Nonexpansive mappings on complex C*-algebras and
their fixed points

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Abstract
A normed space $X$ is said to have the fixed point property, if for each nonexpansive mapping $T : E \to E$ on a nonempty bounded closed convex subset $E$ of $X$ has a fixed point. In this paper, we first show that if $X$ is a locally compact Hausdorff space then the following are equivalent: (i) $X$ is infinite set, (ii) $C_0(X)$ is infinite dimensional, (iii) $C_0(X)$ does not have the fixed point property. We also show that if $A$ is a commutative complex C*-algebra with nonempty carrier space, then the following statements are equivalent: (i) Carrier space of $A$ is infinite, (ii) $A$ is infinite dimensional, (iii) $A$ does not have the fixed point property. Moreover, we show that if $A$ is an infinite dimensional complex C*-algebra (not necessarily commutative), then $A$ does not have the fixed point property.

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1. Introduction and Preliminaries
Let $T : E \to E$ be a self-map on the nonempty set $E$. We denote $\{x \in E : T(x) = x\}$ by Fix($T$) and call the fixed points set of $T$. The symbol $\mathbb{K}$ denote a field that can be either $\mathbb{C}$ or $\mathbb{R}$. Let $(X, \| \cdot \|)$ be a normed linear space over $\mathbb{K}$. A mapping $T : E \subseteq X \to X$ is nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in E$. We say that the normed linear space $(X, \| \cdot \|)$ over $\mathbb{K}$ has the fixed point property (or weak fixed point property) if for every nonempty bounded closed convex (or weakly compact convex, respectively) subset $E$ of $X$ and every nonexpansive mapping $T : E \to E$ we have Fix($T$) $\neq \emptyset$.

One of the central goals in fixed point theory is to find which normed linear spaces over $\mathbb{K}$ have the (weak) fixed point property.

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Theorem 1.1. Let $(X_1, \| \cdot \|_1)$ be a Banach space, $(X_2, \| \cdot \|_2)$ be a normed linear space and there exist a linear isometry from $(X_1, \| \cdot \|_1)$ into $(X_2, \| \cdot \|_2)$ over $\mathbb{K}$. If $(X_1, \| \cdot \|_1)$ does not have the fixed point property then $(X_2, \| \cdot \|_2)$ does not have the fixed point property.

Proof. Let $(X_1, \| \cdot \|_1)$ does not have the fixed point property. Then there exist a nonempty bounded closed convex subset $E$ of $X_1$ and a nonexpansive mapping $T : E \rightarrow E$ such that $\text{Fix}(T) = \emptyset$. Let $\Psi : X_1 \rightarrow X_2$ be a linear isometry from $(X_1, \| \cdot \|_1)$ into $(X_2, \| \cdot \|_2)$ over $\mathbb{K}$. Then $\Psi(E)$ is a nonempty convex subset of $X_2$ and $\Psi(E)$ is bounded in $(X_2, \| \cdot \|_2)$. Moreover, since $E$ is a closed subset of $X_1$ in the Banach space $(X_1, \| \cdot \|_1)$ and $\Psi : X_1 \rightarrow X_2$ is a linear isometry from $(X_1, \| \cdot \|_1)$ into $(X_2, \| \cdot \|_2)$, we deduce that $\Psi(E)$ is a closed subset of $X_2$ in the normed linear space $(X_2, \| \cdot \|_2)$. We define the mapping $S : \Psi(E) \rightarrow \Psi(E)$ by

$$S(\Psi(x)) = \Psi(T(x)) \quad (x \in E).$$

Since for all $x$ and $y$ in $X_1$ we have

$$\|S(\Psi(x)) - S(\Psi(y))\|_2 = \|\Psi(T(x)) - \Psi(T(y))\|_2 = \|T(x) - T(y)\|_1 \leq \|x - y\|_1 = \|\Psi(x) - \Psi(y)\|_2,$$

we conclude that $S : \Psi(E) \rightarrow \Psi(E)$ is a nonexpansive mapping. We claim that $\text{Fix}(S) = \emptyset$. Suppose that $\Psi(x_1) \in \text{Fix}(S)$ where $x_1 \in E$. Then

$$0 = S(\Psi(x_1)) - \Psi(x_1) = \Psi(T(x_1)) - \Psi(x_1) = \Psi(T(x_1) - x_1),$$

and so $0 = T(x_1) - x_1$. This implies that $x_1 \in \text{Fix}(T)$ contradicting to $\text{Fix}(T) = \emptyset$. Hence, our claim is justified. Therefore, $(X_2, \| \cdot \|_2)$ does not have the fixed point property. □

Corollary 1.2. Let $(X, \| \cdot \|)$ be a Banach space and $Y$ be a closed linear subspace of $X$ over $\mathbb{K}$. If $(Y, \| \cdot \|)$ does not have the fixed point property, then $(X, \| \cdot \|)$ does not have the fixed point property.

