x)$ with weight (α, β), the modified degenerate Carlitz's q-Bernoulli polynomials $B_{n,\lambda,q}^{(\alpha,\beta)}(x)$ with weight (α, β), the Stirling numbers of the first kind $S_1(k, n)$ and second kind $S_2(k, n)$.

Let $\alpha \in \mathbb{N}$ and $\lambda, t \in \mathbb{C}_p$ with $0 < |\lambda|_p \leq 1$, $|t|_p \leq p^{-\frac{1}{p-1}}$. For $\alpha, \beta \in \mathbb{N}$, we introduce the modified degenerate Carlitz's q-Bernoulli polynomials $B_{n,\lambda,q}^{(\alpha,\beta)}(x)$ with weight (α, β) by means of the following bosonic p-adic q-integral on $\mathbb{Z}_p$:

$$\int_{\mathbb{Z}_p} q^{-\beta y} (1 + \lambda t)^{\frac{1}{\lambda}[x+y]_{q^\alpha}} d\mu_{q^\beta}(y) = \sum_{n=0}^{\infty} B_{n,\lambda,q}^{(\alpha,\beta)}(x) \frac{t^n}{n!}. \tag{3.1}$$

Upon setting $x = 0$, $B_{n,\lambda,q}^{(\alpha,\beta)}(0) := B_{n,\lambda,q}^{(\alpha,\beta)}$ are termed the modified degenerate Carlitz's q-Bernoulli numbers and are shown by

$$\int_{\mathbb{Z}_p} q^{-\beta y} (1 + \lambda t)^{\frac{1}{\lambda}[y]_{q^\alpha}} d\mu_{q^\beta}(y) = \sum_{n=0}^{\infty} B_{n,\lambda,q}^{(\alpha,\beta)} \frac{t^n}{n!}.$$

Remark 3.1. Some special cases of $B_{n,\lambda,q}^{(\alpha,\beta)}(x)$ are examined as follows:

$B_{n,\lambda,q}^{(1,1)}(x) := B_{n,\lambda,q}(x)$ are called the modified degenerate q-Bernoulli polynomials, see [12],
$\lim_{\lambda\to 0} B_{n,\lambda,q}^{(\alpha,\beta)}(x) := B_{n,q}^{(\alpha,\beta)}(x)$ are called the q-Bernoulli polynomials with weight (α, β) in (2.2),
$\lim_{\lambda\to 0} B_{n,\lambda,q}^{(\alpha,1)}(x) := B_{n,q}^{(\alpha)}(x)$ are called the weighted q-Bernoulli polynomials, see [6],
$\lim_{\lambda\to 0} B_{n,\lambda,q}^{(1,1)}(x) := B_{n,q}(x)$ are called the modified q-Bernoulli polynomials, see [5],
$\lim_{q\to 1^-} B_{n,\lambda,q}^{(1,1)}(x) := B_{n,\lambda}(x)$ are called the degenerate Bernoulli polynomials, see [3],
$\lim_{q\to 1^-} \left(\lim_{\lambda\to 0} B_{n,\lambda,q}^{(\alpha,\beta)}(x) \right) := B_n(x)$ are called the classical Bernoulli polynomials, see [15].

We now investigate some behaviours of the mentioned polynomials.

Note that the falling factorial and a generalized version of the falling factorial are defined, respectively, by

$$(x)_n = x(x-1)(x-2)\cdots(x-(n-1)) \text{ for } n \in \mathbb{N}^*$$

and

$$(x)_n^{(\lambda)} = x(x-\lambda)(x-\lambda)\cdots(x-(n-1)\lambda) \text{ for } n \in \mathbb{N}^*.$$

We give a relation between $B_{n,\lambda,q}^{(\alpha,\beta)}(x)$ and $(x)_n^{(\lambda)}$ as follows.

