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σ-Weak Amenability of Banach Algebras

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Abstract

Let $\mathcal{A}$ be a Banach algebra, σ be continuous homomorphism on $\mathcal{A}$ with $\overline{\sigma(\mathcal{A})} = \mathcal{A}$. The bounded linear map $D : \mathcal{A} \to \mathcal{A}^*$ is $\sigma-$derivation, if

$$D(ab) = D(a) \cdot \sigma(b) + \sigma(a) \cdot D(b) \qquad (a, b \in \mathcal{A}).$$

We say that $\mathcal{A}$ is σ-weakly amenable, when for each bounded derivation $D : \mathcal{A} \to \mathcal{A}^*$, there exists $a^* \in \mathcal{A}$ such that $D(a) = \sigma(a) \cdot a^* - a^* \cdot \sigma(a)$. For a commutative Banach algebra $\mathcal{A}$, we show $\mathcal{A}$ is $\sigma-$weakly amenable if and only if every $\sigma-$derivation from $\mathcal{A}$ into a $\sigma-$symmetric Banach $\mathcal{A}-$bimodule $\mathcal{X}$ is zero. Also, we show that a commutative Banach algebra $\mathcal{A}$ is $\sigma-$weakly amenable if and only if $\mathcal{A}^{\#}$ is $\sigma^{\#}-$weakly amenable, where $\sigma^{\#}(a + \alpha) = \sigma(a) + \alpha$.

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1. Introduction and preliminaries

The notion of amenable Banach algebra was introduced by B.E. Johnson in [5]. This class of Banach algebras arises naturally out of the cohomology theory for Banach algebras, the algebraic version of which was developed by Hochschild [4]. For a comprehensive account on amenability the reader is referred to the books [2,11,12].

Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{X}$ be a Banach $\mathcal{A}-$bimodule. Then a derivation from $\mathcal{A}$ into $\mathcal{X}$ is a bounded linear map $D : \mathcal{A} \to \mathcal{X}$ such that for each $a, b \in \mathcal{A}$,

$$D(ab) = D(a) \cdot b + a \cdot D(b)$$

and if $x \in \mathcal{X}$, the mapping $\delta_x : \mathcal{A} \to \mathcal{X}$ defined by $\delta_x(a) = a \cdot x - x \cdot a$ is a derivation. A derivation of this form is called an inner derivation. The set of all derivations from $\mathcal{A}$ into $\mathcal{X}$ is denoted

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by $\mathcal{Z}^1(\mathcal{A},\mathcal{X})$ and the set of all inner derivations from $\mathcal{A}$ into $\mathcal{X}$ is denoted by $\mathcal{N}^1(\mathcal{A},\mathcal{X})$. Then $\mathcal{H}^1(\mathcal{A},\mathcal{X}) = \frac{\mathcal{Z}^1(\mathcal{A},\mathcal{X})}{\mathcal{N}^1(\mathcal{A},\mathcal{X})}$ is the first Hochschild cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{X}$.

Notice that $\mathcal{H}^1(\mathcal{A},\mathcal{X}) = \{0\}$ if and only if every derivation from $\mathcal{A}$ into Banach $\mathcal{A}-$bimodule $\mathcal{X}$ is an inner derivation.

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a Banach $\mathcal{A}-$bimodule. Then $\mathcal{X}^*$ is the dual of Banach $\mathcal{A}-$bimodule, and is also a Banach $\mathcal{A}-$bimodule as well, if for each $a \in \mathcal{A},\ x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$ we define

$$\langle x, a\cdot x^*\rangle = \langle x\cdot a, x^*\rangle \quad , \quad \langle x, x^*\cdot a\rangle = \langle a\cdot x, x^*\rangle.$$

A Banach algebra $\mathcal{A}$ is amenable if every derivation from $\mathcal{A}$ into every dual Banach $\mathcal{A}-$bimodule is inner, equivalently if $\mathcal{H}^1(\mathcal{A},\mathcal{X}^*) = \{0\}$ for every Banach $\mathcal{A}-$bimodule $\mathcal{X}$, this definition was introduced by Johnson in [5]. The Banach algebra $\mathcal{A}$ is weakly amenable if $\mathcal{H}^1(\mathcal{A},\mathcal{A}^*) = \{0\}$; this definition generalizes that introduced by Bade, Curtis and Dales in [1].

2. $\sigma-$WEAK AMENABILITY OF BANACH ALGEBRAS

Let $\mathcal{A}$ and $\mathcal{B}$ be a Banach algebras. We denote by $Hom(\mathcal{A},\mathcal{B})$ the space of all continuous homomorphisms from $\mathcal{A}$ into $\mathcal{B}$. In particular, the space of all continuous homomorphisms from $\mathcal{A}$ into $\mathcal{A}$ by $Hom(\mathcal{A},\mathcal{A})$ or $Hom(\mathcal{A})$.

