Int. J. Nonlinear Anal. Appl. 8 (2017) No. 2, 263-276
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.11562.1570

Existence and uniqueness of the solution for a general system of operator equations in $b-$metric spaces endowed with a graph

Cristian Chifu*, Gabriela Petrușel

Department of Business, Faculty of Business, Babeș-Bolyai University Cluj-Napoca, Romania

(Communicated by M. Eshaghi)

Abstract

The purpose of this paper is to present some coupled fixed point results on a metric space endowed with two b-metrics. We shall apply a fixed point theorem for an appropriate operator on the Cartesian product of the given spaces endowed with directed graphs. Data dependence, well-posedness and Ulam-Hyers stability are also studied. The results obtained here will be applied to prove the existence and uniqueness of the solution for a system of integral equations.

Keywords: fixed point; coupled fixed point; b-metric space; connected graph; integral equations.
2010 MSC: Primary 47H10; Secondary 54H25, 45G15.

1. Introduction and Preliminaries

In the study of operator equation systems, a very useful concept is that of coupled fixed point. Introduced by Opoitsev (see [15], [16]), the topic knew a fast expansion starting with the papers of Guo and Lakshmikantam [12] and Gnana and Lakshmikantam [10]. For related results regarding coupled fixed point theory see [14, 4, 17, 5, 18].

Regarding the theory of fixed points in metric spaces endowed with a graph, this research area was initiated by Jachymski [13] and Gwóźdź-Lukawska, Jachymski [11]. Other results for single-valued and multivalued operators in such metric spaces were given by Beg et al. [1], Chifu and Petruşel [6], [7], Dehkordi and Ghods [9].

*Corresponding author

Email addresses: cristian.chifu@tbs.ubbcluj.ro (Cristian Chifu), gabi.petrusel@tbs.ubbcluj.ro (Gabriela Petruşel)

Received: June 2017 *Revised:* October 2017

The purpose of this paper is to generalize some of these results, in special those from [7], using the context of two b-metrics spaces endowed with a directed graph.

In what follow we shall recall some essential definitions and results which will be useful throughout this paper.

Definition 1.1. ([8]) Let X be a nonempty set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to [0, \infty)$ is said to be a b-metric with constant s, if all axioms of the metric space take place with the following modification of the triangle axiom:

$$d(x,z) \leq s[d(x,y) + d(y,z)], \text{ for all } x,y,z \in X.$$

In this case the pair (X,d) is called a b-metric space with constant s.

Remark 1.2. Since a b-metric space is a metric space when s=1, the class of b-metric spaces is larger than the class of metric spaces. For more details and examples on b-metric spaces, see e.g. [4].

Example 1.3. Let $X = \mathbb{R}_+$ and $d : X \times X \to \mathbb{R}_+$ such that $d(x,y) = |x-y|^p, p > 1$. It's easy to see that d is a b-metric with $s = 2^p$, but is not a metric.

Let (X,d) and (Y,ρ) be two b-metric spaces, with the same constant $s \geq 1$, and let $Z = X \times Y$. Let us consider the functional $\widetilde{d} : Z \times Z \to [0,\infty)$, defined by

$$\widetilde{d}((x,y),(u,v)) = d(x,u) + \rho(y,v), \text{ for all } (x,y),(u,v) \in Z. \tag{1.1}$$

Lemma 1.4. *If (X,d) and (Y,ρ) are two complete b-metric spaces, with the same constant $s \geq 1$, then $\widetilde{d}$ is a b-metric on $Z = X \times Y$, with the same constant $s \geq 1$, and $\left(Z, \widetilde{d}\right)$ is a complete b-metric space.*

Definition 1.5. A mapping $\varphi : [0,\infty) \to [0,\infty)$ is called a comparison function if it is increasing and $\varphi^n(t) \to 0$, as $n \to \infty$, for any $t \in [0,\infty)$.

Lemma 1.6. ([2]) If $\varphi : [0,\infty) \to [0,\infty)$ is a comparison function, then:

(1) each iterate φ^k of φ, $k \geq 1$, is also a comparison function;

(2) φ is continuous at 0;

(3) $\varphi(t) < t$, for any $t > 0$.

In 1997, V. Berinde [2] introduced the concept of (c)-*comparison function* as follows:

Definition 1.7. ([2]) A function $\varphi : [0,\infty) \to [0,\infty)$ is said to be a (c)-comparison function if

(1) φ is increasing;
(2) there exists $k_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $\varphi^{k+1}(t) \leq a\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0,\infty)$.

In order to give some fixed point results to the class of b-metric spaces, the notion of a (c)-comparison function was extended to (b)-comparison function by V. Berinde [3].

Definition 1.8. ([3]) Let $s \geq 1$ be a real number. A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a (b)-comparison function if the following conditions are fulfilled

(1) φ is monotone increasing;

(2) there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\varphi^{k+1}(t) \leq as^k\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.

It is obvious that the concept of (b)-comparison function reduces to that of (c)-comparison function when $s = 1$.

The following lemma is very important in the proof of our results.

Lemma 1.9. ([4]) If $\varphi : [0, \infty) \to [0, \infty)$ is a (b)-comparison function, then we have the following conclusions:

(1) the series $\sum_{k=0}^{\infty} s^k\varphi^k(t)$ converges for any $t \in [0, \infty)$;
(2) the function $S_b : [0, \infty) \to [0, \infty)$ defined by $S_b(t) = \sum_{k=0}^{\infty} s^k\varphi^k(t),\ t \in [0, \infty)$, is increasing and continuous at 0.

Due to the above lemma, any (b)-comparison function is a comparison function.

Let (X, d) be a b-metric space and Δ be the diagonal of $X \times X$. Let G be a directed graph, such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, where $E(G)$ is the set of the edges of the graph. Assume also that G has no parallel edges and, thus, G can be identified with the pair $(V(G), E(G))$.