Theorem 3.2. *For $n \in \mathbb{N}^*$, $\mathrm{B}_{n,\lambda,q}^{(\alpha,\beta)}(x)$ and $(x)_n^{(\lambda)}$ satisfy the following property*

$$\mathrm{B}_{n,\lambda,q}^{(\alpha,\beta)}(x) = \int_{\mathbb{Z}_p} q^{-\beta y} \left([x+y]_{q^\alpha}\right)_n^{(\lambda)} d\mu_{q^\beta}(y). \tag{3.2}$$

Proof . From (3.1), we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,\lambda,q}^{(\alpha,\beta)} \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} q^{-\beta y} (1+\lambda t)^{\frac{1}{\lambda}[x+y]_{q^\alpha}} d\mu_{q^\beta}(y) \\
&= \int_{\mathbb{Z}_p} q^{-\beta y} \sum_{n=0}^{\infty} \binom{\frac{[x+y]_{q^\alpha}}{\lambda}}{n} \lambda^n t^n d\mu_{q^\beta}(y) \\
&= \int_{\mathbb{Z}_p} q^{-\beta y} \sum_{n=0}^{\infty} \lambda^n \left(\frac{[x+y]_{q^\alpha}}{\lambda}\right)_n d\mu_{q^\beta}(y) \frac{t^n}{n!}.
\end{aligned}$$

Because of $\left([x+y]_{q^\alpha}\right)_n^{(\lambda)} = [x+y]_{q^\alpha}\left([x+y]_{q^\alpha}-\lambda\right)\cdots\left([x+y]_{q^\alpha}-(n-1)\lambda\right)$ and by comparing the coefficients $\frac{t^n}{n!}$ of the both sides above, we acquire the desired result (3.2). □

For $n \in \mathbb{N}^*$, the generating functions of the Stirling numbers of first kind and second kind are defined, respectively, by (see [7] and [10])

$$(\log(1+t))^n = n! \sum_{l=n}^{\infty} S_1(l,n) \frac{t^l}{l!} \text{ and } \left(e^t-1\right)^n = n! \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!}.$$

We also note that $(x)_n = \sum_{l=0}^{n} S_1(n,l) x^l$.

Here we give a correlation among the new and old polynomials and numbers.

Theorem 3.3. *Let $n \in \mathbb{N}^*$. Then we have*

$$\mathrm{B}_{n,\lambda,q}^{(\alpha,\beta)}(x) = \sum_{l=0}^{n} \lambda^{n-l} S_1(n,l) \mathrm{B}_{l,q}^{(\alpha,\beta)}(x). \tag{3.3}$$

Proof . By (2.3), we have

$$\begin{aligned}
B_{n,\lambda,q}^{(\alpha,\beta)}(x) &= \lambda^n \int_{\mathbb{Z}_p} q^{-\beta y} \left(\frac{[x+y]_{q^\alpha}}{\lambda}\right)_n d\mu_{q^\beta}(y) \\
&= \sum_{l=0}^{n} S_1(n,l) \lambda^{n-l} \int_{\mathbb{Z}_p} q^{-\beta y} [x+y]_{q^\alpha}^l d\mu_{q^\beta}(y) \\
&= \sum_{l=0}^{n} S_1(n,l) \lambda^{n-l} B_{l,q}^{(\alpha,\beta)}(x).
\end{aligned}$$

Thus, the proof of this theorem is completed. □

In view of the (2.7) and (3.3), we deduce the following result.

Corollary 3.4. *The following relation*

$$\mathrm{B}_{n,\lambda,q}^{(\alpha,\beta)}(x)=\frac{\alpha}{\beta}\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{l}{k}\frac{S_1(n,l)}{(1-q^{\alpha})^{l}}\frac{k(-q^{\alpha x})^{k}}{[\alpha k/\beta]_{q^{\beta}}}\lambda^{n-l}$$

holds true for $n\in\mathbb{N}^{*}$.

We here provide a relation by the following theorem.