Let $\mathcal{A}$ be a Banach algebra, $\mathcal{X}$ be a Banach $\mathcal{A}-$bimodule and $\sigma \in Hom(\mathcal{A})$. We consider the following module actions on $\mathcal{X}$,

$$a\cdot x := \sigma(a)x \quad , \quad x\cdot a := x\sigma(a) \qquad (a\in\mathcal{A}, x\in\mathcal{X}).$$

We denote the above Banach $\mathcal{A}-$bimodule by $\mathcal{X}_\sigma$.

Definition 1. *Let $\mathcal{A}$ be a Banach algebra, $\mathcal{X}$ be a Banach $\mathcal{A}-$bimodule and $\sigma \in Hom(\mathcal{A})$. The bounded linear map $D : \mathcal{A} \to \mathcal{X}$ is $\sigma-$derivation, if*

$$D(ab) = D(a)\cdot\sigma(b) + \sigma(a)\cdot D(b) \qquad (a,b\in\mathcal{A}).$$

For example every ordinary derivation of Banach algebra $\mathcal{A}$ into Banach $\mathcal{A}-$bimodule $\mathcal{X}$ is $id_{\mathcal{A}}-$derivation, where $id_{\mathcal{A}}$ is the identity map on $\mathcal{A}$. Derivations of this from are studied in [7,8,10]. A bounded linear map $D : \mathcal{A} \to \mathcal{X}$ is called $\sigma-$inner derivation, if there exists $x \in \mathcal{X}$ such that

$$D(a) = \delta_x^\sigma(a) = \sigma(a)\cdot x - x\cdot\sigma(a) \qquad (a\in\mathcal{A}).$$

We denote the set of all $\sigma-$derivations from $\mathcal{A}$ into $\mathcal{X}$ by $\mathcal{Z}_\sigma^1(\mathcal{A},\mathcal{X})$ and the set of all $\sigma-$inner derivations from $\mathcal{A}$ into $\mathcal{X}$ by $\mathcal{N}_\sigma^1(\mathcal{A},\mathcal{X})$. We define the space $\mathcal{H}_\sigma^1(\mathcal{A},\mathcal{X})$ as the quotient space $\frac{\mathcal{Z}_\sigma^1(\mathcal{A},\mathcal{X})}{\mathcal{N}_\sigma^1(\mathcal{A},\mathcal{X})}$. The space $\mathcal{H}_\sigma^1(\mathcal{A},\mathcal{X})$ is called the first $\sigma-$cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{X}$. Notice that $\mathcal{H}_\sigma^1(\mathcal{A},\mathcal{X}) = \{0\}$ if and only if every $\sigma-$derivation from $\mathcal{A}$ into Banach $\mathcal{A}-$bimodule $\mathcal{X}$ is an $\sigma-$inner derivation.

Definition 2. *The Banach algebra $\mathcal{A}$ is called $\sigma-$amenable if every $\sigma-$derivation from $\mathcal{A}$ into Banach $\mathcal{A}-$bimodule $\mathcal{X}^*$ be an $\sigma-$inner derivation for each Banach $\mathcal{A}-$bimodule $\mathcal{X}$, equivalently if $\mathcal{H}_\sigma^1(\mathcal{A},\mathcal{X}^*) = \{0\}$ for every Banach $\mathcal{A}-$bimodule $\mathcal{X}$.*

Definition 3. *The Banach algebra $\mathcal{A}$ is called $\sigma-$weakly amenable if every $\sigma-$derivation from $\mathcal{A}$ into $\mathcal{A}^*$ be $\sigma-$inner derivation (i.e. $\mathcal{H}_\sigma^1(\mathcal{A},\mathcal{A}^*) = \{0\}$). For example the Banach algebra $\mathcal{A}$ is weakly amenable if and only if $\mathcal{A}$ is $id_{\mathcal{A}}-$weakly amenable.*

Theorem 4. *Let $\mathcal{A}$ be a $\sigma-$amenable Banach algebra. Then $\mathcal{A}$ has a bounded approximate identity for $\sigma(\mathcal{A})$; i.e., there exists a bounded net $\{e_\alpha\}$ in $\mathcal{A}$ such that*

$$\sigma(a)e_\alpha \to \sigma(a) \quad , \quad e_\alpha\sigma(a) \to \sigma(a) \qquad (a \in \mathcal{A})$$