Definition 1.10. We say that G has the transitivity property if and only if, for all $x, y, z \in X$,

$$(x, z) \in E(G), (z, y) \in E(G) \Rightarrow (x, y) \in E(G).$$

Let us denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus,

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

Remark 1.11. If G has the transitivity property, then G^{-1} has the same property.

Throughout the paper we shall say that G with the above mentioned properties *satisfies standard conditions.*

Definition 1.12. ([5]) Let (X, d) be a b-metric space, with constant $s \geq 1$, and G be a directed graph. We say that the triple (X, d, G) has the property (A_1), if for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to x,$ as $n \to \infty$, and $(x_n, x_{n+1}) \in E(G)$, for $n \in \mathbb{N}$, we have that $(x_n, x) \in E(G)$.

Definition 1.13. ([5]) Let (X, d) be a b-metric space, with constant $s \geq 1$, and G be a directed graph. We say that the triple (X, d, G) has the property (A_2) if for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to x,$ as $n \to \infty$, and $(x_n, x_{n+1}) \in E(G^{-1})$, for $n \in \mathbb{N}$, we have that $(x_n, x) \in E(G^{-1})$.

2. Existence and uniqueness results

Let (X,d) be a b-metric space with constant $s \geq 1$, endowed with a directed graph G_1 satisfying the standard conditions, and let (Y,ρ) be a b-metric space, with the same constant $s \geq 1$, endowed with a directed graph G_2, also satisfying the standard conditions.

We shall consider a graph G on $X \times Y$ such that

$$((x,y),(u,v)) \in E(G) \Leftrightarrow (x,u) \in E(G_1), (y,v) \in E\left(G_2^{-1}\right).$$

Let $F_1 : X \times Y \to X$ and $F_2 : X \times Y \to Y$ be two operators.

Throughout the paper the following notations will be used: $Z := X \times Y$ and $F := (F_1, F_2) : Z \to Z, F(x,y) = (F_1(x,y), F_2(x,y))$, for all $(x,y) \in Z$.

Definition 2.1. We say that the operator F has the property (P) if:

(i) $x,u \in X$ such that $(x,u) \in E(G_1)$, then

$$(F_1(x,y), F_1(u,y)) \in E(G_1), (F_2(x,y), F_2(u,y)) \in E\left(G_2^{-1}\right), \forall y \in Y.$$

(ii) $y,v \in Y$ such that $(y,v) \in E\left(G_2^{-1}\right)$, then

$$(F_1(x,y), F_1(x,v)) \in E(G_1), (F_2(x,y), F_2(x,v)) \in E\left(G_2^{-1}\right), \forall x \in X.$$

Proposition 2.2. *If the operator F has the property (P), then if $x,u \in X$ and $y,v \in Y$ are such that $((x,y),(u,v)) \in E(G)$, then*

$$((F_1(x,y), F_2(x,y)), (F_1(u,v), F_2(u,v))) \in E(G),$$

or

$$(F(x,y), F(u,v)) \in E(G).$$

Proof . If $((x,y),(u,v)) \in E(G)$, then $(x,u) \in E(G_1), (y,v) \in E\left(G_2^{-1}\right)$.

If $(x,u) \in E(G_1)$, from property (P) we have

$$(F_1(x,y), F_1(u,y)) \in E(G_1), \tag{2.1}$$

$$(F_2(x,y), F_2(u,y)) \in E\left(G_2^{-1}\right), \forall y \in Y. \tag{2.2}$$

If $(y,v) \in E\left(G_2^{-1}\right)$, from property (P) we have that

$$(F_1(x,y), F_1(x,v)) \in E(G_1), \tag{2.3}$$

$$(F_2(x,y), F_2(x,v)) \in E\left(G_2^{-1}\right), \forall x \in X. \tag{2.4}$$

Considering $x = u$ in (2.3), then $(F_1(u,y), F_1(u,v)) \in E(G_1)$. Now from (2.1) and the transitivity of G_1 we have

$$(F_1(x,y), F_1(u,v)) \in E(G_1). \tag{2.5}$$

In we consider $y = v$ in (2.2), then $(F_2(x,v), F_2(u,v)) \in E\left(G_2^{-1}\right)$. From (2.4) and the transitivity of G_2^{-1} we have

$$(F_2(x,y), F_2(u,v)) \in E\left(G_2^{-1}\right). \tag{2.6}$$

From (2.5) and (2.6) we obtain

$$((F_1(x,y), F_2(x,y)), (F_1(u,v), F_2(u,v))) \in E(G).$$

□

Proposition 2.3. *If the operator F has property (P), then if $x,u\in X$ and $y,v\in Y$ are such that $((x,y),(u,v))\in E(G)$, then*

$$(F^n(x,y),F^n(u,v))\in E(G).$$

Proof . From Proposition 2.2 we have that if $x,u\in X$ and $y,v\in Y$ are such that $((x,y),(u,v))\in E(G)$, then (2.5) and (2.6) take place. Using these relations and the fact that $F=(F_1,F_2)$ has property (P), we obtain:
For $(x,u)\in E(G_1)$,

$$(F_1(F_1(x,y),y_1),F_1(F_1(u,v),y_1))\in E(G_1) \tag{2.7}$$

$$(F_2(F_1(x,y),y_1),F_2(F_1(u,v),y_1))\in E\left(G_2^{-1}\right),\forall y_1\in Y. \tag{2.8}$$

For $(y,v)\in E\left(G_2^{-1}\right)$,

$$(F_1(x_1,F_2(x,y)),F_1(x_1,F_2(u,v)))\in E(G_1) \tag{2.9}$$

$$(F_2(x_1,F_2(x,y)),F_2(x_1,F_2(u,v)))\in E\left(G_2^{-1}\right),\forall x_1\in X. \tag{2.10}$$

