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On the fine spectrum of generalized upper triangular double-band matrices Δ^{uv} over the sequence spaces c_o and c

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Abstract

The main purpose of this paper is to determine the fine spectrum of the generalized upper triangular double-band matrices Δ^{uv} over the sequence spaces c_0 and c. These results are more general than the spectrum of upper triangular double-band matrices of Karakaya and Altun[V. Karakaya, M. Altun, Fine spectra of upper triangular double-band matrices, Journal of Computational and Applied Mathematics. 234(2010) 1387-1394].

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1. Preliminaries, Background and Notations

In functional analysis, the spectrum of an operator generalizes the notion of eigenvalues for matrices. The spectrum of an operator over a Banach space is partitioned into three parts, which are the point spectrum, the continuous spectrum and the residual spectrum. The calculation of three parts of the spectrum of an operator is called calculating the fine spectrum of the operator.

Several authors have studied the spectrum and fine spectrum of linear operators defined by some particular limitation matrices over some sequence spaces. We introduce knowledge in the existing literature concerning the spectrum and the fine spectrum. The fine spectrum of the Cesaro operator on the sequence space ℓ_p for $(1 < p < \infty)$ has been studied by Gonzalez [14]. Also, Wenger [22] examined the fine spectrum of the integer power of the Cesaro operator over c, and Rhoades [19] generalized this result to the weighted mean methods. Reade [18] worked the spectrum of the Cesaro

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operator over the sequence space c_0. Okutoyi [17] computed the spectrum of the Cesaro operator over the sequence space bv. The fine spectrum of the Rhally operators on the sequence spaces c_0 and c is studied by Yildirim [24]. The fine spectra of the Cesaro operator over the sequence spaces c_0 and bv_p have determined by Akhmedov and Basar [1, 4]. Akhmedov and Basar [2, 3]. have studied the fine spectrum of the difference operator Δ over the sequence spaces ℓ_p, and bv_p, where $(1 \leq p < \infty)$. The fine spectrum of the Zweier matrix as an operator over the sequence spaces ℓ_1 and bv_1 have been examined by Altay and Karakus [6].. Altay and Basar [5, 9]. have determined the fine spectrum of the difference operator Δ over the sequence spaces c_0, c and ℓ_p, where $(0 < p < 1)$. The fine spectrum of the difference operator Δ over the sequence spaces ℓ_1 and bv is investigated by Kayaduman and Furkan [6]. Altun and Karakaya [7, 8]. has been studied the fine spectra of Lacunary Matrices and Fine spectra of upper triangular triangular double-band matrices. Also, the fine spectrum of the operator Δ_{uv} over the sequence space c_0 has been examined by Fathi and Lashkaripour [10] recently, Fathi and Lashkaripour [10, 11]. has been studied the fine spectrum of generalized upper triangular double-band matrices Δ^v and Δ^{uv} over the sequence ℓ_1.

In this work, our purpose is to determine the fine spectra of the generalized upper triangular double-band matrices Δ^{uv} as an operator over the sequence spaces c_0 and c.

By w, we denote the space of all real or complex valued sequences. Any vector subspace of w is called a sequence space. Let μ and ν be two sequence spaces and $A = (a_{n,k})$ be an infinite matrix operator of real or complex numbers $a_{n,k}$, where $n, k \in \{0, 1, 2, \ldots\}$. We say that A defines a matrix mapping from μ into ν and denote it by $A : \mu \longrightarrow \nu$, if for every sequence $x = \big(x_k\big) \in \mu$ the sequence $Ax = \big((Ax)_n\big)$, the A-transform of x, is in ν, where $(Ax)_n = \sum_{k=0}^{\infty} a_{n,k} x_k$.

Let X and Y be Banach spaces and $T : X \longrightarrow Y$, also be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \{y \in Y : y = Tx, x \in X\}.$$

By $B(X)$, we denote the set of all bounded linear operator on X into itself. If X is any Banach space and $T \in B(X)$ then the *adjoint* T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\psi)(x) = \psi(Tx)$ for all $\psi \in X^*$ and $x \in X$ with $\|T\| = \|T^*\|$.

Let $X \neq \Theta$ be a complex normed space and $T : \mathbb{D}(T) \longrightarrow X$, also be a bounded linear operator with domain $\mathbb{D} \subseteq X$. With T, we associate the operator $T_\lambda = T - \lambda I$, where λ is a complex number and I is the identity operator on $\mathcal{D}(T)$, if T_λ has an inverse, which is linear, we denote it by T_λ^{-1}, that is $T_\lambda^{-1} = (T - \lambda I)^{-1}$ and call it the *resolvent* operator of T.