Let $A$ be a complex algebra and let $A_{e} := A \times \mathbb{C}$. Then $A_{e}$ is a complex algebra with unit $e = (0, 1)$ whenever algebra operations are defined by

$$(f, \lambda) + (g, \mu) = (f + g, \lambda + \mu), \quad \alpha(f, \lambda) = (\alpha f, \alpha \lambda), \quad (f + \lambda)(g, \mu) = (fg + \mu f + \lambda g, \lambda \mu),$$

for $f, g \in A$, $\lambda, \mu, \alpha \in \mathbb{C}$. We say that $A_{e}$ is the unitisation of $A$. Clearly, $A_{e}$ is commutative if $A$ is commutative. Moreover, if $\| \cdot \|$ is an algebra norm on $A$ then $A_{e}$ is a normed algebra under the norm $\| \cdot \|$ defined by

$$\|(f, \lambda)\| = \|f\| + |\lambda| \quad (f \in A, \lambda \in \mathbb{C}).$$

Note that $(A_{e}, \| \cdot \|)$ is a unital Banach algebra if $(A, \| \cdot \|)$ is a Banach algebra.

Let $A$ be a complex algebra with unit $e$ and let $G(A)$ be the set of all invertible elements of $A$. We define the spectrum of an element $f \in A$ to be the set $\{\lambda \in \mathbb{C} : \lambda e - f \notin G(A)\}$ and denote it by $\sigma_{A}(f)$.

Let $A$ be a complex algebra and $A_{e}$ be the unitisation of $A$. For $f \in A$, the set $\sigma_{A_{e}}(f, 0)$ is called the spectrum of $f$ and denoted by $\sigma_{A}(f)$.

Let $A$ be a complex normed algebra and let $f \in A$. The spectral radius of $f$ is denoted by $r_{A}(f)$ and defined by
It is known\footnote{\cite[Lemma 1.2.5 and Theorem 1.2.8]{5}} that
\[ r_A(f) = \inf \left\{ \|f^n\|^{\frac{1}{n}} : n \in \mathbb{N} \right\}. \]

Let $A$ be a complex algebra. A \textit{character} on $A$ is a nonzero multiplicative linear functional on $A$. We denote by $\Delta(A)$ the set of all characters on $A$. If $A$ has the unite $e$, then $\phi(e) = 1$ for all $\phi \in \Delta(A)$. Note that each $\phi \in \Delta(A)$ has a unique extension $\hat{\phi} \in \Delta(A_c)$ given by
\[ \hat{\phi}(f + \lambda e) = \phi(f) + \lambda \quad (f \in A, \lambda \in \mathbb{C}). \]

It is known that if $A$ is complex Banach algebra and $\phi \in \Delta(A)$, then $\phi$ is bounded and $\|\phi\| \leq 1$. In particular, $\|\phi\| = 1$ if $A$ is unital. Moreover, if $A$ is a unital commutative complex Banach algebra, then $\Delta(A) \neq \emptyset$ and $\sigma_A(f) = \{\phi(f) : \phi \in \Delta(A)\}$ for all $f \in A$. If $A$ is without unit, it is possible $\Delta(A) = \emptyset$. (See\footnote{\cite[Example 2.1.6 and Example 2.1.7]{5}})

Let $A$ be a commutative complex Banach algebra with $\Delta(A) \neq \emptyset$. For each $f \in A$, we define $\hat{f} : \Delta(A) \longrightarrow \mathbb{C}$ by $\hat{f}(\phi) = \phi(f)$ and say that $\hat{f}$ is the \textit{Gelfand transform} of $f$. We denote $\{\hat{f} : f \in A\}$ by $\hat{A}$. Then $\hat{A}$ strongly separates the points of $\Delta(A)$. Moreover, the following statements are equivalent:

(i) $\hat{A}$ is self-adjoint.

