Int. J. Nonlinear Anal. Appl. 7 (2016) No. 1, 53-62
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2015.294

Some fixed point theorems for weakly subsequentially continuous and compatible of type (E) mappings with an application

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(Communicated by M. Eshaghi Gordji)

Abstract

In this paper, we will establish some fixed point results, for two pairs of self mappings satisfying generalized contractive condition, by using a new concept as weak subsequential continuity, with compatibility of type (E) in metric spaces, as an application the existence of unique solution for a systems of functional equations arising in system programming is proved.

Keywords: generalized contractive condition; weakly subsequentially continuous; compatible of type (E); functional equation.
2010 MSC: Primary 47H10; Secondary 54H25.

1. Introduction

Jungck [11] introduced the notion of commuting maps, in order to prove a common fixed point theorem, self maps A and S of a metric space (X, d) are commuting, if $ASx = SAx$ for all $x \in X$. Later, Sessa [25] defined A and S to be weakly commuting if for all $x \in X$, $d(ASx, SAx) \leq d(Sx, Ax)$. Jungck [12] generalized the last notion of weakly commute to the following definition: A and S are compatible, if $\lim\limits_{n\to\infty} d(ASx_n, SAx_n) = 0$, where $\{x_n\}$ is a sequence in X, such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = t$, for some $t \in X$.

It is easy to show commuting implies weakly commuting, implies compatible and there are examples in the literature, which verifying that the inclusions are proper.

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Received: October 2014 *Revised:* January 2015

Jungck et al [13] defined A and S to be compatible mappings of type (A) if

$$\lim_{n\to\infty} d(ASx_n, S^2x_n) = 0 \text{ and } \lim_{n\to\infty} d(SAx_n, A^2x_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X, such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to+\infty} Sx_n = t$, for some $t \in X$.

In [21], Pathak and Khan defined: S and T to be compatible mappings of type (B), which is a generalization of compatible mappings of type (A), if

$$\lim_{n\to\infty} d(SAx_n, A^2x_n) \leq \frac{1}{2}\Big[\lim_{n\to+\infty} d(St, SAx_n) + \lim_{n\to+\infty} d(St, S^2x_n)\Big] \text{ and}$$

$$\lim_{n\to\infty} d(ASx_n, S^2x_n) \leq \frac{1}{2}\Big[\lim_{n\to+\infty} d(ASx_n, At) + \lim_{n\to+\infty} d(At, A^2x_n)\Big],$$

whenever $\{x_n\}$ is a sequence in X, such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to+\infty} Sx_n = t$, for some $t \in X$. It is clear that compatible mappings of type (A) are compatible mappings of type (B), but the converse is not true (see [21]). However, compatibility, compatibility of type (A) and compatibility of type (B) are equivalent, if S and T are continuous (see [21]).

Pathak et al. [22] defined: A and S to be compatible mappings of type (P), if

$$\lim_{n\to\infty} d(A^2x_n, S^2x_n) = 0,$$

where $\{x_n\}$ is a sequence in X, such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = t$, for some $t \in X$.

Pathak et al. [23] defined: A and S to be compatible mappings of type (C), if

$$\lim_{n\to\infty} d(ASx_n, S^2x_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(ASx_n, At) + \lim_{n\to\infty} d(At, S^2x_n) + \lim_{n\to\infty} d(At, A^2x_n)]$$

and

$$\lim_{n\to\infty} d(SAx_n, A^2x_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(SAx_n, St) + \lim_{n\to\infty} d(St, S^2x_n) + \lim_{n\to\infty} d(St, A^2x_n)],$$

whenever $\{x_n\}$ is a sequence in X, such that $\lim\limits_{n\to+\infty} Ax_n = \lim\limits_{n\to+\infty} Sx_n = t$, for some $t \in X$.

Notice that, compatible of type (A) implies compatible of type (C), however compatible (compatible of type (A), compatible of type (B) and compatible of type (C)) are equivalent under the continuity of A and S. Jungck and Rhoades [14] defined: two self maps A, S of space metric (X, d) are weakly compatible, if they commute at their coincidence points; i.e., if $Au = Su$ for some $u \in X$, then $ASu = SAu$.

2. Preliminaries

Singh and Mahendra Singh [29] introduced the notion of compatibility of type (E), and gave some properties about this type as follows:

Definition 2.1. Self maps A and S of a metric space (X, d) are said to be compatible of type (E), if $\lim\limits_{n\to\infty} S^2x_n = \lim\limits_{n\to\infty} SAx_n = At$ and $\lim\limits_{n\to\infty} A^2x_n = \lim\limits_{n\to+\infty} ASx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = t$, for some $t \in X$.

Remark 2.2. If $At = St$, then compatible of type (E) implies compatible (compatible of type (A), compatible of type (B), compatible of type (C), compatible of type (P)), however the converse may be not true. Generally compatibility of type (E) implies compatibility of type (B).