The name resolvent is appropriate, since T_λ^{-1} helps to solve the equation $T_\lambda x = y$. Thus, $x = T_\lambda^{-1} y$ provided T_λ^{-1} exists. More important, the investigation of properties of T_λ^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all λ in the complex plane such that T_λ^{-1} exists. Boundedness of T_λ^{-1} is another property that will be essential. We shall also ask for what λ the domain of T_λ^{-1} is dense in X, to name just a few aspects. For our investigation of T, T_λ and T_λ^{-1}, we shall need some basic concepts in spectral theory which are given as follows (see [13], pp. 370-371).

Definition 1.1. Let $X \neq \emptyset$ be a complex normed space and $T : \mathbb{D}(T) \longrightarrow X$, be a linear operator with domain $\mathbb{D} \subseteq X$. A *regular* value of T is a complex number λ such that
$(R1)$ T_λ^{-1} exists,

$(R2)$ T_λ^{-1} is bounded,
$(R3)$ T_λ^{-1} is defined on a set which is dense in X.

The *resolvent* set $\rho(T,X)$ of T is the set of all *regular* value λ of T. Its complement $\sigma(T,X) = \mathbb{C} - \rho(T,X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into three disjoint sets as follows:
The *point spectrum* $\sigma_p(T,X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} dose not exist. The element of $\sigma_p(T,X)$ is called *eigenvalue* of T.
The *continuous spectrum* $\sigma_c(T,X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} exists and satisfies $(R3)$ but not $(R2)$, that is, T_λ^{-1} is unbounded.
The *residual spectrum* $\sigma_r(T,X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} exists but do not satisfy $(R3)$, that is, the domain of T_λ^{-1} is not dense in X. The condition $(R2)$ may or may not holds good.

Goldberg's classification of operator $T_\lambda = (T - \lambda I)$ (see [13], PP. 58-71): Let X be a Banach space and $T_\lambda = (T - \lambda I) \in B(X)$, where λ is a complex number. Again let $R(T_\lambda)$ and T_λ^{-1} be denote the range and inverse of the operator T_λ, respectively. Then following possibilities may occur:
(A) $R(T_\lambda) = X$,
(B) $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$,
(C) $\overline{R(T_\lambda)} \neq X$,
and
(1) T_λ is injective and T_λ^{-1} is continuous,
(2) T_λ is injective and T_λ^{-1} is discontinuous,
(3) T_λ is not injective.
If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2$ *and* C_3. If λ is a complex number such that $T_\lambda \in A_1$ *or* $T_\lambda \in B_1$, then λ is in the resolvent set $\rho(T,X)$ of T on X. The other classifications give rise to the fine spectrum of T. We use $\lambda \in B_2\sigma(T,X)$ means the operator $T_\lambda \in B_2$, i.e. $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$ and T_λ is injective but T_λ^{-1} is discontinuous, similarly others.

Lemma 1.2. ([13], p.59). A linear operator T has a dense range if and only if the adjoint T^* is one to one.

Lemma 1.3. ([13], p.60). The adjoint operator T^* is onto if and only if T has a bounded inverse.

Lemma 1.4. ([23], Theorem 1.3.6). The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c)$ from c to itself if and only if
(1) the rows of A in ℓ_1 and their ℓ_1 norms are bounded.
(2) the columns of A are in c.
(3) the sequence of row sums of A is in c.

The operator norm of T is the supremum of the ℓ_1 norms of the rows.

Lemma 1.5. ([23], Example 8.4.5 A). The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c_0)$ from c_0 to itself if and only if
(1) the rows of A in ℓ_1 and their ℓ_1 norms are bounded.
(2) the columns of A are in c_0.

The operator norm of T is the supremum of the ℓ_1 norms of the rows.

Lemma 1.6. ([21], Lemma 1). Let (a_n) be a bounded sequence of complex numbers such that $\lim_{n\to\infty}(a_{n+1}+\alpha a_n)$ exists and is finite, say a, for some $\alpha \in \mathbb{C}$. Then
(a) If $|\alpha| \neq 1$, the sequence (a_n) is convergent. Moreover, in this case $\lim_{n\to\infty}(a_n) = \frac{a}{1+\alpha}$.
(b) For each $|\alpha| = 1$, the sequence (a_n) can be divergent.