If in (2.9) we consider $x_1=F_1(u,v)$ and in (2.7) we consider $y_1=F_2(x,y)$, then we shall have

$$(F_1(F_1(u,v),F_2(x,y)),F_1(F_1(u,v),F_2(u,v)))\in E(G_1) \tag{2.11}$$

$$(F_1(F_1(x,y),F_2(x,y)),F_1(F_1(u,v),F_2(x,y)))\in E(G_1). \tag{2.12}$$

From (2.11) and (2.12), using the transitivity of G_1 we obtain

$$(F_1(F_1(x,y),F_2(x,y)),F_1(F_1(u,v),F_2(u,v)))\in E(G_1). \tag{2.13}$$

In the same way we shall obtain

$$(F_2(F_1(x,y),F_2(x,y)),F_2(F_1(u,v),F_2(u,v)))\in E\left(G_2^{-1}\right). \tag{2.14}$$

(2.13) and (2.14) are equivalent with

$$(F_1(F(x,y)),F_1(F(u,v)))\in E(G_1) \tag{2.15}$$

$$(F_2(F(x,y)),F_2(F(u,v)))\in E\left(G_2^{-1}\right). \tag{2.16}$$

From (2.15) and (2.16), using Proposition 2.2, we have

$$\left(F^2(x,y),F^2(u,v)\right)\in E(G).$$

By induction we reach the conclusion. □

Let us consider the set denoted by Z^F and defined as:

$$Z^F=\left\{(x,y)\in Z:(x,F_1(x,y))\in E(G_1)\ \text{ and }\ (y,F_2(x,y))\in E\left(G_2^{-1}\right)\right\}.$$

Consider the sequence $(x_n)_{n\in\mathbb{N}}$ in X and $(y_n)_{n\in\mathbb{N}}$ in Y defined by

$$x_{n+1}=F_1(x_n,y_n),\quad y_{n+1}=F_2(x_n,y_n),\ \text{for all } n\in\mathbb{N}. \tag{2.17}$$

Proposition 2.4. *Suppose that the operator F has property (P) and $(x_0,y_0)\in Z^F$. Then for any sequence $(z_n)_{n\in\mathbb{N}}$, $z_n=(x_n,y_n)$ in Z, with $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ defined as above, we have $(z_n,z_{n+1})\in E(G)$, for all $n\in\mathbb{N}$.*

Proof . From the fact that $(x_0, y_0) \in Z^F$ it follows that $(x_0, F_1(x_0, y_0)) \in E(G_1)$ and $(y_0, F_2(x_0, y_0)) \in E\left(G_2^{-1}\right)$ which is equivalent with $(x_0, x_1) \in E(G_1)$ and $(y_0, y_1) \in E\left(G_2^{-1}\right)$.

Now, from Proposition 2.2 we have

$$\begin{aligned} &(F_1(x_0, y_0), F_1(x_1, y_1)) \in E(G_1), \\ &(F_2(x_0, y_0), F_2(x_1, y_1)) \in E\left(G_2^{-1}\right), \end{aligned}$$

which is equivalent with $(x_1, x_2) \in E(G_1)$ and $(y_1, y_2) \in E\left(G_2^{-1}\right)$.

By induction we shall obtain that $(x_n, x_{n+1}) \in E(G_1)$ and $(y_n, y_{n+1}) \in E\left(G_2^{-1}\right)$ which is equivalent with $((x_n, y_n), (x_{n+1}, y_{n+1})) \in E(G)$, i.e. $(z_n, z_{n+1}) \in E(G)$. □

Remark 2.5. It can be proved that $x_n = F_1^n(x_0, y_0)$ and $y_n = F_2^n(x_0, y_0)$ and thus, $z_n = F^n(z_0)$, for all $n \in \mathbb{N}$, where $z_0 = (x_0, y_0)$.

Definition 2.6. The operator $F = (F_1, F_2) : Z \to Z$ is called (φ, G)-contraction of type (b) if:

i. F has property (P);
ii. there exists $\varphi : [0, \infty) \to [0, \infty)$ a (b)-comparison function such that

$$\begin{aligned} &d(F_1(x, y), F_1(u, v)) + \rho(F_2(x, y), F_2(u, v)) \leq \varphi(d(x, u) + \rho(y, v)), \\ &\text{for all } (x, u) \in E(G_1), (y, v) \in E(G_2^{-1}). \end{aligned}$$

In what follows we shall consider the $b-$metric $\widetilde{d}$ defined by (1.1).

Lemma 2.7. *Let (X, d) be a b-metric space, with constant $s \geq 1$, endowed with a directed graph G_1 satisfying the standard conditions and (Y, ρ) be a b-metric space, with the same constant $s \geq 1$, endowed with a directed graph G_2 also satisfying the standard conditions. Let $F : Z \to Z$ be a (φ, G)-contraction of type (b). Consider the sequence $(z_n)_{n\in\mathbb{N}}$ as above. Then, if $(x_0, y_0) \in Z^F$, there exists $r(x_0, y_0) \geq 0$ such that*

$$\widetilde{d}(z_n, z_{n+1}) \leq \varphi^n(r(x_0, y_0)), \text{ for all } n \in \mathbb{N}.$$

Proof . Let $(x_0, y_0) \in Z^F$. From Proposition 2.3 we have that $(z_n, z_{n+1}) \in E(G)$ which is $(x_n, x_{n+1}) \in E(G_1)$ and $(y_n, y_{n+1}) \in E\left(G_2^{-1}\right)$ for all $n \in \mathbb{N}$.