(ii) For each $f \in A$, there exists an element $g \in A$ such that $\phi(g) = \overline{\phi(f)}$ for all $\phi \in \Delta(A)$.

We endow $\Delta(A)$ with the Gelfand topology, the weakest topology on $\Delta(A)$ for which every $\hat{f} \in \hat{A}$ is continuous. $\Delta(A)$ with the Gelfand topology is called the \textit{carrier space} of $A$. We know\footnote{\cite[Theorem 2.2.3]{5}} that

(i) $\Delta(A)$ is a locally compact Hausdorff space,

(ii) $\Delta(A_c)$ is one-point compactification of $\Delta(A),$

(iii) $\Delta(A)$ is compact if $A$ has the unit element.

Fupinwong studied the fixed point property of commutative complex Banach algebras in\footnote{\cite{3}} and obtained the following result.

\textbf{Theorem 1.3.} (See\footnote{\cite[Theorem 3.1]{3}}) Let $A$ be an infinite dimensional commutative complex Banach algebra with $\Delta(A) \neq \emptyset$ satisfying each of the following:

(i) $\hat{A}$ is self-adjoint,

(ii) if $f, g \in A$ such that $|\phi(f)| \leq |\phi(g)|$ for all $\phi \in \Delta(A)$, then $\|f\| \leq \|g\|$, \[ \inf \{r_A(f) : f \in A, \|f\| = 1\} > 0. \]

Then $A$ does not have the fixed point property.

Fupinwong and Dhompongsa were obtained the mentioned result in\footnote{\cite[Theorem 4.3]{4}} whenever $A$ is unital. (See\footnote{\cite[Theorem 4.3]{4}})

Let $X$ be a locally compact Hausdorff space. We denote by $C(X)$ and $C_0(X)$ the set of all complex-valued continuous functions on $X$ and the set of all functions in $C(X)$ which vanish at infinity, respectively. Then $C(X)$ is a commutative complex algebra with unit $1_X$ and $C_0(X)$ is a complex subalgebra of $C(X)$. Moreover, $C_0(X) = C(X)$ if $X$ is compact. It is known that $C_0(X)$ under the uniform norm on $X$ defined by
is a commutative complex Banach algebra. Moreover, $C_0(X)$ is without unit if $X$ is not compact.

Applying the concept of peak points, it is shown \[1\] that certain uniformly closed subalgebras of $C(X)$ do not have the fixed point property, where $X$ is a compact Hausdorff space.

Dhompongsa, Fupinwong and Lawton studied the fixed point property and weak fixed point property of complex $C^*$-algebras in \[2\].

In Section \[2\] applying Theorem \[1.3\] we study the fixed point property of $C_0(X)$ and certain its uniformly closed subalgebras.

In Section \[3\] we show that a commutative complex $C^*$-algebra $A$ does not have the fixed point property if and only if $A$ is infinite dimensional. We also prove that if $A$ is an infinite dimensional complex $C^*$-algebra (not necessarily commutative), then $A$ does not have the fixed point property.

2. The fixed point property of $C_0(X)$

Applying Urysohn’s lemma \[3\] Theorem 2.12], Theorem \[1.3\] and Schauder–Tychonoff fixed point theorem \[8\] Theorem 5.28], we study the fixed point property of $C_0(X)$ whenever $X$ is a locally compact Hausdorff space.

Theorem 2.1. Let $X$ be a locally compact Hausdorff space and $A = C_0(X)$. Then the following statements are equivalent:

(i) $X$ is infinite set.

(ii) $A$ is infinite dimensional.

(iii) $(A, \| \cdot \|_X)$ does not have the fixed point property.