In this paper, we introduce a class of a generalized upper triangular double-band matrices Δ^{uv} over sequence spaces c_o and c. Let (u_k) be a sequence of positive real numbers such that $u_k \neq 0$ for each $k \in \mathbb{N}$ with $u = \lim_{k\to\infty} u_k \neq 0$ and (v_k) is either constant or strictly decreasing sequence of positive real numbers with $v = \lim_{k\to\infty} v_k \neq 0$, and $\sup_k v_k < u + v$. We define the operator Δ^{uv} on sequence space c_0 as follows:

$$\Delta^{uv}x = \Delta^{uv}(x_n) = (v_n x_n + u_{n+1}x_{n+1})_{n=0}^{\infty}.$$

It is easy to verify that the operator Δ^{uv} can be represented by the matrix,

$$\Delta^{uv} = \begin{bmatrix} v_0 & u_1 & 0 & 0 & 0 & \cdots \\ 0 & v_1 & u_2 & 0 & 0 & \cdots \\ 0 & 0 & v_2 & u_3 & 0 & \cdots \\ 0 & 0 & 0 & v_3 & u_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

2. The fine spectra of Δ^{uv} over c_0

In this section, we compute spectrum, the point spectrum, the continuous spectrum and the residual spectrum of the generalized upper triangular double-band matrices Δ^{uv} over the sequence space c_0.

Theorem 2.1. *The operator* $\Delta^{uv} : c_0 \longrightarrow c_0$ *is a bounded linear operator and*

$$\|\Delta^{uv}\| = \sup_k(|v_k| + |u_{k+1}|).$$

Proof . It is elementary. □

Theorem 2.2. *Point spectrum of the operator* Δ^{uv} *over* c_0 *is given by*

$$\sigma_p(\Delta^{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - v| < u\} \cup M_1.$$

where

$$M_1 = \left\{\lambda \in \mathbb{C} : |\lambda - v| = u, \lim_{k\to\infty}\left(\prod_{i=1}^{k}\frac{v_{i-1}-\lambda}{u_i}\right) = 0\right\}.$$

Proof . Consider $\Delta^{uv}x = \lambda x$, for $x \neq \mathbf{0} = (0,0,0,\ldots)$ in c_0, which gives

$$\begin{aligned} v_0x_0 + u_1x_1 &= \lambda x_o \\ v_1x_1 + u_2x_2 &= \lambda x_1 \\ v_2x_2 + u_3x_3 &= \lambda x_2 \\ &\vdots \\ v_kx_k + u_{k+1}x_{k+1} &= \lambda x_k \\ &\vdots \end{aligned}$$

if $x_0 = 0$, then $x_k = 0$ for all k. Hence $x_0 \neq 0$. Solving this equations, we get

$$x_n = \prod_{i=1}^{n} \left(\frac{\lambda - v_{i-1}}{u_i} \right) x_0 \qquad for \quad all \quad n \in \mathbb{N}.$$

Now suppose $\lambda \in \mathbb{C}$ with $|\lambda - v| < u$, then $\lim_{n\to\infty} \left| \frac{v_{n-1}-\lambda}{u_n} \right| < 1$. This means that $\left| \frac{\lambda - v_{i-1}}{u_n} \right| < 1$ for large n, and consequently

$$\lim_{n\to\infty} |x_n| = 0.$$

Also, it can be proved that $M_1 \subseteq \sigma_p(\Delta^{uv}, c_0)$. Thus

$$\{\lambda \in \mathbb{C} : |\lambda - v| < u\} \cup M_1 \subseteq \sigma_p(\Delta^{uv}, c_0).$$

Conversely, if $\lambda \in \sigma_p(\Delta^{uv}, c_0)$, then there exists $x = (x_0, x_1, x_2, ...) \neq 0$ in c_0, $\Delta^{uv} x = \lambda x$. Then, $x_{k+1} = \frac{\lambda - v_k}{u_{k+1}} x_k$, $k \in N$ and $\lim_{k\to\infty} x_k$ exist. Therefore

$$\lim_{k\to\infty} \left| \frac{x_{k+1}}{x_k} \right| = \left| \frac{\lambda - v}{u} \right| \leq 1.$$

(In case $|\lambda - v| = u$, $\lambda \in M_1$) this completes the proof. □

If $T : c_0 \longrightarrow c_0$ is a bounded linear operator with matrix A, then it is known that the adjoint operator $T^* : c_0^* \longrightarrow c_0^*$ is defined by the transpose of the matrix A. The dual space of c_0 is isomorphic to ℓ_1, the space of all absolutely summable sequences, with the norm $\|x\| = \sum_{k=0}^{\infty} |x_k|$. We now obtain The point spectrum of the dual operator $(\Delta^{uv})^*$ of Δ^{uv} over the space c_0^*.