Since F is a (φ, G)-contraction of type (b), we shall obtain

$$\begin{aligned} \widetilde{d}(z_n, z_{n+1}) &= d(F_1(x_{n-1}, y_{n-1}), F_1(x_n, y_n)) + \rho(F_2(x_{n-1}, y_{n-1}), F_2(x_n, y_n)) \\ &\leq \varphi(d(F_1(x_{n-2}, y_{n-2}), F_1(x_{n-1}, y_{n-1})) + \rho(F_2(x_{n-2}, y_{n-2}), F_2(x_{n-1}, y_{n-1}))) \\ &\leq \ldots \leq \varphi^n(d(x_0, x_1) + \rho(y_0, y_1)) = \varphi^n(d(x_0, F_1(x_0, y_0)) + \rho(y_0, F_2(x_0, y_0))). \end{aligned}$$

If we consider $r(x_0, y_0) := d(x_0, F_1(x_0, y_0)) + \rho(y_0, F_2(x_0, y_0))$, then

$$\widetilde{d}(z_n, z_{n+1}) \leq \varphi^n(r(x_0, y_0)), \text{ for all } n \in \mathbb{N}.$$

□

Lemma 2.8. *Let (X, d) be a complete b-metric space, with constant $s \geq 1$, endowed with a directed graph G_1 satisfying the standard conditions and (Y, ρ) be a complete b-metric space, with the same constant $s \geq 1$, endowed with a directed graph G_2 also satisfying the standard conditions. Let $F : Z \to Z$ be a (φ, G)-contraction of type (b). Consider the sequence $(z_n)_{n\in\mathbb{N}}$ as above. Then, if $(x_0, y_0) \in Z^F$, there exists $z^* = (x^*, y^*) \in Z$, such that $(z_n)_{n\in\mathbb{N}}$ converges to z^*, as $n \to \infty$.*

Proof . Let $(x_0, y_0) \in Z^F$. From Lemma 2.7 we know that

$$\widetilde{d}(z_n, z_{n+1}) \leq \varphi^n(r(x_0, y_0)), \text{ for all } n \in \mathbb{N}.$$

Now we shall prove that $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. We have

$$\begin{aligned}\widetilde{d}(z_n, z_{n+p}) &\leq s\widetilde{d}(z_n, z_{n+1}) + s^2\widetilde{d}(z_{n+1}, z_{n+2}) + \cdots + s^{p-1}\widetilde{d}(z_{n+p-2}, z_{n+p-1}) \\ &+ s^{p-1}\widetilde{d}(z_{n+p-1}, z_{n+p}) \leq s\varphi^n(r(x_0, y_0)) + s^2\varphi^{n+1}(r(x_0, y_0)) \\ &+ \cdots + s^{p-1}\varphi^{n+p-2}(r(x_0, y_0)) + s^p\varphi^{n+p-1}(r(x_0, y_0)) \\ &\leq \frac{1}{s^{n-1}}\sum_{k=n}^{n+p-1} s^k\varphi^k(r(x_0, y_0)).\end{aligned}$$

Let $S_n = \sum_{k=0}^{n} s^k\varphi^k(r(x_0, y_0))$. Hence we have

$$\widetilde{d}(z_n, z_{n+p}) \leq \frac{1}{s^{n-1}}(S_{n+p-1} - S_{n-1}) \leq \frac{1}{s^{n-1}}\sum_{k=0}^{\infty} s^k\varphi^k(r(x_0, y_0)).$$

From Lemma 1.9 we have that the series is convergent. In this way, we shall obtain

$$\widetilde{d}(z_n, z_{n+p}) \leq \frac{1}{s^{n-1}}\sum_{k=0}^{\infty} s^k\varphi^k(r(x_0, y_0)) \to 0, \text{ as } n \to \infty.$$

In conclusion the sequence (z_n) is a Cauchy sequence. Since $\left(Z, \widetilde{d}\right)$ is a complete $b-$metric, there exists $z^* \in Z$, such that $z_n \to z^*$, as $n \to \infty$. □

Remark 2.9. $z_n \to z^*$ means that there exist $x^* \in X$ and $y^* \in Y$ such that $x_n \to x^*$ and $y_n \to y^*$, as $n \to \infty$.

Let us now consider the following operator equation system

$$\begin{cases} x = F_1(x, y) \\ y = F_2(x, y) \end{cases}. \tag{2.18}$$

Theorem 2.10. *Let (X, d) be a complete b-metric space, with constant $s \geq 1$, endowed with a directed graph G_1 satisfying the standard conditions and (Y, ρ) be a complete b-metric space, with the same constant $s \geq 1$, endowed with a directed graph G_2 also satisfying the standard conditions. Let $F : Z \to Z$ be a (φ, G)-contraction of type (b). Suppose that the triple (X, d, G_1) has property (A_1) and the triple (Y, ρ, G_2) has property (A_2). If there exists $(x_0, y_0) \in Z^F$, then the system (2.18) has at least one solution.*

Proof . From Lemma 2.8, there exists $z^* \in Z$, such that $z_n \to z^*$, as $n \to \infty$. We shall prove that $F(z^*) = z^*$. From Remark 2.9, we have that $x^* \in X$ and $y^* \in Y$ such that $z^* = (x^*, y^*) \in Z$,

$$\begin{aligned}\widetilde{d}(z^*, F(z^*)) &= d(x^*, F_1(x^*, y^*)) + \rho(y^*, F_2(x^*, y^*)) \leq s[d(x^*, x_{n+1}) + \rho(y^*, y_{n+1})] \\ &+ s[d(F_1(x_n, y_n), F_1(x^*, y^*)) + \rho(F_2(x_n, y_n), F_2(x^*, y^*))] \\ &\leq s[d(x^*, x_{n+1}) + \rho(y^*, y_{n+1})] + s\varphi(d(x_n, x^*) + \rho(y_n, y^*)) \to 0, \text{ as } n \to \infty.\end{aligned}$$

Hence $F(z^*) = z^*$, i.e.,

$$\begin{cases} x^* = F_1(x^*, y^*) \\ y^* = F_2(x^*, y^*) \end{cases}.$$

□

Let us suppose now that for every $(x,y),(u,v) \in Z$, there exists $(t,w) \in Z$ such that

$$(x,t) \in E(G_1), (y,w) \in E(G_2^{-1}), \quad (u,t) \in E(G_1), (v,w) \in E(G_2^{-1}). \tag{2.19}$$