Proof . Consider $\mathcal{X} = \mathcal{A}^*$ as a Banach $\mathcal{A}-$bimodule with the trivial left action, that is:

$$a \cdot x := 0 \quad , \quad x \cdot a := xa \qquad (a \in \mathcal{A} \ , \ x \in \mathcal{X})$$

Then $D : \mathcal{A} \to \mathcal{X}^*$ defined by $D(a) = \widehat{\sigma(a)}$ is a $\sigma-$derivation. Since $\mathcal{A}$ is $\sigma-$amenable, there exists $E \in \mathcal{X}^* = \mathcal{A}^{**}$ such that $D = \delta_E^\sigma$. Hence

$$\widehat{\sigma(a)} = \sigma(a) \cdot E - E \cdot \sigma(a) = \sigma(a) \cdot E \qquad (a \in \mathcal{A})$$

Now, let $\{e_\alpha\}$ be a bounded net in $\mathcal{A}$ such that $\widehat{e_\alpha} \overset{w^*}{\to} E$. Then

$$\widehat{\sigma(a)e_\alpha} \overset{w^*}{\to} \sigma(a) \cdot E = \widehat{\sigma(a)} \qquad (a \in \mathcal{A})$$

Hence $\sigma(a)e_\alpha \overset{w}{\to} \sigma(a)$ for each $a \in \mathcal{A}$. Which shows that $\{e_\alpha\}$ is a weakly right approximate identity for $\sigma(\mathcal{A})$. So $\sigma(\mathcal{A})$ has a bounded right approximate identity and similarly, has a bounded left approximate identity in $\mathcal{A}$. Thus $\sigma(\mathcal{A})$ has a bounded approximate identity in $\mathcal{A}$. □

Corollary 5. *Let $\mathcal{A}$ be a $\sigma-$amenable Banach algebra and $\overline{\sigma(\mathcal{A})} = \mathcal{A}$. Then $\mathcal{A}$ has a bounded approximate identity.*

We recall that a Banach algebra $\mathcal{A}$ is called essential, if the linear span of $\{a \cdot b; \ a, b \ \in \mathcal{A}\}$ is dense in $\mathcal{A}$.

Theorem 6. *Let $\mathcal{A}$ be a $\sigma-$weakly amenable Banach algebra such that $\overline{\sigma(\mathcal{A})} = \mathcal{A}$. Then $\mathcal{A}$ is essential.*

Proof . Let $\overline{span}\ \mathcal{A} \cdot \mathcal{A} \neq \mathcal{A}$. By Hahn-Banach Theorem there is a nonzero $a^* \in \mathcal{A}^*$ such that $\langle ab, a^* \rangle = 0$ for each $a, b \in \mathcal{A}$. Define $D : \mathcal{A} \to \mathcal{A}^*$ by $D(a) = \langle \sigma(a), a^* \rangle a^*$. It is routinely checked that D is a $\sigma-$derivation. So there is $b^* \in \mathcal{A}^*$ such that

$$D(a) = \sigma(a)b^* - b^*\sigma(a) \qquad (a \in \mathcal{A})$$

Hence $\langle \sigma(a), D(a) \rangle = \langle \sigma(a^2) - \sigma(a^2), b^* \rangle = 0$ for each $a \in \mathcal{A}$ and thus $\langle \sigma(a), a^* \rangle \langle \sigma(a), a^* \rangle = 0$ for each $a \in \mathcal{A}$. So $a^* = 0$ on $\sigma(\mathcal{A})$ and since $\overline{\sigma(\mathcal{A})} = \mathcal{A}$, we have $a^* = 0$. This is a contradiction. □

Let $\mathcal{A}$ be a Banach algebra. We recall that when $\mathcal{A}$ has a bounded right (left) approximate identity and $\mathcal{X}$ be a Banach $\mathcal{A}-$bimodule such that $a \cdot x = 0$ ($x \cdot a = 0$) for each $a \in \mathcal{A}$ and $x \in \mathcal{X}$, then $\mathcal{H}^1(\mathcal{A}, \mathcal{X}^*) = \{0\}$ [12, Proposition 2.1.3]. Similarly if $\mathcal{A}$ has a bounded right approximate identity and $\sigma(a) \cdot x = 0$ for each $a \in \mathcal{A}$ and $x \in \mathcal{X}$, then $\mathcal{H}^1_\sigma(\mathcal{A}, \mathcal{X}^*) = \{0\}$.