Theorem 2.3. *The point spectrum of the operator Δ^{uv} over c_0^* is*

$$\sigma_p((\Delta^{uv})^*, c_0^*) = \emptyset$$

Proof . The proof of this theorem is divided into two cases:

Case 1. Suppose (v_k) is a constant sequence, say $v_k = v$ for all k. Consider $(\Delta^{uv})^* f = \lambda f$, for $f \neq \mathbf{0} = (0, 0, 0, \ldots)$ in $c_0^* \cong \ell_1$, where

$$(\Delta^{uv})^* = \begin{bmatrix} v_0 & 0 & 0 & 0 & 0 & \cdots \\ u_1 & v_1 & 0 & 0 & 0 & \cdots \\ 0 & u_2 & v_2 & 0 & 0 & \cdots \\ 0 & 0 & u_2 & v_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad and \quad f = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{bmatrix}$$

this gives

$$\begin{aligned} & v_0 f_0 = \lambda f_o \\ & u_1 f_0 + v_1 f_1 = \lambda f_1 \\ & u_2 f_1 + v_2 f_2 = \lambda f_2 \\ & \vdots \\ & u_k f_{k-1} + v_k f_k = \lambda f_k \\ & \vdots \end{aligned}$$

Let f_m be the first non-zero entry of the sequence (f_n). So we get $u_m f_{m-1} + v f_m = \lambda f_m$ which implies $\lambda = v$ and from the equation $u_{m+1} f_m + v f_{m+1} = \lambda f_{m+1}$ we get $f_m = 0$, which is a contradiction to our assumption. Therefore,

$$\sigma_p((\Delta^{uv})^*, c_0^*) = \emptyset.$$

Case 2. Suppose (v_k) is a strictly decreasing sequence. Consider

$$(\Delta^{uv})^* f = \lambda f,$$

for $f \neq \mathbf{0} = (0,0,0,\dots)$ in $c_0^* \cong \ell_1$, which gives above system of equations. Hence, for all

$$\lambda \notin \{v_0, v_1, v_2, \dots\},$$

we have $f_k = 0$ for all k, which is a contradiction. So $\lambda \notin \sigma_p((\Delta^{uv})^*, c_0^*)$. This shows that

$$\sigma_p((\Delta^{uv})^*, c_0^*) \subseteq \{v_0, v_1, v_2, \dots\}.$$

Let $\lambda = v_m$ for some m. Then $f_0 = f_1 = \dots = f_{m-1} = 0$. Now if $f_m = 0$, then $f_k = 0$ for all k, which is a contradiction. Also if $f_m \neq 0$, then

$$f_{k+1} = \frac{u_{k+1}}{v_m - v_{k+1}} f_k, \quad for \quad all \quad k \geq m,$$

and hence,

$$\lim_{k\to\infty} \left| \frac{f_{k+1}}{f_k} \right| = \lim_{k\to\infty} \left| \frac{u_{k+1}}{v_m - v_{k+1}} \right| = \left| \frac{u}{v_m - v} \right| > 1 \quad for \quad all \quad k \geq m,$$

since $v_m < v + u$. Then, $f \notin c_0^*$. Thus

$$\sigma_p((\Delta^{uv})^*, c_0^*) = \emptyset.$$

□

Theorem 2.4. *For any $\lambda \in \mathbb{C}$, $\Delta_\lambda^{uv} : c_0 \longrightarrow c_0$ has a dence range.*

Proof . By Theorem 2.3, $\sigma_p((\Delta^{uv})^*, c_0^*) = \emptyset$. Hence $(\Delta^{uv})^* - \lambda I$ is one to one for all λ. By applying Lemma 1.2, we get the result. □

Corollary 2.5. Residual spectrum $\sigma_r(\Delta^{uv}, c_0)$ of operator Δ^{uv} over c_0 is $\sigma_r(\Delta^{uv}, c_0) = \emptyset$.