Theorem 2.11. *Adding the condition* (2.19) *to the hypotheses of Theorem* 2.10*, we obtain the uniqueness of the solution of the system* (2.18)*.*

Proof . Let us suppose that there exist $(x^*,y^*),(u^*,v^*) \in Z$ two solutions of the system (2.18). From (2.19) we have that there exists $(z,w) \in Z$ such that

$$\begin{aligned} &(x^*,z) \in E(G_1), (y^*,w) \in E(G_2^{-1}),\\ &(u^*,z) \in E(G_1), (v^*,w) \in E(G_2^{-1}). \end{aligned}$$

Using Lemma 2.7 we shall have

$$\begin{aligned} d(x^*,u^*) + \rho(y^*,v^*) &= d(F_1^n(x^*,y^*), F_1^n(u^*,v^*)) + \rho(F_2^n(x^*,y^*), F_2^n(u^*,v^*)) \\ &\le s\,[d(F_1^n(x^*,y^*), F_1^n(z,w)) + \rho(F_2^n(x^*,y^*), F_2^n(z,w))] + \\ &\quad + s\,[d(F_1^n(z,w), F_1^n(u^*,v^*)) + \rho(F_2^n(z,w), F_2^n(u^*,v^*))] \\ &\le s\,[\varphi^n(d(x^*,z) + \rho(y^*,w)) + \varphi^n d(u^*,z) + \rho(v^*,w))] \to 0, \text{ as } n \to \infty. \end{aligned}$$

Hence $d(x^*,u^*) + \rho(y^*,v^*) = 0$ and thus we obtain that $x^* = u^*$ and $y^* = v^*$. □

Theorem 2.12. *Let (X,d) be a complete b-metric space, with constant $s \ge 1$, endowed with a directed graph G_1 satisfying the standard conditions and (Y,ρ) be a complete b-metric space, with the same constant $s \ge 1$, endowed with a directed graph G_2 also satisfying the standard conditions. Let us consider $F = (F_1,F_2) : Z \to Z, H = (H_1,H_2) : Z \to Z$ two operators. Suppose that*

(i) F *satisfies the conditions from Theorem* 2.11;

(ii) there exists at least $(u^*,v^*) \in Z$ such that

$$H(u^*,v^*) = (u^*,v^*) \text{ and } (x^*,u^*) \in E(G_1), (y^*,v^*) \in E\left(G_2^{-1}\right),$$

where (x^*,y^*) is a unique solution of the system (2.18)*.*

(iii) there exist $\eta_1, \eta_2 > 0$, such that

$$\begin{aligned} d(F_1(x,y), H_1(x,y)) &\le \eta_1, \\ \rho(F_2(x,y), H_2(x,y)) &\le \eta_2. \end{aligned}$$

(iv) $t - s\varphi(t) \ge 0$*, for all* $t \ge 0$ *and* $\lim_{t\to\infty}(t - s\varphi(t)) = \infty$.

In these conditions we have the following estimation:

$$d(x^*,u^*) + \rho(y^*,v^*) \le \sup\{t \ge 0 |\, t - s\varphi(t) \le s(\eta_1 + \eta_2)\}.$$

Proof . From (i) there exists a unique pair $(x^*, y^*) \in Z$ such that $F(x^*, y^*) = (x^*, y^*)$. Let $(u^*, v^*) \in Z$ such that $H(u^*, v^*) = (u^*, v^*)$.

$$\begin{aligned} d(x^*, u^*) + \rho(y^*, v^*) &= d\left(F_1(x^*, y^*), H_1(u^*, v^*)\right) + \rho\left(F_2(x^*, y^*), H_2(u^*, v^*)\right) \\ &\leq s\left[d\left(F_1(x^*, y^*), F_1(u^*, v^*)\right) + d\left(F_1(u^*, v^*), H_1(u^*, v^*)\right)\right] \\ &\quad + s\left[\rho\left(F_2(x^*, y^*), F_2(u^*, v^*)\right) + \rho\left(F_2(u^*, v^*), H_2(u^*, v^*)\right)\right] \\ &\leq s\varphi\left(d(x^*, u^*) + \rho(y^*, v^*)\right) + s\left(\eta_1 + \eta_2\right). \end{aligned}$$

Hence

$$d(x^*, u^*) + \rho(y^*, v^*) - s\varphi\left(d(x^*, u^*) + \rho(y^*, v^*)\right) \leq s\left(\eta_1 + \eta_2\right).$$

Finally, we obtain that

$$d(x^*, u^*) + \rho(y^*, v^*) \leq \sup\left\{t \geq 0 | \, t - s\varphi(t) \leq s\left(\eta_1 + \eta_2\right)\right\}.$$

□

3. Well-posedness and Ulam-Hyers stability

Let us consider the operator equation system (2.18)

$$\begin{cases} x = F_1(x, y) \\ y = F_2(x, y) \end{cases}.$$

Definition 3.1. By definition, the operator equation system (2.18) is said to be well-posed if:

(i) there exists a unique pair $(x^*, y^*) \in Z$ such that

$$\begin{cases} x^* = F_1(x^*, y^*) \\ y^* = F_2(x^*, y^*) \end{cases}.$$

(ii) for any sequence $(x_n, y_n)_{n\in\mathbb{N}} \in Z$ for which

$$d\left(x_n, F_1(x_n, y_n)\right) \to 0, \qquad \rho\left(y_n, F_2(x_n, y_n)\right) \to 0$$

as $n \to \infty$, we have that $x_n \to x^*$ and $y_n \to y^*$, as $n \to \infty$.