Proof. (i) $\Rightarrow$ (ii). Let $X$ be infinite set. We can choose a sequence $\{x_n\}_{n=1}^\infty$ in $X$ such that $x_i \neq x_j$ if $i, j \in \mathbb{N}$ and $i \neq j$. By Urysohn’s lemma, we obtain a sequence $\{f_n\}_{n=1}^\infty$ of functions in $C_0(X)$ such that $f_1(x_1) = 1$ and

$$f_n(x_1) = \ldots = f_n(x_{n-1}) = 0, \quad f_n(x_n) = 1 \quad (n \in \mathbb{N}, n \geq 2).$$

To prove that $C_0(X)$ is infinite dimensional, it is sufficient we show that the set $\{f_1, \ldots, f_n\}$ is a linearly independent set in $C_0(X)$ for all $n \in \mathbb{N}$. Since $f_1(x_1) = 1$, we deduce that $\{f_1\}$ is a linearly independent set in $C_0(X)$. Suppose that $n \in \mathbb{N}$ with $n \geq 2$. Let

$$\alpha_1 f_1 + \cdots + \alpha_n f_n = 0$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Since $f_1(x_1) = 1$ and $f_2(x_1) = \cdots = f_n(x_1) = 0$, we conclude that $\alpha_1 = 0$ by (2.1). Suppose that $j \in \{1, \cdots, n\}$ such that $\alpha_1 = \cdots = \alpha_{j-1} = 0$. Then

$$\alpha_j f_j + \cdots + \alpha_n f_n = 0,$$

by (2.1). If $j = n$, then $\alpha_1 = \cdots = \alpha_{n-1} = 0$ and $\alpha_n f_n = 0$ by (2.2). Thus $\alpha_n = 0$ since $f_n(x_n) = 1$. If $j \in \{1, \cdots, n-1\}$, then by (2.2) we have $\alpha_j = 0$ since $f_j(x_j) = 1$ and $f_{j+1}(x_j) = \cdots = f_n(x_j) = 0$. Therefore, $\alpha_j = 0$ for all $j \in \{1, \cdots, n\}$ and so the set $\{f_1, \cdots, f_n\}$ is a linearly independent set in $C_0(X)$. Hence, (ii) holds.

(ii) $\Rightarrow$ (iii). Let $A$ be infinite dimensional. It is known \[6\] Theorem 2.3] that

$$\Delta(A) = \{e_x : x \in X\}$$

where $e_x : A \rightarrow \mathbb{C}$ is defined by $e_x(f) = f(x)$ for all $f \in A$. Let $F \in \hat{A}$. Then there exists $f \in A$ such that $F = \hat{f}$, the Gelfand transform of $f$. Since $\hat{f} \in A$ and
\[ F(e_x) = F(e_x) = f(e_x) = e_x(f) = f(x) = e_x(f) = \hat{f}(e_x), \]

for all \( x \in X \), we deduce that \( F = \hat{f} \) by (2.3). Hence, \( F \in \hat{A} \) and so \( \hat{A} \) is self-adjoint.

Let \( f, g \in A \) such that \( |\phi(f)| \leq |\phi(g)| \) for all \( \phi \in \Delta(A) \). Since \( e_x \in \Delta(A) \) for each \( x \in X \), we have

\[ |f(x)| = |e_x(f)| \leq |e_x(g)| = |g(x)| \]

for each \( x \in X \). Therefore, \( \|f\|_X \leq \|g\|_X \).

Let \( f \in A \) such that \( \|f\|_X = 1 \). Since \( \|g^n\|_X = (\|g\|_X)^n \) for all \( g \in A \) and for each \( n \in \mathbb{N} \), we have \( \|f^n\|_X = 1 \) for each \( n \in \mathbb{N} \), and so

\[ r_A(f) = \lim_{n \to \infty} (\|f^n\|_X)^{\frac{1}{n}} = 1. \]

Therefore,

\[ \inf\{r_A(f) : f \in A, \|f\|_X = 1\} = \inf\{1\} = 1 > 0. \]

Hence, \( A \) does not have the fixed point property by Theorem 1.3.

(iii) \( \implies \) (i). Let \( X \) be a nonempty finite set. Let \( x \in X \). Since \( X \setminus \{x\} \) is a finite set and \( X \) is a Hausdorff space, we deduce that \( X \setminus \{x\} \) is closed in \( X \) and so \( \{x\} \) is an open set in \( X \). Hence, the given topology on \( X \) is \( \mathcal{P}(X) \), the power set of \( X \). This implies that there exists a linear isometry from \((C_0(X), \|\cdot\|_X)\) onto \( \mathbb{C}^n \) with the Euclidean norm, where \( n \) is the cardinal number of \( X \). Hence, every bounded closed subset \( E \) of \( C_0(X) \) is compact in \((C_0(X), \|\cdot\|_X)\). Let \( E \) be a nonempty bounded closed convex subset of \( C_0(X) \) and \( T : E \to E \) be a nonexpansive mapping. Then \( E \) is a nonempty compact convex subset of the Banach space \((C_0(X), \|\cdot\|_X)\) and \( T : E \to E \) is continuous. Hence, \( T \) has a fixed point by Schauder–Tychonoff fixed point theorem [8, Theorem 5.28]. Therefore, \((C_0(X), \|\cdot\|_X)\) has the fixed point property and so (iii) does not hold. \( \square \)