Theorem 2.6. *The spectrum of Δ^{uv} on c_0 is given by*

$$\sigma(\Delta^{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - v| \leq u\}.$$

Proof . Let $f \in \ell_1$ and consider $(\Delta_\lambda^{uv})^* x = f$. Then we have the linear system of equations

$$\begin{aligned} &(v_0-\lambda)x_0 = f_o\\ &u_1x_0 + (v_1-\lambda)x_1 = f_1\\ &u_2x_1 + (v_2-\lambda)x_2 = f_2\\ &\vdots\\ &u_kf_{k-1} + (v_k-\lambda)x_k = f_k\\ &\vdots \end{aligned}$$

solving the equations, for $x = (x_k)$ in terms of f, we get

$$x_0 = \frac{1}{v_0-\lambda} f_0, \quad and \quad x_k = \frac{1}{v_k-\lambda}\left[\sum_{i=0}^{k-1}\prod_{j=i}^{k-1}\left(\frac{u_{j+1}}{\lambda - v_j}\right) f_i + f_k\right], \quad for \quad k \geq 1.$$

Then $\sum_k |x_k| \leq \sum_k S_k |f_k|, \quad where$

$$S_k = \frac{1}{|v_k-\lambda|} + \frac{u_{k+1}}{|v_k-\lambda||v_{k+1}-\lambda|} + \frac{u_{k+1}u_{k+2}}{|v_k-\lambda||v_{k+1}-\lambda||v_{k+2}-\lambda|} + \ldots, \quad for \quad all \quad k.$$

Let

$$\begin{aligned} S_{n,k} &= \frac{1}{|v_k-\lambda|} + \frac{u_{k+1}}{|v_k-\lambda||v_{k+1}-\lambda|} + \frac{u_{k+1}u_{k+2}}{|v_k-\lambda||v_{k+1}-\lambda||v_{k+2}-\lambda|}\\ &+ \ldots + \frac{u_{k+1}u_{k+2}\ldots u_{k+n+1}}{|v_k-\lambda||v_{k+1}-\lambda|\ldots|v_{k+n+1}-\lambda|} \quad for \quad all \quad k,n. \end{aligned}$$

Then

$$S_n = \lim_{k\to\infty} S_{n,k} = \frac{1}{|v-\lambda|} + \frac{u}{|v-\lambda|^2} + \frac{u^2}{|v-\lambda|^3} + \ldots + \frac{u^{n+1}}{|v-\lambda|^{n+2}}.$$

Now for $u < |\lambda - v|$, we can see that

$$S = \lim_{n\to\infty} S_n = \frac{1}{|v-\lambda|} + \frac{u}{|v-\lambda|^2} + \frac{u^2}{|v-\lambda|^3} + \ldots < \infty,$$

hence (S_k) is a sequence of positive real numbers which has a lim S. Therefore, (S_k) is bounded and $\sup_k S_k < \infty$. Thus

$$\sum_k |x_k| \leq \sup_k S_k \sum_k |f_k| < \infty.$$

This shows that $x \in \ell_1$. Hence, for $u < |\lambda - v|$, $(\Delta_\lambda^{uv})^*$ is onto, and by Lemma 1.3, Δ_λ^{uv} has a bounded inverse. This means that

$$\sigma_c(\Delta^{uv}, c_0) \subseteq \{\lambda \in \mathbb{C} : |\lambda - v| \leq u\}.$$

Combining this with Theorem 2.2 and Corollary 2.5, we get

$$\{\lambda \in \mathbb{C} : |\lambda - v| < u\} \subseteq \sigma(\Delta^{uv}, c_0) \subseteq \{\lambda \in \mathbb{C} : |\lambda - v| \leq u\}.$$

Since the spectrum of any bounded operator is closed, we have

$$\sigma(\Delta^{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - v| \leq u\}.$$

□

Theorem 2.7. *Continuous spectrum $\sigma_c(\Delta^{uv}, c_0)$ of operator Δ^{uv} over c_0 is*

$$\sigma_c(\Delta^{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - v| = u\} \setminus M_1.$$

Proof . Since $\sigma_r(\Delta^{uv}, c_0) = \emptyset$, $\sigma_p(\Delta^{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - v| < u\} \cup M_1$ and $\sigma(\Delta^{uv}, c_0)$ is the disjoint union of the parts $\sigma_p(\Delta^{uv}, c_0)$, $\sigma_r(\Delta^{uv}, c_0)$ and $\sigma_c(\Delta^{uv}, c_0)$, we deduce that

$$\sigma_c(\Delta^{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - v| = u\} \setminus M_1.$$

□

Theorem 2.8. *If $|\lambda - v| < u$, then $\lambda \in A_3\sigma(\Delta^{uv}, c_0)$.*