Theorem 3.5. *We have*

$$\mathrm{B}_{n,q}^{(\alpha,\beta)}(x)=\sum_{l=0}^{n}\lambda^{n-l}\mathrm{B}_{l,\lambda,q}^{(\alpha,\beta)}(x)\,S_2(n,l)\,. \tag{3.4}$$

Proof . By replacing t by $\frac{1}{\lambda}\left(e^{\lambda t}-1\right)$ in (3.1), we have

$$\begin{aligned}\int_{\mathbb{Z}_p}q^{-\beta y}e^{t[x+y]_{q^{\alpha}}}d\mu_{q^{\beta}}(y)&=\sum_{m=0}^{\infty}B_{m,\lambda,q}^{(\alpha,\beta)}\lambda^{-m}\frac{\left(e^{\lambda t}-1\right)^{m}}{m!}\\&=\sum_{m=0}^{\infty}B_{m,\lambda,q}^{(\alpha,\beta)}\lambda^{-m}\sum_{n=m}^{\infty}S_2(n,m)\frac{\lambda^{n}t^{n}}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}B_{m,\lambda,q}^{(\alpha,\beta)}\lambda^{n-m}S_2(n,m)\right)\frac{t^{n}}{n!}.\end{aligned}$$

Comparing (2.1) and (3.5) yields to the asserted result (3.4). □

The following theorem includes a correlation among $B_{n,q}^{(\alpha,\beta)}$, $S_1(n,m)$ and $B_{n,\lambda,q}^{(\alpha,\beta)}(x)$.

Theorem 3.6. *We have*

$$\mathrm{B}_{n,\lambda,q}^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}\sum_{l=0}^{k}\left([x]_{q^{\alpha}}\right)_{n-k}^{(\lambda)}\lambda^{k-l}q^{\alpha xl}\mathrm{B}_{l,q}^{(\alpha,\beta)}S_1(k,l)\,. \tag{3.5}$$

Proof . We firstly note that

$$\begin{aligned}q^{-\beta y}(1+\lambda t)^{\frac{1}{\lambda}[x+y]_{q^{\alpha}}}&=q^{-\beta y}(1+\lambda t)^{\frac{1}{\lambda}[x]_{q^{\alpha}}}(1+\lambda t)^{\frac{1}{\lambda}q^{\alpha x}[y]_{q^{\alpha}}}\\&=q^{-\beta y}\left(\sum_{m=0}^{\infty}\left([x]_{q^{\alpha}}\right)_{m}^{(\lambda)}\frac{t^{m}}{m!}\right)\sum_{l=0}^{\infty}\lambda^{-n}q^{\alpha xl}[y]_{q^{\alpha}}^{l}\frac{(\log(1+\lambda t))^{l}}{l!}\\&=\left(\sum_{m=0}^{\infty}\left([x]_{q^{\alpha}}\right)_{m}^{(\lambda)}\frac{t^{m}}{m!}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}\lambda^{k-l}q^{\alpha xl-\beta y}[y]_{q^{\alpha}}^{l}S_1(k,l)\right)\frac{t^{k}}{k!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\left([x]_{q^{\alpha}}\right)_{n-k}^{(\lambda)}\lambda^{k-l}q^{\alpha xl-\beta y}[y]_{q^{\alpha}}^{l}S_1(k,l)\right)\frac{t^{n}}{n!}.\end{aligned}$$

From above computation, we derive

$$\sum_{n=0}^{\infty} B_{n,\lambda,q}^{(\alpha,\beta)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\left([x]_{q^\alpha}\right)_{n-k}^{(\lambda)}\lambda^{k-l}q^{\alpha xl}\int_{\mathbb{Z}_p} q^{-\beta y}[y]_{q^\alpha}^{l}\,d\mu_{q^\beta}(y)\,S_1(k,l)\right)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\left([x]_{q^\alpha}\right)_{n-k}^{(\lambda)}\lambda^{k-l}q^{\alpha xl}B_{l,q}^{(\alpha,\beta)}S_1(k,l)\right)\frac{t^n}{n!}.$$

By equating the coefficients $\frac{t^n}{n!}$ of both sides above, we procure the relation (3.5). □