Definition 7. *Let $\mathcal{A}$ be a commutative Banach algebra and $\sigma \in Hom(\mathcal{A})$. Banach $\mathcal{A}-$bimodule $\mathcal{X}$ is called $\sigma-$symmetric, if*

$$\sigma(a) \cdot x = x \cdot \sigma(a) \qquad (a \in \mathcal{A} \ , \ x \in \mathcal{X}).$$

Note that when $\mathcal{A}$ is a commutative Banach algebra and $\mathcal{X}$ is a $\sigma-$symmetric Banach $\mathcal{A}-$bimodule. Then every $\sigma-$inner derivation from $\mathcal{A}$ into $\mathcal{X}$ is zero. Clearly, every symmetric Banach $\mathcal{A}-$bimodule is a $\sigma-$symmetric Banach $\mathcal{A}-$bimodule.

Theorem 8. *Let $\mathcal{A}$ be a commutative Banach algebra, $\sigma \in Hom(\mathcal{A})$ and $\overline{\sigma(\mathcal{A})} = \mathcal{A}$. Then $\mathcal{A}$ is $\sigma-$weakly amenable if and only if for each $\sigma-$symmetric Banach $\mathcal{A}-$bimodule $\mathcal{X}$, every $\sigma-$derivation from $\mathcal{A}$ into $\mathcal{X}$ is zero.*

Proof . Suppose that $\mathcal{A}$ is $\sigma-$weakly amenable, $\mathcal{X}$ is a $\sigma-$symmetric Banach $\mathcal{A}-$bimodule and $D : \mathcal{A} \to \mathcal{X}$ be an arbitrary $\sigma-$derivation. By Theorem 6, $\mathcal{A}$ is essential. Assume that $D \neq 0$. So there are a_0 and b_0 in $\mathcal{A}$ such that

$$D(a_0b_0) = D(a_0)\sigma(b_0) + \sigma(a_0)D(b_0) \neq 0.$$

Without lose of generality we can assume that $\sigma(a_0)D(b_0) \neq 0$. Then

$$\exists\ x^* \in \mathcal{X}^* \qquad s.t. \qquad \langle \sigma(a_0)D(b_0), x^*\rangle = 1$$

The mapping $d : \mathcal{A} \to \mathcal{A}^*$ defined by $d(a) = D(a)x^*$ is a $\sigma-$derivation, and hence $d = 0$. On the other hand,

$$0 = \langle \sigma(a_0), d(b_0)\rangle = \langle \sigma(a_0), D(b_0)x^*\rangle = \langle \sigma(a_0)D(b_0), x^*\rangle = 1$$

This is a contradiction and so $D = 0$.

The converse is trivial. □

Let $\mathcal{A}$ be a Banach algebra and $\sigma \in Hom(\mathcal{A})$. The Banach algebra $\mathcal{A}^{\#} = \mathcal{A} \oplus \mathbb{C}$, is the unitization of $\mathcal{A}$ with $\| a + \alpha \| = \| a \| + | \alpha |$ for $a \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. Define $\sigma^{\#} : \mathcal{A}^{\#} \to \mathcal{A}^{\#}$ by $\sigma^{\#}(a + \alpha) = \sigma(a) + \alpha$ for each $a \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. It is easy too see that $\sigma^{\#}$ is a bounded endomorphism of $\mathcal{A}^{\#}$.

We know that $(\mathcal{A}^{\#})^* = \mathcal{A}^* \oplus \mathbb{C}e^*$, where

$$\langle a + \beta, a^* + \alpha e^*\rangle = a^*(a) + \alpha\beta \quad for \quad a \in \mathcal{A},\ a^* \in \mathcal{A}^*,\ \alpha, \beta \in \mathbb{C}$$

In particular $e^*(a + \alpha) = \alpha$. Note that $\| a^* + \alpha e^* \| = max\ \{\| a^* \|, | \alpha |\}$ for $a^* \in \mathcal{A}^*$ and $\alpha \in \mathbb{C}$.

Now for $a, b \in \mathcal{A}$, $a^* \in \mathcal{A}^*$ and $\alpha, \beta \in \mathbb{C}$ we have

$$(a + \alpha) \cdot (a^* + \beta e^*) = aa^* + \alpha a^* + (a^*(a) + \alpha\beta)e^*$$

$$(a^* + \beta e^*) \cdot (a + \alpha) = a^*a + \alpha a^* + (a^*(a) + \beta\alpha)e^*$$

Note that $\mathbb{C}e^*$ is a closed $\mathcal{A}^{\#}-$submodule of $(\mathcal{A}^{\#})^*$, but $\mathcal{A}^*$ is not a $\mathcal{A}^{\#}-$submodule of $(\mathcal{A}^{\#})^*$. The projection map $P_2 : (\mathcal{A}^{\#})^* \to \mathbb{C}e^*$ is not a $\mathcal{A}^{\#}-$bimodule homomorphism.