Proof . Let $|\lambda - v| < u$. Then by Theorem 2.2, $\lambda \in (3)$ it remains to prove that Δ^{uv}_λ is surjective when $|\lambda - v| < u$. Let $y = (y_0, y_1, y_2, \ldots) \in c_0$ and consider the equation $\Delta^{uv}_\lambda x = y$. Then we have the linear system of equations

$$\begin{aligned}
&(v_0 - \lambda)x_0 + u_1x_1 = y_o \\
&(v_1 - \lambda)x_1 + u_2x_2 = y_1 \\
&(v_2 - \lambda)x_2 + u_3x_3 = y_2 \\
&\vdots \\
&(v_k - \lambda)x_k + u_{k+1}x_{k+1} = y_k \\
&\vdots
\end{aligned}$$

Now, set $x_0 = 0$ and by solving these equations, we get $x_1 = \frac{1}{u_1}y_0$ and

$$x_k = \frac{1}{u_k}\left(\sum_{i=0}^{k-2}\left[\prod_{j=i+1}^{k-1}(\frac{\lambda - v_j}{u_j})\right]y_i + y_{k-1}\right) \qquad for \quad all \quad k \geq 2.$$

By the above equation, x_k satisfies

$$x_k = \frac{\lambda - v_{k-1}}{u_k}x_{k-1} + \frac{1}{u_k}y_{k-1} \qquad for \quad k \geq 1.$$

To complete the proof we need to show that $x \in c_0$. Since $|\lambda - v| < u$, we have

$$\alpha = \lim_{k\to\infty}\left|\frac{\lambda - v_{k-1}}{u_k}\right| < 1.$$

On the other hand

$$\lim_{k\to\infty}(x_k - \alpha x_{k-1}) \quad = \quad \lim_{k\to\infty}\left(x_k - \frac{\lambda - v_{k-1}}{u_k}x_{k-1}\right) = \lim_{k\to\infty}\frac{y_{k-1}}{v_k} = 0.$$

Since $\alpha < 1$, by Lemma 1.6 $\lim_{k\to\infty} x_k = 0$. Hence $x \in c_0$. □

Theorem 2.9. *Let (v_k) and (u_k) be a constant sequences, say $v_k = v$ and $u_k = u$ for all k, and $|\lambda - v| = u$. Then $\lambda \in B_2\sigma(\Delta^{uv}, c_0)$.*

Proof . By Theorem 2.7 $\lambda \in A_2 \cup B_2$. To prove $\lambda \in B_2$, we need to show that Δ^{uv} is not surjective when λ satisfies $|\lambda - v| = u$. Define $y = (y_0, y_1, y_2, \ldots) \in c_0$ by

$$y_k = \left(\frac{\lambda - v}{u}\right)^k \frac{1}{k+1}.$$

Suppose $x \in c_0$ with $\Delta_\lambda^{uv} x = y$. Then we have the linear system equations

$$\begin{aligned}
&(v-\lambda)x_0 + ux_1 = 1\\
&(v-\lambda)x_1 + ux_2 = \left(\frac{\lambda - v}{u}\right)\frac{1}{2}\\
&(v-\lambda)x_2 + ux_3 = \left(\frac{\lambda - v}{u}\right)^2\frac{1}{3}\\
&\vdots
\end{aligned}$$

solving x_n by means of x_0, we get

$$x_n - \left(\frac{\lambda - v}{u}\right)^n x_0 = \frac{1}{u}\left(\frac{\lambda - v}{u}\right)^{n-1}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right).$$

Now, By taking absolute value of both sides and using the triangle inequality we get

$$\frac{1}{u}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) \leq |x_1| + |x_n|.$$

Then we have $\lim_{n\to\infty} |x_n| = \infty$, which contradicts the fact that $x \in c_0$. Hence, there is no $x \in c_0$ satisfying $\Delta_\lambda^{uv} x = y$. So, Δ_λ^{v} is not surjective. □

3. The fine spectra of Δ^{uv} over c

In this section, we compute spectrum, the point spectrum, the continuous spectrum and the residual spectrum of the generalized upper triangular double-band matrices Δ^{uv} over the sequence space c.

Theorem 3.1. *The operator $\Delta^{uv} : c \longrightarrow c$ is a bounded linear operator and*

$$\|\Delta^{uv}\| = \sup_k (|v_k| + |u_{k+1}|).$$

Proof . It is elementary.□

Theorem 3.2. *Point spectrum of the operator Δ^{uv} over c is given by*

$$\sigma_p(\Delta^{uv}, c) = \{\lambda \in \mathbb{C} : |\lambda - v| < u\} \cup M_2$$

where

$$M_2 = \left\{\lambda \in \mathbb{C} : |\lambda - v| = u, \lim_{k\to\infty}\left(\prod_{i=1}^{k}\frac{v_{i-1} - \lambda}{u_i}\right) \text{ exist}\right\}.$$