Let \( X \) be a locally compact Hausdorff space and \( K \) be a compact subset of \( X \). We denote by \( C_0Z(X, K) \) the set of all \( f \in C_0(X) \) such that \( f|_K = 0 \). It is easy to see that \( C_0Z(X, K) \) is a self-adjoint uniformly closed subalgebra of \( C_0(X) \). Moreover, \( C_0Z(X, K) = C_0(X) \) if and only if \( K = \emptyset \).

**Theorem 2.2.** Let \( X \) be a locally compact Hausdorff space and \( K \) be a compact subset of \( X \) such that \( X \setminus K \neq \emptyset \). Then the following statements hold.

(i) If \( f \in C_0Z(X, K) \) and \( g = f|_{X \setminus K} \), then \( g \in C_0(X \setminus K) \).

(ii) If \( g \in C_0(X \setminus K) \) and the function \( g_0 : X \to \mathbb{C} \) defines by

\[ g_0(x) = \begin{cases} g(x) & x \in X \setminus K, \\ 0 & x \in K. \end{cases} \]

then \( g_0 \in C_0Z(X, K) \).

(iii) The map \( \Phi : C_0Z(X, K) \to C_0(X \setminus K) \) defined by \( \Phi(f) = f|_{X \setminus K} \), is an isometrical isomorphism from \((C_0Z(X, K), \|\cdot\|_X)\) onto \((C_0(X \setminus K), \|\cdot\|_{X \setminus K})\).

(iv) \( C_0Z(X, K) \) strongly separates the points of \( X \setminus K \).

(v) For each \( x \in X \setminus K \), \( e_x \in \Delta(C_0Z(X, K)) \).

(vi) If \( x, y \in X \setminus K \) with \( x \neq y \), then \( e_x \neq e_y \).
(vii) $\Delta (C_0Z(X,K)) = \{e_x : x \in X \setminus K\}$.

**Proof.** (i) Let $f \in C_0Z(X,K)$ and $g = f|_{X \setminus K}$. Clearly, $g \in C(X \setminus K)$. Let $\varepsilon > 0$ and

$$H = \{x \in X : |f(x)| \geq \varepsilon\}.$$

Then $H$ is a closed set in $X$ and $H \subseteq X \setminus K$. Since $f \in C_0(X)$, there exists a compact subset $L$ of $X$ such that

$$f(X \setminus L) \subseteq \{z \in \mathbb{C} : |z| < \varepsilon\}.$$

Set $E = H \cap L$. Then $E$ is a compact set in $X$ and $E \subseteq X \setminus K$. Thus $E$ is a compact set in $X \setminus K$. Moreover,

$$|g(x)| = |f(x)| < \varepsilon,$$

for all $x \in (X \setminus K) \setminus E$. Therefore, $g \in C_0(X \setminus K)$ and so (i) holds.

(ii) Let $g \in C_0(X \setminus K)$ and the function $g_0 : X \rightarrow \mathbb{C}$ defines as above. To prove $g_0 \in C_0Z(X,K)$, it is sufficient we show that $g_0 \in C_0(X)$ since $g_0|_K = 0$. Let $x_0 \in X \setminus K$ and choose $\varepsilon > 0$. The continuity of $g : X \setminus K \rightarrow \mathbb{C}$ at $x_0$ implies that there exists a neighborhood $U_0$ of $x_0$ in $X \setminus K$ such that

$$|g(x) - g(x_0)| < \varepsilon \quad (\forall x \in U_0).$$

(2.4)

Since $X \setminus K$ is an open set in $X$, we deduce that $U_0$ is a neighborhood of $x_0$ in $X$. Since $g_0(x_0) = 0 = g(x_0)$ and $g_0|_{X \setminus K} = g$, we conclude that

$$|g_0(x) - g_0(x_0)| < \varepsilon \quad (\forall x \in U_0),$$

by (2.4). Therefore, $g_0$ is continuous at $x_0$.