We know that $\mathcal{A}^{\#}$ is a Banach $\mathcal{A}-$bimodule with the following module actions:

$$a \cdot (b + \beta) = ab + \beta a \quad , \quad (b + \beta) \cdot a = ba + \beta a \qquad (a, b \in \mathcal{A}\ ,\ \beta \in \mathbb{C}).$$

So $(\mathcal{A}^{\#})^*$ is a Banach $\mathcal{A}-$bimodule with the following module actions:

$$a \cdot (a^* + \alpha e^*) = aa^* + a^*(a)e^* \quad , \quad (a^* + \alpha e^*) \cdot a = a^*a + a^*(a)e^*$$

for $a \in \mathcal{A}$, $a^* \in \mathcal{A}^*$ and $\alpha \in \mathbb{C}$.

The projection $P_1 : (\mathcal{A}^{\#})^* \to \mathcal{A}^*$ defined by $P_1(a^* + \alpha e^*) = a^*$, is a Banach $\mathcal{A}-$bimodule homomorphism.

Definition 9. *Let $\mathcal{A}$ be unital (with unit e) Banach algebra and $\sigma \in Hom(\mathcal{A})$. Banach $\mathcal{A}-$bimodule $\mathcal{X}$ is called $\sigma-$unital, if*

$$\sigma(e) \cdot x = x \cdot \sigma(e) = x \qquad (x \in \mathcal{X})$$

We know that if $\mathcal{A}$ is a unital (with unit e) Banach algebra and $\overline{\sigma(\mathcal{A})} = \mathcal{A}$, then $\sigma(e) = e$.

Lemma 10. *Let $\mathcal{A}$ be a unital (with unit e) Banach algebra, $\sigma \in Hom(\mathcal{A})$. If $\mathcal{X}$ is a $\sigma-$unital Banach $\mathcal{A}-$bimodule and $D : \mathcal{A} \to \mathcal{X}$ is a $\sigma-$derivation, then $D(e) = 0$.*

Proof . $D(e) = D(e \cdot e) = D(e) \cdot \sigma(e) + \sigma(e) \cdot D(e) = 2D(e)$. So we have $D(e) = 0$. □

Theorem 11. *Let $\mathcal{A}$ be a commutative Banach algebra, $\sigma \in Hom(\mathcal{A})$ and $\overline{\sigma(\mathcal{A})} = \mathcal{A}$. If $\mathcal{A}^{\#}$ is $\sigma^{\#}-$weakly amenable. Then $\mathcal{A}$ is $\sigma-$weakly amenable.*

Proof . Clearly $\mathcal{A}^{\#}$ is a commutative Banach algebra and $\overline{\sigma^{\#}(\mathcal{A}^{\#})} = \mathcal{A}^{\#}$. Suppose that and $d : \mathcal{A} \to \mathcal{X}$ be an arbitrary $\sigma-$derivation, where $\mathcal{X}$ is a $\sigma-$symmetric Banach $\mathcal{A}-$bimodule. Note that $\mathcal{X}$ is a $\sigma^{\#}-$symmetric Banach $\mathcal{A}^{\#}-$bimodule with the following module actions:

$$(a+\alpha) \cdot x = ax + \alpha x \quad , \quad x \cdot (a+\alpha) = xa + \alpha x \qquad (a+\alpha \in \mathcal{A}^{\#} \ , \ x \in \mathcal{X})$$

Define $D : \mathcal{A}^{\#} \to \mathcal{X}$ by $D(a+\alpha) = d(a)$. Clearly D is a bounded linear map. Also we have

$$\begin{aligned} D\left((a+\alpha)\cdot(b+\beta)\right) &= d(ab) + \alpha d(b) + \beta d(a) \\ &= d(a)\sigma(b) + \beta d(a) + \sigma(a) d(b) + \alpha d(b) \\ &= d(a) \cdot (\sigma(b) + \beta) + (\sigma(a) + \alpha) \cdot d(b) \\ &= D(a+\alpha) \cdot \sigma^{\#}(b+\beta) + \sigma^{\#}(a+\alpha) \cdot D(b+\beta) \end{aligned}$$

Thus D is a $\sigma^{\#}-$derivation and therefore $D = 0$. So $d = 0$ and $\mathcal{A}$ is $\sigma-$weakly amenable. □

Theorem 12. *Let $\mathcal{A}$ be a Banach algebra, $\sigma \in Hom(\mathcal{A})$ and $\overline{\sigma(\mathcal{A})} = \mathcal{A}$. If $\mathcal{A}$ is $\sigma-$weakly amenable. Then $\mathcal{A}^{\#}$ is $\sigma^{\#}-$weakly amenable.*