Proof . Consider $\Delta^{uv}x = \lambda x$, for $x \neq \mathbf{0} = (0,0,0,\ldots)$ in c, which gives

$$\begin{aligned} &v_0x_0 + u_1x_1 = \lambda x_o \\ &v_1x_1 + u_2x_2 = \lambda x_1 \\ &v_2x_2 + u_3x_3 = \lambda x_2 \\ &\vdots \\ &v_kx_k + u_{k+1}x_{k+1} = \lambda x_k \\ &\vdots \end{aligned}$$

if $x_0 = 0$, then $x_k = 0$ for all k. Hence $x_0 \neq 0$. Solving this equations, we get

$$x_n = \prod_{i=1}^{n} \left(\frac{\lambda - v_{i-1}}{u_i} \right) x_0 \qquad for \quad all \quad n \in \mathbb{N}.$$

Now suppose $\lambda \in \mathbb{C}$ with $|\lambda - v| < u$, then $\lim_{n\to\infty} \left| \frac{v_{n-1}-\lambda}{u_n} \right| < 1$. This means that $\left| \frac{\lambda - v_{i-1}}{u_n} \right| < 1$ for large n, and consequently $\lim_{n\to\infty} |x_n| = 0$. Also, it can be proved that $M_2 \subseteq \sigma_p(\Delta^{uv}, c)$. Thus

$$\{\lambda \in \mathbb{C} : |\lambda - v| < u\} \cup M_2 \subseteq \sigma_p(\Delta^{uv}, c).$$

Conversely, if $\lambda \in \sigma_p(\Delta^{uv}, c)$, then there exists $x = (x_0, x_1, x_2, \ldots) \neq 0$ in c, $\Delta^{uv}x = \lambda x$. Then, $x_{k+1} = \frac{\lambda - v_k}{u_{k+1}} x_k$, $k \in N$ and $\lim_{k\to\infty} x_k$ exist. Therefore

$$\lim_{k\to\infty} \left| \frac{x_{k+1}}{x_k} \right| = \left| \frac{\lambda - v}{u} \right| \leq 1.$$

(In case $|\lambda - v| = u$, $\lambda \in M$) this completes the proof. □

If $T : c \longrightarrow c$ is a bounded linear operator represented with matrix A, then the adjoint operator $T^* : c^* \longrightarrow c^*$ acting on $\mathbb{C} \oplus \ell_1$ has a matrix representation of the form

$$\begin{bmatrix} \chi & 0 \\ b & A^t \end{bmatrix},$$

where the χ is the limit of the sequence of row sums of A minus the sum of the limits of the columns of A, and b is the column vector whose k-th entry is the limit of the k-th column of A for each $k \in \mathbb{N}$. For $\Delta^{uv}_\lambda : c \longrightarrow c$, the matrix $(\Delta^{uv}_\lambda)^*$ is of the form

$$\begin{bmatrix} u+v-\lambda & 0 \\ 0 & (\Delta^{uv}_\lambda)^t \end{bmatrix} = \begin{bmatrix} u+v-\lambda & 0 & 0 & 0 & \cdots \\ 0 & v_0 - \lambda & 0 & 0 & \cdots \\ 0 & u_1 & v_1 - \lambda & 0 & \cdots \\ 0 & 0 & u_2 & v_2 - \lambda & \cdots \\ 0 & 0 & 0 & u_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We now obtain The point spectrum of the dual operator $(\Delta^{uv})^*$ of Δ^{uv} over the space c^*.

Theorem 3.3. *The point spectrum of the operator* Δ^{uv} *over* c_0^* *is*

$$\sigma_p((\Delta^{uv})^*, \mathbb{C} \oplus \ell_1) = \{u + v\}$$

Proof . Suppose that λ is eigenvalue of the operator $(\Delta^{uv})^* : \mathbb{C} \oplus \ell_1 \to \mathbb{C} \oplus \ell_1$. Then there exists $0 \neq f \in \ell_1$ satisfying the system of equations

$$\begin{aligned}
&(u+v)f_0 = \lambda f_0\\
&v_0 f_1 = \lambda f_1\\
&u_1 f_1 + v_1 f_2 = \lambda f_2\\
&u_2 f_2 + v_2 f_3 = \lambda f_3\\
&\vdots\\
&u_k f_k + v_k f_{k+1} = \lambda f_{k+1}.\\
&\vdots
\end{aligned}$$

From above we can see that $\lambda = u+v$ is an eigenvalue corresponding to the eigenvector $(1, 0, 0, 0, ...)$. Now, the proof of this theorem is divided into two cases.