Let $x_0 \in K$ and choose $\varepsilon > 0$. Since $g \in C_0(X \setminus K)$, there exists a compact subset $H$ in $X \setminus K$ such that

$$g((X \setminus K) \setminus H) \subseteq \{z \in \mathbb{C} : |z| < \varepsilon\}.$$

(2.5)

The compactness of $H$ in $X \setminus K$ implies that $H$ is a compact set in $X$ and so $H$ is closed in $X$. Set $U = X \setminus K$. Then $U$ is an open set in $X$ and $K \subseteq U$. Hence, $U$ is a neighborhood of $x_0$ in $X$. If $x \in K$, then

$$|g_0(x) - g_0(x_0)| = 0 < \varepsilon.$$

Suppose that $x \in U \setminus K$. Then $x \in (X \setminus H) \setminus K = (X \setminus K) \setminus H$ and so $|g(x)| < \varepsilon$ by (2.5). Hence, $|g_0(x) - g_0(x_0)| = |g(x) - 0| = |g(x)| < \varepsilon$.

Therefore, $g_0$ is continuous at $x_0$. So, $g \in C(X)$.

Let $\varepsilon > 0$ be given. Since $g \in C_0(X \setminus K)$, there exists a compact set $H$ of $X \setminus K$ such that

$$g((X \setminus K) \setminus H) \subseteq \{z \in \mathbb{C} : |z| < \varepsilon\}.$$

Clearly, $H$ is a compact set in $X$. Set $L = K \cup H$. Then $H$ is a compact set in $X$ and $X \setminus L = (X \setminus K) \setminus H$. So

$$|g_0(x)| = |g(x)| < \varepsilon,$$
for all \( x \in X \setminus L \). Therefore, \( g_0 \in C_0(X) \) and so (ii) holds.

(iii) By (i), \( \Phi \) is well-defined. Clearly, \( \Phi \) is an algebra homomorphism. Let \( f \in C_0Z(X, K) \). Then \( f \in C_0(X) \) and \( f|_K = 0 \). Thus \( \|f\|_X = \|f\|_{X\setminus K} \) and so

\[
\|\Phi(f)\|_{X\setminus K} = \|f|_{X\setminus K}\|_{X\setminus K} = \|f\|_{X\setminus K} = \|f\|_X.
\]

Therefore, \( \Phi \) is an isometry.

Let \( g \in C_0(X \setminus K) \). We define the function \( g_0 : X \to \mathbb{C} \) by

\[
g_0(x) = \begin{cases} g(x) & x \in X \setminus K, \\ 0 & x \in K. \end{cases}
\]

Then \( g_0 \in C_0Z(X, K) \) by (ii) and \( g_0|_K = g \). Therefore, \( \Phi \) is surjective.

(iv) Let \( x_1, x_2 \in X \setminus K \) such that \( x_1 \neq x_2 \). By Urysohn’s lemma, there exists a function \( f_0 \in C_0(X) \) such that \( f_0(x_1) = 1 \) and \( f_0(x) = 0 \) for all \( x \in K \cup \{x_2\} \). So \( f_0 \in C_0Z(X, K) \) and \( f_0(x_1) \neq f_0(x_2) \). Therefore, \( C_0Z(X, K) \) separates the points of \( X \setminus K \).

Let \( x \in X \setminus K \). By Urysohn’s lemma, there exists a function \( f_1 \in C_0(X) \) such that \( f_1(x) = 1 \) and \( f_1(y) = 0 \) for all \( y \in K \). So \( f_1 \in C_0Z(X, K) \) and \( f_1(x) \neq 0 \). Therefore, \( \psi \) is surjective.

(v) Let \( x \in X \setminus K \). Clearly, \( e_x \) is a multiplicative complex linear functional on \( C_0Z(X, K) \). By (iv), there exists a function \( f_1 \in C_0Z(X, K) \) such that \( f_1(x) \neq 0 \) and so \( e_x(f_1) \neq 0 \). Therefore, \( e_x \in \Delta(C_0Z(X, K)) \).