Theorem 3.2. *Suppose that all the hypotheses of Theorem 2.11 holds. If the $(b)-$comparison function $\varphi : [0, \infty) \to [0, \infty)$ is such that $\varphi(t) < \frac{t}{s}, \forall t > 0$ and for any sequence $(x_n, y_n)_{n\in\mathbb{N}} \in Z$ for which*

$$d\left(x_n, F_1(x_n, y_n)\right) \to 0, \qquad \rho\left(y_n, F_2(x_n, y_n)\right) \to 0$$

as $n \to \infty$, we have that $(x_n, x^*) \in E(G_1)$ and $(y_n, y^*) \in E\left(G_2^{-1}\right)$, then the operator equation system (2.18) is well-posed.

Proof . From Theorem 2.11 we obtain that there exists a unique pair $(x^*, y^*) \in Z$ such that

$$\begin{cases} x^* = F_1(x^*, y^*) \\ y^* = F_2(x^*, y^*) \end{cases}.$$

Let $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence in Z such that $d\left(x_n, F_1(x_n, y_n)\right) \to 0$ and $\rho\left(y_n, F_2(x_n, y_n)\right) \to 0$ as $n \to \infty$. In this way we have that $(x_n, x^*) \in E(G_1)$ and $(y_n, y^*) \in E\left(G_2^{-1}\right)$.

It follows that

$$\begin{aligned} d(x_n,x^*)+\rho(y_n,y^*) &\le s\left[d(x_n,F_1(x_n,y_n))+d(F_1(x_n,y_n),x^*)\right]+ \\ &\quad + s\left[\rho(y_n,F_2(x_n,y_n))+\rho(F_2(x_n,y_n),y^*)\right] \\ &= s\left[d(F_1(x_n,y_n),F_1(x^*,y^*))+\rho(F_2(x_n,y_n),F_2(x^*,y^*))\right] \\ &\quad + s\left[d(x_n,F_1(x_n,y_n))+\rho(y_n,F_2(x_n,y_n))\right] \\ &\le s\varphi\left(d(x_n,x^*)+\rho(y_n,y^*)\right)+s\left[d(x_n,F_1(x_n,y_n))+\rho(y_n,F_2(x_n,y_n))\right]. \end{aligned}$$

Hence we have the following inequality

$$\begin{aligned} d(x_n,x^*)+\rho(y_n,y^*) &\le s\varphi\left(d(x_n,x^*)+\rho(y_n,y^*)\right) \\ &\quad + s\left(d(x_n,F_1(x_n,y_n))+\rho(y_n,F_2(x_n,y_n))\right). \end{aligned} \tag{3.1}$$

Suppose that there exists $\delta>0$ such that $d(x_n,x^*)+\rho(y_n,y^*)\to\delta$, as $n\to\infty$. If in (3.1), $n\to\infty$, we shall have

$$\delta \le s\varphi(\delta) < \delta,$$

which is a contradiction. Thus, $\delta=0$ and hence $d(x_n,x^*)+\rho(y_n,y^*)\to 0$, as $n\to\infty$. From here we obtain the conclusion. □

Definition 3.3. By definition, the operator equation system (2.18) is said to be generalized Ulam-Hyers stable if and only if there exists $\psi:\mathbb{R}_+^2\to\mathbb{R}_+$, increasing, continuous in 0 with $\psi(0,0)=0$, such that for each $\varepsilon_1,\varepsilon_2>0$ and for each solution $(\overline{x},\overline{y})\in Z$ of the inequality system

$$\begin{cases} d(x,F_1(x,y))\le\varepsilon_1 \\ \rho(y,F_2(x,y))\le\varepsilon_2 \end{cases},$$

there exists a solution $(x^*,y^*)\in Z$ of the operator equation system (2.18) such that

$$d(\overline{x},x^*)+\rho(\overline{y},y^*)\le\psi(\varepsilon_1,\varepsilon_2). \tag{3.2}$$

Theorem 3.4. *Suppose that all the hypotheses of Theorem* 2.11 *holds and the* $(b)-$*comparison function* φ *is such that* $\varphi(t)<\frac{t}{s},\forall t>0$. *If there exists a function* $\beta:[0,\infty)\to[0,\infty)$, $\beta(r):=r-s\varphi(r)$ *strictly increasing and onto, then the operator equation system* (2.18) *is Ulam-Hyers stable.*

Proof . From Theorem 3.2 we obtain that there exists a unique pair $(x^*,y^*)\in Z$ such that

$$\begin{cases} x^*=F_1(x^*,y^*) \\ y^*=F_2(x^*,y^*) \end{cases}.$$

Let $\varepsilon_1,\varepsilon_2>0$ and let $(\overline{x},\overline{y})\in Z$ such that

$$\begin{cases} d(\bar{x},F_1(\overline{x},\overline{y}))\le\varepsilon_1 \\ \rho(\bar{y},F_2(\overline{x},\overline{y}))\le\varepsilon_2 \end{cases},$$

where $(\overline{x},x^*)\in E(G_1)$, $(\overline{y},y^*)\in E\left(G_2^{-1}\right)$. We have

$$\begin{aligned} d(\overline{x},x^*)+\rho(\overline{y},y^*) &= d(\overline{x},F_1(x^*,y^*))+\rho(\overline{y},F_2(x^*,y^*)) \\ &\le s\left[d(\bar{x},F_1(\overline{x},\overline{y}))+\rho(\bar{y},F_2(\overline{x},\overline{y}))\right] \\ &\quad + s\left[d(F_1(\overline{x},\overline{y}),F_1(x^*,y^*))+\rho(F_2(\overline{x},\overline{y}),F_2(x^*,y^*))\right] \\ &\le s(\varepsilon_1+\varepsilon_2)+s\varphi\left(d(\overline{x},x^*)+\rho(\overline{y},y^*)\right). \end{aligned}$$

Hence, we have

$$d(\overline{x},x^*)+\rho(\overline{y},y^*)-s\varphi(d(\overline{x},x^*)+\rho(\overline{y},y^*))\leq s(\varepsilon_1+\varepsilon_2),$$

which is

$$\beta(d(\overline{x},x^*)+\rho(\overline{y},y^*))\leq s(\varepsilon_1+\varepsilon_2).$$