Case 1. Suppose (v_k) is a constant sequence, say $v_k = v$ for all k. Now, suppose that $\lambda \neq u + v$. Let f_m be the first non-zero entry of the sequence (f_n). So we get $u_m f_{m-1} + v f_m = \lambda f_m$ which implies $\lambda = v$ and from the equation $u_{m+1} f_m + v f_{m+1} = \lambda f_{m+1}$ we get $f_m = 0$, which is a contradiction to our assumption. Therefore,

$$\sigma_p((\Delta^{uv})^*, \mathbb{C} \oplus \ell_1) = \{u + v\}$$

Case 2. Suppose (v_k) is a strictly decreasing sequence. Consider $(\Delta^{uv})^* f = \lambda f$, for $f \neq \mathbf{0} = (0, 0, 0, \dots)$ in ℓ_1, which gives above system of equations. Hence, for all $\lambda \notin \{u+v, v_0, v_1, v_2, \dots\}$, we have $f_k = 0$ for all k, which is a contradiction. So $\lambda \notin \sigma_p((\Delta^{uv})^*, c^*)$. This shows that

$$\sigma_p((\Delta^{uv})^*, \mathbb{C} \oplus \ell_1) \subseteq \{u + v, v_0, v_1, v_2, \dots\}.$$

Let $\lambda = v_m$ for some m and $\lambda \neq u + v$. Then $f_0 = f_1 = \ldots = f_{m-1} = 0$. Now if $f_m = 0$, then $f_k = 0$ for all k, which is a contradiction. Also if $f_m \neq 0$, then

$$f_{k+1} = \frac{u_k}{v_m - v_k} f_k, \quad for \ \ all \quad k \geq m,$$

and hence,

$$\lim_{k\to\infty} \left| \frac{f_{k+1}}{f_k} \right| = \lim_{k\to\infty} \left| \frac{u_k}{v_m - v_k} \right| = \left| \frac{u}{v_m - v} \right| > 1 \quad for \ \ all \quad k \geq m,$$

since $v_m < v + u$. Then, $f \notin \ell_1$. Thus

$$\sigma_p((\Delta^{uv})^*, \mathbb{C}\oplus\ell_1) = \{u + v\}$$

□

Theorem 3.4. *For any $\lambda \in \mathbb{C}$, $\Delta^{uv}_\lambda : c \longrightarrow c$ has a dence range if and only if $\lambda \neq u + v$*

Proof . By Theorem 3.3, $\sigma_p((\Delta^{uv})^*, \mathbb{C} \oplus \ell_1) = \{u + v\}$ Hence. $(\Delta^{uv})^* - \lambda I$ is one to one for all λ. By applying Lemma 1.2, we get the result. □

Corollary 3.5. Residual spectrum $\sigma_r(\Delta^{uv}, c)$ of operator Δ^{uv} over c is

$$\sigma_r(\Delta^{uv}, c) = \emptyset$$

Since the fine spectrum of the operator Δ^{uv} on c can be obtained using arguments similar to those used in the case of the space c_0, we omit the details and give the results without proof.

Theorem 3.6.

(1) $\sigma(\Delta^{uv}, c) = \{\lambda \in \mathbb{C} : |\lambda - v| \leq u\}$.

(2) $\sigma_c(\Delta^{uv}, c) = \{\lambda \in \mathbb{C} : |\lambda - v| = u\} \setminus M_2$.

Theorem 3.7. If $|\lambda - v| < u$, then $\lambda \in A_3\sigma(\Delta^{uv}, c)$.

Theorem 3.8. Let (v_k) and (u_k) be a constant sequences, say $v_k = v$ and $u_k = u$ for all k, and $\lambda \in \{\lambda \in \mathbb{C} : |\lambda - v| = u\} \setminus M_2$. Then $\lambda \in B_2\sigma(\Delta^{uv}, c)$.

4. Conclusion

In the present work, as a natural continuation of Karakaya and Altun [8] and Fathi and Lashkaripour [12] we have determined the spectrum and the fine spectrum of the double sequential band matrix Δ^{uv} on the sequence spaces c_0 and c. These results are more general than the spectrum of upper triangular double-band matrices of Karakaya and Altun [8] over the sequence spaces c_0 and c. Indeed, if the sequences (v_k) and (u_k) are taken such that $v_k = r$ and $u_k = s$ for all $k \in N$, then the operator Δ^{uv} reduces to the operator $U(r, s)$.

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