Proof . Let $D : \mathcal{A}^{\#} \to (\mathcal{A}^{\#})^*$ be an arbitrary $\sigma^{\#}-$derivation. Since $(\mathcal{A}^{\#})^*$ is a $\sigma^{\#}-$unital Banach $\mathcal{A}-$bimodule and $\sigma^{\#}(1) = 1$, so $D(1) = 0$. Let $P_1 : (\mathcal{A}^{\#})^* \to \mathcal{A}^*$ defined by $P_1(a^* + \alpha e^*) = a^*$ and $P_2 : (\mathcal{A}^{\#})^* \to \mathbb{C}$ defined by $P_2(a^* + \alpha e^*) = \alpha$ for each $a^* \in \mathcal{A}^*$ and $\alpha \in \mathbb{C}$. Define $d : \mathcal{A} \to \mathcal{A}^*$ by $d(a) = P_1\left(D(a)\right)$ and $\Phi : \mathcal{A} \to \mathbb{C}$ by $\Phi(a) = P_2\left(D(a)\right)$. Clearly d and Φ are bounded linear maps, furthermore $D(a) = d(a) + \Phi(a)e^*$ for each $a \in \mathcal{A}$. Also for each $a, b \in \mathcal{A}$ we have:

$$\begin{aligned} d(ab) &= P_1\left(D(ab)\right) \\ &= P_1\left[D(a)\sigma^{\#}(b) + \sigma^{\#}(a) D(b)\right] \\ &= P_1\left[\left(d(a) + \Phi(a)e^*\right)\sigma(b) + \sigma(a)\left(d(b) + \Phi(b)e^*\right)\right] \\ &= P_1\left[d(a)\sigma(b) + \langle \sigma(b), d(a)\rangle e^* + \sigma(a) d(b) + \langle \sigma(a), d(b)\rangle e^*\right] \\ &= d(a)\sigma(b) + \sigma(a) d(b). \end{aligned}$$

So d is a $\sigma-$derivation. Hence there is $a_0^* \in \mathcal{A}^*$ such that

$$d(a) = \sigma(a) a_0^* - a_0^* \sigma(a) \qquad (a \in \mathcal{A})$$

Now by Theorem 6, $\mathcal{A}$ is essential and for each $a, b \in \mathcal{A}$ we have

$$\begin{aligned}
\Phi(ab) &= P_2\left(D(ab)\right) \\
&= \langle \sigma(a), d(b)\rangle + \langle \sigma(b), d(a)\rangle \\
&= \langle \sigma(a), \sigma(b)a_0^* - a_0^*\sigma(b)\rangle + \langle \sigma(b), \sigma(a)a_0^* - a_0^*\sigma(a)\rangle \\
&= \langle \sigma(a)\sigma(b) - \sigma(b)\sigma(a) + \sigma(b)\sigma(a) - \sigma(a)\sigma(b), a_0^*\rangle = 0
\end{aligned}$$

Consequently $\Phi = 0$.

$$D(a + \alpha\cdot 1) = D(a) = d(a) + \Phi(a)e^* = d(a) = \sigma(a)a_0^* - a_0^*\sigma(a)$$

We have also

$$\begin{aligned}
\delta_{a_0^*}^{\sigma^\#}(a + \alpha\cdot 1) &= \sigma^\#(a + \alpha\cdot 1)\cdot a_0^* - a_0^*\cdot\sigma^\#(a + \alpha\cdot 1) \\
&= (\sigma(a) + \alpha)\cdot a_0^* - a_0^*\cdot(\sigma(a) + \alpha) \\
&= \sigma(a)a_0^* + \alpha a_0^* + \langle \sigma(a), a_0^*\rangle e^* - a_0^*\sigma(a) - \alpha a_0^* - \langle \sigma(a), a_0^*\rangle e^* \\
&= \sigma(a)a_0^* - a_0^*\sigma(a) \\
&= D(a + \alpha\cdot 1) \qquad (a + \alpha\cdot 1 \in \mathcal{A}^\#)
\end{aligned}$$

Thus $D = \delta_{a_0^*}^{\sigma^\#}$ is an $\sigma^\#-$inner derivation and $\mathcal{A}^\#$ is $\sigma^\#-$weakly amenable. □

Proposition 13. *Let $\mathcal{A}$ be a Banach algebra and $\sigma \in Hom(\mathcal{A})$. If $\mathcal{A}$ is $\sigma-$weakly amenable then it is $\tau o\sigma-$weakly amenable, for each continuous homomorphism τ on $\mathcal{A}$.*