Hence

$$d(\overline{x},x^*)+\rho(\overline{y},y^*)\leq \beta^{-1}(s(\varepsilon_1+\varepsilon_2)).$$

Follows that the operator equation system (2.18) is Ulam-Hyers stable, where

$$\psi(\varepsilon_1,\varepsilon_2)=\beta^{-1}(s(\varepsilon_1+\varepsilon_2)).$$

□

4. An application

In what follows we shall give an application for Theorem 2.10. Let us consider the following problem:

$$\left\{\begin{array}{c} x''(t)=f(t,x(t),y(t)) \\ y''(t)=g(t,x(t),y(t)) \\ x(0)=x'(1)=y(0)=y'(1) \end{array}\right. ,t\in[0,1]. \tag{4.1}$$

Notice now that the problem (4.1) is equivalent with the following integral system

$$\left\{\begin{array}{l} x(t)=\int_0^1 K(t,s)\,f(s,x(s),y(s))ds \\ y(t)=\int_0^1 K(t,s)\,g(s,x(s),y(s))ds \end{array}\right. ,t\in[0,1], \tag{4.2}$$

where

$$K(t,s)=\left\{\begin{array}{l} t,t\leq s \\ s,t>s \end{array}\right.$$

The purpose of this section is to give existence results for the solution of the system (4.2), using Theorem 2.10.

Let us consider $X:=C([0,1],\mathbb{R}^n)$ endowed with the following b-metric with $s=2^p, p>1$,

$$d(x,y)=\max_{t\in[0,1]}|x(t)-y(t)|^p.$$

Let $Y:=C([0,1],\mathbb{R}^m)$ endowed with the following b-metric with $s=2^q, q>1$,

$$\rho(x,y)=\max_{t\in[0,1]}|x(t)-y(t)|^q.$$

Suppose that $p<q$. Consider also the graphs G_1 and G_2 defined by the partial order relation, i.e.,

$$G_1: x,u\in X, x\leq u\Leftrightarrow x(t)\leq u(t),\text{ for any } t\in[0,1],$$
$$G_2: y,v\in Y, y\leq v\Leftrightarrow y(t)\leq v(t),\text{ for any } t\in[0,1].$$

Hence (X,d) is a complete $b-$metric space endowed with a directed graph G_1 and (Y,ρ) is a complete $b-$metric space endowed with a directed graph G_2.

If we consider $E(G_1)=\{(x,u)\in X\times X: x\leq u\}$ and $E(G_2)=\{(y,v)\in Y\times Y: y\leq v\}$, then the diagonal Δ_1 of $X\times X$ is included in $E(G_1)$ and the diagonal Δ_2 of $Y\times Y$ is included in $E(G_2)$. On the other hand $E(G_1^{-1})=\{(x,u)\in X\times X: u\leq x\}$ and $E(G_2^{-1})=\{(y,v)\in Y\times Y: v\leq y\}$.

Moreover (X,d,G_1) has the property (A_1) and (Y,ρ,G_2) has the property (A_2). In this case $Z^F=\{(x,y)\in Z: x\leq F_1(x,y) \text{ and } F_2(x,y)\leq y\}$ where $Z=X\times Y$.

Theorem 4.1. *Consider the system* (4.1). *Suppose:*

(i) $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $g : [0,1] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ *are continuous;*

(ii) for all $x, u \in \mathbb{R}^n$ *with* $x \leq u$ *we have* $f(t,x,y) \leq f(t,u,y)$ *and* $g(t,x,y) \geq g(t,u,y)$, *for all* $y \in \mathbb{R}^m$ *and* $t \in [0,1]$;

(iii) for all $y, v \in \mathbb{R}^m$ *with* $v \leq y$ *we have* $f(t,x,y) \leq f(t,x,v)$ *and* $g(t,x,y) \geq g(t,x,v)$, *for all* $x \in \mathbb{R}^n$ *and* $t \in [0,1]$;

(iv) there exists $\widetilde{\varphi}, \widetilde{\psi} : [0,\infty) \to [0,\infty)$, (b)*-comparison functions and* $\alpha, \beta, \gamma, \delta \in (0,\infty)$, *with* $\max\{\alpha, \beta\} < 1$, *and* $\max\{\gamma, \delta\} < 1$ *such that*

$$\begin{aligned} &(f(t,x,y) - f(t,u,v))^p \leq \widetilde{\varphi}\left(\alpha\,|x-u|^p + \beta\,|y-v|^p\right), \\ &\qquad \textit{for each } t \in [0,1], x, u \in \mathbb{R}^n, y, v \in \mathbb{R}^m, x \leq u, v \leq y. \\ &|g(t,x,y) - g(t,u,v)|^q \leq \widetilde{\psi}\left(\gamma\,|x-u|^q + \delta\,|y-v|^q\right), \\ &\qquad \textit{for each } t \in [0,1], x, u \in \mathbb{R}^n, y, v \in \mathbb{R}^m, x \leq u, v \leq y. \end{aligned}$$

(v) there exists $(x_0, y_0) \in X \times Y$ *such that*

$$\begin{array}{l} x_0(t) \leq \int_0^1 K(t,s)\, f(s, x_0(s), y_0(s))ds \\ y_0(t) \geq \int_0^1 K(t,s)\, g(s, x_0(s), y_0(s))ds \end{array}, t \in [0,1].$$

Then, there exists a unique solution of the integral system (4.2).

Proof . Let $F_1 : Z \to X$, and $F_2 : Z \to Y$, defined as

$$F_1(x,y)(t) = \int_0^1 K(t,s)\, f(s, x(s), y(s))ds, t \in [0,1],$$

$$F_2(x,y)(t) = \int_0^1 K(t,s)\, g(s, x(s), y(s))ds, t \in [0,1]$$

In this way, the system (4.2) can be written as

$$\left\{ \begin{array}{l} x = F_1(x,y) \\ y = F_2(x,y) \end{array} \right. . \tag{4.3}$$

It can be seen, from (4.3), that a solution of this system is a coupled fixed point of the mapping F. We shall verify if the conditions of Theorem 2.10 are fulfilled.