(vi) Let \( x, y \in X \setminus K \) such that \( x \neq y \). By (iv), there exists a function \( f_0 \in C_0Z(X, K) \) such that \( f_0(x) \neq f_0(y) \). Hence, \( e_x(f_0) \neq e_y(f_0) \) and so \( e_x \neq e_y \).

(vii) By (v), we have

\[
\{e_x : x \in X \setminus K\} \subseteq \Delta(C_0Z(X, K)). \tag{2.6}
\]

Let \( \psi \in \Delta(C_0Z(X, K)) \). By (iii), the map \( \Phi : C_0Z(X, K) \to C_0(X \setminus K) \) defined by

\[
\Phi(f) = f|_{X\setminus K} \quad (f \in C_0Z(X, K))
\]

is an isometrical algebra isomorphism from \( (C_0Z(X, K), \|\cdot\|_X) \) onto \( (C_0(X \setminus K), \|\cdot\|_{X\setminus K}) \). Therefore, \( \psi \circ \Phi^{-1} \) is a multiplicative complex linear function on \( C_0(X \setminus K) \). On the other hand, there exists a function \( f_0 \in C_0Z(X, K) \) such that \( \psi(f_0) \neq 0 \). Set \( g_0 = \Phi(f_0) \). Then \( g_0 \in C_0(X \setminus K) \) and \( f_0 = \Phi^{-1}(g_0) \). Thus \( \psi \circ \Phi^{-1} \) \( (g_0) \neq 0 \) and so \( \psi \circ \Phi^{-1} \in \Delta(C_0(X \setminus K)) \). Since \( \Delta(C_0(X \setminus K)) = \{e_x : x \in X \setminus K\} \), there exists \( y \in X \setminus K \) such that \( \psi \circ \Phi^{-1} = e_y \) on \( C_0(X \setminus K) \). Let \( f \in C_0Z(X, K) \). Then \( \Phi(f) \in C_0(X \setminus K) \) and so

\[
(\psi \circ \Phi^{-1})(\Phi(f)) = e_y(f).
\]

This implies that \( \psi(f) = e_y(f) \). Therefore, \( \psi = e_y \) on \( C_0Z(X, K) \) and so

\[
\Delta(C_0Z(X, K)) \subseteq \{e_x : x \in X \setminus K\}. \tag{2.7}
\]

From (2.6) and (2.7), we have

\[
\Delta(C_0Z(X, K)) = \{e_x : x \in X \setminus K\}.
\]

Therefore, (vii) holds. \( \square \)

**Theorem 2.3.** Let \( X \) be a locally compact Hausdorff space and \( K \) be a compact subset of \( X \). If \( X \setminus K \) is infinite set, then \( (C_0Z(X, K), \|\cdot\|_X) \) does not have the fixed point property.
By Gelfand–Naimark theorem, the Gelfand homomorphism

Proof. Since $X \setminus K$ is an open subset of $X$, we deduce that $X \setminus K$ with the relative topology is a locally compact Hausdorff space. Since $X \setminus K$ is infinite set, $(C_0(X \setminus K), \| \cdot \|_{X \setminus K})$ does not have the fixed point property by Theorem 2.1.

By part (iii) of Theorem 2.2 the map $\Phi: C_0(Z(X, K)) \rightarrow C_0(X \setminus K)$ defined by

$$\Phi(f) = f|_{X \setminus K} \quad (f \in C_0(Z(X, K)))$$

is a linear isometry from $(C_0(Z(X, K)), \| \cdot \|_Z)$ onto $(C_0(X \setminus K), \| \cdot \|_{X \setminus K})$. Hence, $\Phi^{-1}: C_0(X \setminus K) \rightarrow C_0(Z(X, K))$ is a linear isometry from $(C_0(X \setminus K), \| \cdot \|_{X \setminus K})$ onto $(C_0(Z(X, K)), \| \cdot \|_Z)$. Therefore, $(C_0(Z(X, K)), \| \cdot \|_Z)$ does not have the fixed point property by Theorem 1.1 □

Remark 2.4. Let $X$ be a locally compact Hausdorff space and $K$ be a compact subset of $X$ such that $X \setminus K$ is an infinite set. By part (iii) and part (vii) of Theorem 2.2 we can show that the commutative complex Banach algebra $(C_0(Z(X, K), \| \cdot \|_X)$ satisfies in all conditions of Theorem 1.3 and so does not have the fixed point property.