Proof . Let $D : \mathcal{A} \to \mathcal{A}^*$ be an arbitrary $\tau o\sigma-$derivation. Then $D : \mathcal{A} \to \mathcal{A}_\tau^*$ is a $\sigma-$derivation, because

$$\begin{aligned}
\langle c, D(ab)\rangle &= \langle c, D(a)\tau o\sigma(b) + \tau o\sigma(a)D(b)\rangle \\
&= \langle \tau(\sigma(b))c, D(a)\rangle + \langle c\tau(\sigma(a)), D(b)\rangle \\
&= \langle \sigma(b)\cdot c, D(a)\rangle + \langle c\cdot\sigma(a), D(b)\rangle \\
&= \langle c, D(a)\cdot\sigma(b) + \sigma(a)\cdot D(b)\rangle \qquad (a, b, c \in \mathcal{A}).
\end{aligned}$$

So there is $a^* \in \mathcal{A}_\tau^*$ such that

$$D(a) = \delta_{a^*}^{\sigma}(a) = \sigma(a)\cdot a^* - a^*\cdot\sigma(a) \qquad (a \in \mathcal{A})$$

So for arbitrary member $b \in \mathcal{A}_\tau$, we have

$$\begin{aligned}
\langle b, D(a)\rangle &= \langle b, \sigma(a)\cdot a^* - a^*\cdot\sigma(a)\rangle \\
&= \langle b\cdot\sigma(a), a^*\rangle - \langle \sigma(a)\cdot b, a^*\rangle \\
&= \langle b\tau(\sigma(a)), a^*\rangle - \langle \tau(\sigma(a))b, a^*\rangle \\
&= \langle b, \tau o\sigma(a)a^* - a^*\tau o\sigma(a)\rangle \\
&= \langle b, \delta_{a^*}^{\tau o\sigma}(a)\rangle \qquad (a, b \in \mathcal{A})
\end{aligned}$$

Thus D is an $\tau o\sigma-$inner derivation. □

Corollary 14. *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}$ is weakly amenable, then $\mathcal{A}$ is $\sigma-$weakly amenable for each continuous homomorphism σ on $\mathcal{A}$.*

Proposition 15. *Let $\mathcal{A}$ be a Banach algebra and σ be an epimorphism on $\mathcal{A}$. If $\mathcal{A}$ is $\sigma-$weakly amenable, then $\mathcal{A}$ is weakly amenable.*

Proof . Let $D : \mathcal{A} \to \mathcal{A}^*$ be an arbitrary derivation. Set $d := Do\sigma : \mathcal{A} \to \mathcal{A}^*$. Then d is a $\sigma-$derivation, because

$$\begin{aligned} d(ab) &= Do\sigma(ab) \\ &= D(\sigma(a)\sigma(b)) \\ &= D(\sigma(a))\sigma(b) + \sigma(a)D(\sigma(b)) \\ &= Do\sigma(a)\sigma(b) + \sigma(a)Do\sigma(b) \\ &= d(a)\sigma(b) + \sigma(a)d(b) \qquad (a, b \in \mathcal{A}) \end{aligned}$$

Since $\mathcal{A}$ is $\sigma-$weakly amenable so d is an $\sigma-$inner derivation. Thus there exists $a^* \in \mathcal{A}^*$ such that

$$d(a) = \delta^{\sigma}_{a^*}(a) = \sigma(a)a^* - a^*\sigma(a) \qquad (a \in \mathcal{A})$$

Let $b \in \mathcal{A}$ be an arbitrary, there is $a \in \mathcal{A}$ such that $\sigma(a) = b$. We have

$$\begin{aligned} D(b) = Do\sigma(a) = d(a) &= \sigma(a)a^* - a^*\sigma(a) \\ &= ba^* - a^*b \\ &= \delta_{a^*}(b) \end{aligned}$$

Therefor D is an inner derivation and so $\mathcal{A}$ is weakly amenable. □

Proposition 16. *Let $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras, $\sigma \in Hom(\mathcal{B})$ and $\varphi : \mathcal{A} \to \mathcal{B}$ be an epimorphisms. If $\mathcal{A}$ be commutative and weakly amenable, then $\mathcal{B}$ is $\sigma-$weakly amenable.*

Proof . Let $D : \mathcal{B} \to \mathcal{B}^*$ be an arbitrary $\sigma-$derivation. Then $\mathcal{B}^*$ becomes a Banach $\mathcal{A}-$bimodule with the following module actions:

$$a \cdot b^* = \sigma(\varphi(a))b^* \quad , \quad b^* \cdot a = b^*\sigma(\varphi(a)) \qquad (a \in \mathcal{A}\ ,\ b^* \in \mathcal{B}^*)$$

The bounded linear mapping $Do\varphi : \mathcal{A} \to \mathcal{B}^*$ is a derivation.