Let $r\in\mathbb{N}$. We now introduce the modified degenerate Carlitz's q-Bernoulli polynomials of order r with weight (α,β) by the following p-adic q-integral on $\mathbb{Z}_p$:

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-\beta(x_1+\cdots+x_r)}(1+\lambda t)^{\frac{1}{\lambda}[x_1+\cdots+x_r+x]_{q^\alpha}}\,d\mu_{q^\beta}(x_1)\cdots d\mu_{q^\beta}(x_r) = \sum_{n=0}^{\infty} B_{n,\lambda,q}^{(\alpha,\beta:r)}(x)\frac{t^n}{n!}. \tag{3.6}$$

The immediate result for the mentioned polynomials is given in the following theorem.

Theorem 3.7. *For $r\in\mathbb{N}$, we have*

$$\mathrm{B}_{n,\lambda,q}^{(\alpha,\beta:r)}(x) = \sum_{m=0}^{n}\lambda^{n-m}\mathrm{B}_{m,q}^{(\alpha,\beta:r)}(x)\,S_1(n,m). \tag{3.7}$$

Proof . Using (2.8), we readily obtain

$$\sum_{n=0}^{\infty} B_{n,\lambda,q}^{(\alpha,\beta:r)}(x)\frac{t^n}{n!}$$
$$= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-\beta(x_1+\cdots+x_r)}(1+\lambda t)^{\frac{1}{\lambda}[x_1+\cdots+x_r+x]_{q^\alpha}}\,d\mu_{q^\beta}(x_1)\cdots d\mu_{q^\beta}(x_r)$$
$$= \sum_{m=0}^{\infty}\lambda^{-m}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-\beta(x_1+\cdots+x_r)}[x_1+\cdots+x_r+x]_{q^\alpha}^{m}\,d\mu_{q^\beta}(x_1)\cdots d\mu_{q^\beta}(x_r)\frac{(\log(1+\lambda t))^m}{m!}$$
$$= \sum_{m=0}^{\infty}\lambda^{-m}B_{m,q}^{(\alpha,\beta:r)}(x)\sum_{n=m}^{\infty}S_1(n,m)\frac{\lambda^n t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}B_{m,q}^{(\alpha,\beta:r)}(x)\,S_1(n,m)\right)\frac{t^n}{n!}.$$

Hence, we have the claimed result (3.7). □

At last, we give the following theorem.

Theorem 3.8. *For $r\in\mathbb{N}$, we have*

$$\mathrm{B}_{n,q}^{(\alpha,\beta:r)}(x) = \sum_{m=0}^{n}\mathrm{B}_{m,\lambda,q}^{(\alpha,\beta:r)}(x)\,\lambda^{n-m}S_2(n,m).$$

Proof . If we write $\frac{1}{\lambda}\left(e^{\lambda t}-1\right)$ in place of t in (3.6), we readily get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-\beta(x_1+\cdots+x_r)}e^{t[x_1+\cdots+x_r+x]_{q^\alpha}}d\mu_{q^\beta}(x_1)\cdots d\mu_{q^\beta}(x_r)\\
&=\sum_{n=0}^{\infty}B_{n,\lambda,q}^{(\alpha,\beta:r)}(x)\,\lambda^{-n}\frac{\left(e^{\lambda t}-1\right)^n}{n!}\\
&=\sum_{m=0}^{\infty}B_{m,\lambda,q}^{(\alpha,\beta:r)}(x)\,\lambda^{-m}\sum_{n=m}^{\infty}S_2(n,m)\frac{\lambda^n t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}B_{m,\lambda,q}^{(\alpha,\beta:r)}(x)\,\lambda^{n-m}S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned}$$

In the light of (2.8), we have the desired result. □

As a consequence of the Theorem 3.8, we state the following result.

Corollary 3.9.

$$B_{n,q}^{(\alpha,\beta:r)}=\sum_{m=0}^{n}B_{m,\lambda,q}^{(\alpha,\beta:r)}\lambda^{n-m}S_2(n,m).$$

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