3. Fixed point property of $C^*$–algebras

Applying Gelfand–Naimark theorem [5, Theorem 2.4.5], Theorem 1.1 and Theorem 2.1, we study the fixed point property of commutative complex $C^*$–algebras.

Theorem 3.1. Let $(A, \| \cdot \|)$ be a commutative complex $C^*$–algebra with $\Delta(A) \neq \emptyset$. Then the following statements are equivalent:

(i) $\Delta(A)$ is infinite set.

(ii) $A$ is infinite dimensional.

(iii) $(A, \| \cdot \|)$ does not have the fixed point property.

Proof. By Gelfand–Naimark theorem, the Gelfand homomorphism $x \mapsto \hat{x}: A \rightarrow C_0(\Delta(A))$ is an isometric $*$–isomorphism from $(A, \| \cdot \|)$ onto $(C_0(\Delta(A)), \| \cdot \|_{\Delta(A)})$. Hence, $(A, \| \cdot \|)$ does not have the fixed point property if and only if $(C_0(\Delta(A)), \| \cdot \|_{\Delta(A)})$ does not have the fixed point property by Theorem 1.1. Therefore, the proof is complete by Theorem 2.1 □

Corollary 3.2. Let $X$ be a locally compact Hausdorff space such that $X$ is an infinite set. If $A$ is an infinite dimensional self–adjoint uniformly closed complex subalgebra of $C_0(X)$, then $(A, \| \cdot \|_X)$ does not have the fixed point property.

Proof. By hypotheses, $(A, \| \cdot \|_X)$ is a commutative complex $C^*$–algebra under the natural involution $f \mapsto \bar{f}: A \rightarrow A$. Since $A$ is infinite dimensional, $(A, \| \cdot \|_X)$ does not have the fixed point property by Theorem 3.1 □

Example 3.3. Let $m \in \mathbb{N}$ and define the function $g_m: \mathbb{C} \rightarrow \mathbb{C}$ by

$$g_m(z) = \exp(-m|z|).$$

Let $A_m$ be the complex subalgebra of $C_0(\mathbb{C})$ generated by $g_m$ and $B_m$ be the uniform closure of $A_m$ in $(C_0(\mathbb{C}), \| \cdot \|_\mathbb{C})$. Then $B_m$ is a uniformly closed self–adjoint complex subalgebra of $C_0(\mathbb{C})$. Since for each $n \in \mathbb{N}$ the set $\{ (g_m)^k : k \in \{1, \ldots, n\} \}$ is a linearly independent set in $B_m$, we deduce that $B_m$ is infinite dimensional. Therefore, $(B_m, \| \cdot \|_\mathbb{C})$ does not have the fixed point property, by Corollary 3.1.
Remark 3.4. Let $X$ be a locally compact Hausdorff space and $K$ be a compact subset of $X$ such that $X \setminus K$ is an infinite set. Then $C_0Z(X,K)$ satisfies in conditions of Corollary 3.2. Therefore, $(C_0Z(X,K), \| \cdot \|_X)$ does not have the fixed point property.

The following result was given by Ogasawara in [7].

Theorem 3.5. (See [7, Theorem 1]) Let $(A, \| \cdot \|)$ be an infinite dimensional complex $C^*$–algebra with the algebra involution $\star$. Then there exists a commutative infinite dimensional complex subalgebra $B$ of $A$ such that $x^* \in B$ for each $x \in B$ and $(B, \| \cdot \|)$ is a complex $C^*$–algebra with the algebra involution $\star$.

Applying Ogasawara’s theorem (Theorem 3.5), Theorem 3.1, and Corollary 1.2 we obtain the following result.

Theorem 3.6. Let $(A, \| \cdot \|)$ be a complex $C^*$–algebra with the algebra involution $\star$. If $A$ is infinite dimensional, then $(A, \| \cdot \|)$ does not have the fixed point property.

Proof. Let $A$ is infinite dimensional. By Theorem 3.5 there exists a commutative infinite dimensional complex subalgebra $B$ of $A$ such that $x^* \in B$ for each $x \in B$ and $(B, \| \cdot \|)$ is a complex $C^*$–algebra with the algebra involution $\star$. Therefore, $(B, \| \cdot \|)$ does not have the fixed point property by Theorem 3.1 and so $(A, \| \cdot \|)$ does not have the fixed point property by Corollary 1.2.

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References