It is easy to see that $\mathcal{B}$ is commutative and therefore $\mathcal{B}^*$ is a symmetric Banach $\mathcal{B}-$bimodule. Hence $\mathcal{B}^*$ is a symmetric Banach $\mathcal{A}-$bimodule. Now $\mathcal{A}$ is weakly amenable, thus $\mathcal{H}^1(\mathcal{A}, \mathcal{B}^*) = \{0\}$. So $Do\varphi = 0$. Consequently $D = 0$ and $\mathcal{B}$ is $\sigma-$weakly amenable. □

Suppose that $\sigma : \mathcal{A} \to \mathcal{A}$ is an epimorphism and $\mathcal{I}$ is a closed ideal of $\mathcal{A}$ such that $\sigma(\mathcal{I}) \subseteq \mathcal{I}$. Then one can define the map $\widehat{\sigma} : \frac{\mathcal{A}}{\mathcal{I}} \to \frac{\mathcal{A}}{\mathcal{I}}$ by $\widehat{\sigma}(a + \mathcal{I}) = \sigma(a) + \mathcal{I}$ for each $a \in \mathcal{A}$. We notice the mapping $\widehat{\sigma}$ is an epimorphism.

Corollary 17. *Let $\sigma : \mathcal{A} \to \mathcal{A}$ be an epimorphism and $\mathcal{I}$ be a closed ideal of $\mathcal{A}$ such that $\sigma(\mathcal{I}) \subseteq \mathcal{I}$. If $\mathcal{A}$ be commutative and weakly amenable, then $\frac{\mathcal{A}}{\mathcal{I}}$ is $\widehat{\sigma}-$weakly amenable.*

Proposition 18. *Let $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras, $\sigma \in Hom(\mathcal{A})$ such that $\overline{\sigma(\mathcal{A})} = \mathcal{A}$ and $\tau \in Hom(\mathcal{B})$. Also suppose that $\varphi : \mathcal{A} \to \mathcal{B}$ is an epimorphism such that $\varphi o\sigma = \tau o\varphi$. If $\mathcal{A}$ be commutative and $\sigma-$weakly amenable, then $\mathcal{B}$ is $\tau-$weakly amenable.*

Proof . Let $D : \mathcal{B} \to \mathcal{B}^*$ be an arbitrary $\tau-$derivation. Then $\mathcal{B}^*$ is a symmetric Banach $\mathcal{A}-$bimodule with the following module actions:

$$a \cdot b^* = \varphi(a)b^* \quad , \quad b^* \cdot a = b^*\varphi(a) \qquad (a \in \mathcal{A}\ ,\ b^* \in \mathcal{B}^*)$$

Set $d := Do\varphi : \mathcal{A} \to \mathcal{B}^*$. Then d is a $\sigma-$derivation, since

$$\begin{aligned} \langle c, d(ab)\rangle &= \langle c, D(\varphi(a)\varphi(b))\rangle \\ &= \langle c, Do\varphi(a)\tau o\varphi(b) + \tau o\varphi(a)Do\varphi(b)\rangle \\ &= \langle c, Do\varphi(a)\varphi(\sigma(b)) + \varphi(\sigma(a))Do\varphi(b)\rangle \\ &= \langle c, Do\varphi(a) \cdot \sigma(b) + \sigma(a) \cdot Do\varphi(b)\rangle \\ &= \langle c, d(a) \cdot \sigma(b) + \sigma(a) \cdot d(b)\rangle \qquad (a, b \in \mathcal{A}\ ,\ c \in \mathcal{B}) \end{aligned}$$

Since $\mathcal{A}$ is $\sigma-$weakly amenable so $d = 0$. Now for $b \in \mathcal{B}$ there is $a \in \mathcal{A}$ such that $\varphi(a) = b$. Therefor we have

$$\begin{aligned} D(b) = D(\varphi(a)) = d(a) = 0 &= \sigma(a) \cdot b^* - b^* \cdot \sigma(a) \\ &= \varphi o\sigma(a)b^* - b^*\varphi o\sigma(a) \\ &= \tau(\varphi(a))b^* - b^*\tau(\varphi(a)) \\ &= \tau(b)b^* - b^*\tau(b) \\ &= \delta^{\tau}_{b^*}(b) \end{aligned}$$

Thus D be a $\tau-$inner derivation and so $\mathcal{B}$ is $\tau-$weakly amenable. □

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