Let $x, u \in X$ such that $x \leq u$.

$$\begin{aligned} F_1(x,y)(t) &= \int_0^1 K(t,s)\, f(s, x(s), y(s))ds \leq \int_0^1 K(t,s)\, f(s, u(s), y(s))ds \\ &= F_1(u,y)(t), \text{ for each } y \in \mathbb{R}^m, t \in [0,1]. \\ F_2(x,y)(t) &= \int_0^1 K(t,s)\, g(s, x(s), y(s))ds \geq \int_0^1 K(t,s)\, g(s, u(s), y(s))ds \\ &= F_2(u,y)(t), \text{ for each } y \in \mathbb{R}^m, t \in [0,1]. \end{aligned} \tag{4.4}$$

Let now $y, v \in Y$ such that $v \leq y$,

$$\begin{aligned} F_1(x,y)(t) &= \int_0^1 K(t,s)\, f(s,x(s),y(s))ds \leq \int_0^1 K(t,s)\, f(s,x(s),v(s))ds \\ &= F_1(x,v)(t), \text{ for each } x \in \mathbb{R}^n, t \in [0,1]. \\ F_2(x,y)(t) &= \int_0^1 K(t,s)\, g(s,x(s),y(s))ds \geq \int_0^1 K(t,s)\, g(s,x(s),v(s))ds \\ &= F_2(x,v)(t), \text{ for each } x \in \mathbb{R}^n, t \in [0,1]. \end{aligned} \tag{4.5}$$

From (4.4) and (4.5), we have that the operator $F = (F_1, F_2)$ has the property (P).

On the other hand, by Cauchy-Buniakovski-Schwarz inequality, we have

$$\begin{aligned} |F_1(x,y)(t) - F_1(u,v)(t)|^p &\leq \left[\int_0^1 |K(t,s)|\,(f(s,x(s),y(s)) - f(s,u(s),v(s))\,ds\right]^p \\ &\leq \int_0^1 K^p(t,s)\,ds \int_0^1 |f(s,x(s),y(s)) - f(s,u(s),v(s))|^p\,ds, \text{ for each } t \in [0,1]. \end{aligned}$$

We have

$$\int_0^1 K^p(t,s)\,ds = \int_0^t s^p ds + \int_t^1 t^p ds = t^p\left(1 - \frac{p}{p+1}t\right) \leq \frac{1}{p+1}, \text{ for each } t \in [0,1].$$

Hence

$$\begin{aligned} |F_1(x,y)(t) - F_1(u,v)(t)|^p &\leq \frac{1}{p+1}\int_0^1 |f(s,x(s),y(s)) - f(s,u(s),v(s))|^p\,ds \\ &\leq \frac{1}{p+1}\int_0^1 \widetilde{\varphi}(\alpha\,|x(s) - u(s)|^p + \beta\,|y(s) - v(s)|^p)ds \\ &\leq \frac{1}{p+1}\widetilde{\varphi}(\alpha d(x,u) + \beta\rho(y,v)) \leq\leq \frac{1}{p+1}\widetilde{\varphi}(\max\{\alpha,\beta\}(d(x,u) + \rho(y,v))). \end{aligned}$$

Hence

$$d(F_1(x,y), F_1(u,v)) \leq \frac{1}{p+1}\widetilde{\varphi}(\max\{\alpha,\beta\}(d(x,u) + \rho(y,v))), x \leq u, v \leq y. \tag{4.6}$$

In a similar way, for F_2 we obtain

$$\rho(F_2(x,y), F_2(u,v)) \leq \frac{1}{q+1}\widetilde{\psi}(\max\{\gamma,\delta\}(d(x,u) + \rho(y,v))), x \leq u, v \leq y. \tag{4.7}$$

By (4.6) and (4.7), we have

$$\begin{aligned} &d(F_1(x,y), F_1(u,v)) + \rho(F_2(x,y), F_2(u,v)) \\ &\quad\leq \frac{1}{p+1}\widetilde{\varphi}(\max\{\alpha,\beta\}(d(x,u) + \rho(y,v))) + \frac{1}{q+1}\widetilde{\psi}(\max\{\gamma,\delta\}(d(x,u) + \rho(y,v))) \\ &\quad\leq \frac{1}{p+1}\left[\widetilde{\varphi}(\max\{\alpha,\beta\}(d(x,u) + \rho(y,v))) + \widetilde{\psi}(\max\{\gamma,\delta\}(d(x,u) + \rho(y,v)))\right], x \leq u, v \leq y. \end{aligned}$$

Let us consider the function $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(t) = \frac{1}{p+1}\left(\widetilde{\varphi}(kt) + \widetilde{\psi}(lt)\right), 0 \le k,l < 1$, which is a (b)-comparison function. Then, we have

$$d(F_1(x,y), F_1(u,v)) + \rho(F_2(x,y), F_2(u,v)) \le \varphi(d(x,u) + \rho(y,v)), x \le u, v \le y.$$

Thus we have that $F = (F_1, F_2) : Z \to Z$ is a (φ, G)-contraction of type (b).

Condition (iv) from Theorem 4.1, shows that there exists $(x_0, y_0) \in Z$ such that $x_0 \le F_1(x_0, y_0)$ and $F_2(x_0, y_0) \le y_0$ which implies that $Z^F \ne \varnothing$. On the other hand, (X, d, G_1) and (Y, ρ, G_2) have the properties (A_1) and (A_2), so (ii) from Theorem 2.10 is fulfilled. In this way, we have that $F_1 : Z \to X$ and $F_2 : Z \to Y$ defined by (4.3), verify the conditions of Theorem 2.10. Thus, there exists $(x^*, y^*) \in Z$ solution of the problem (4.2). $\square$

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