Definition 2.3. Two self maps A and S of a metric space (X, d) are A-compatible of type (E), if $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = At$, for some $t \in X$. Also, the pair $\{A, S\}$ is said to be S-compatible of type (E), if $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = At$, for some $t \in X$.

Notice that if A and S are compatible of type (E), then they are A-compatible and S-compatible of type (E), but the converse is not true.

Pant[18] introduced the notion of reciprocal continuity as follows:

Definition 2.4. Self maps A and S of a metric space (X, d) are said to be reciprocally continuous, if $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n = St$, whenever $\{x_n\}$ is a sequence in X, such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

H.Bouhadjera and C. Godet Thobie [7] introduced the concept of subsequential continuity:

Definition 2.5. ([7]) Two self maps A and S of a metric space (X, d) is called to be subsequentially continuous if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$ and satisfy $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n) = St$.

Clearly that continuous, or reciprocally continuous maps are subsequentially continuous, but the converse may be not.

Example 2.6. Let $X = [0, \infty)$ endowed with the euclidian metric, we define A, S as follows:

$$Ax = \begin{cases} 2 + x, & 0 \le x \le 2 \\ \frac{x+2}{2}, & x > 2 \end{cases}, \quad Sx = \begin{cases} 2 - x, & 0 \le x < 2 \\ 2x - 2, & x \ge 2 \end{cases}$$

Clearly that A and S are discontinuous at 2. Consider a sequence $\{x_n\}$, such that for each $n \ge 1$: $x_n = \dfrac{1}{n}$,
clearly that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 2$, also we have:

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A(2 - \frac{1}{n}) = 4 = A(2), \quad \lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S(2 + \frac{1}{n}) = 2 = S(2),$$

then $\{A, S\}$ is subsequentially continuous.

On other hand, let $\{y_n\}$ be a sequence, which defined for each $n \ge 1$ by: $y_n = 2 + \dfrac{1}{n}$, we have $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = 2$, but

$$\lim_{n\to\infty} ASy_n = \lim_{n\to\infty} A(2 + \frac{2}{n}) = 2 \ne A(2), \quad \lim_{n\to\infty} SAy_n = \lim_{n\to\infty} S(4 + \frac{1}{n}) = 6 \ne S(2),$$

then A and S are never reciprocally continuous.

We denote $\mathbb{R}^+_A = [0, A)$, where $A \in [0, \infty)$, let $\mathcal{F}[0, A)$ be a set of all the functions $F : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following assumptions:

1. $F(x) = 0$ if and only if $x = 0$.
2. F is nondecreasing in each variable.

3. F is continuous.

The following examples are given in [4, 30] too:

Example 2.7. Let $F(t) = t$, then $F \in \mathcal{F}[0, A)$.

Example 2.8. For nonnegative, Lebesgue integrable function φ on $[0, A)$, and satisfies:

$$\int_0^{\varepsilon} \varphi(t)dt > 0, \text{ for each } \varepsilon \in (0, A),$$

define $F : t \mapsto \int_0^t \varphi(t)dt$ is in $\mathcal{F}[0, A)$.

Example 2.9. Let ϕ be a nonnegative, Lebesgue integrable function on $[0, A)$ satisfies:

$$\int_0^{\varepsilon} \phi(t)dt > 0, \text{ for each } \varepsilon \in (0, A),$$

and let φ be a nonnegative, Lebesgue integrable function on $[0, \int_0^A \phi(s)ds)$ satisfies:

$$\int_0^{\varepsilon} \varphi(t)dt > 0, \text{ for each } \varepsilon \in (0, \int_0^A \phi(s)ds)),$$

define $F(t) = \int_0^{\int_0^t \phi(s)ds} \varphi(u)du$, then $F \in \mathcal{F}$.

Lemma 2.10. [30] Let ε_n be a sequence in $\mathbb{R}_A^+$, if $\lim\limits_{n\to\infty} F(\varepsilon_n) = 0$ then $\lim\limits_{n\to\infty} F(\varepsilon_n) = 0$, where $F \in \mathcal{F}[0, A)$.

The following theorem is proved in [30].

Theorem 2.11. Let (X, d) be a complete metric space and let $D = \sup\{d(x, y)x, y \in X\}$. Set $A = D$ if $D = \infty$ and $A > D$ if $D < \infty$. $A, B : X \to X$ are two self-mappings, which satisfying for all $x, y \in X$ the inequality:

$$F(d(Ax, By)) \leq \psi\Big(F(\max(d(x, y), d(Ax, x), d(By, y), \frac{d(Ax, y) + d(By, x)}{2}\Big),$$

where $F \in \mathcal{F}[0, A)$ and $\psi \in \Psi(0, F(A - 0))$, then A and B have a common fixed point, moreover for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ converges to the common fixed point of T and S.

The aim of this paper is to improve Theorem 2.11, for two pairs of self mappings in metric spaces. Without completeness of the space and by using the notion of a weak subsequential continuity and compatibility of type (E), we give an example and an application, for the existence of the solution of functional equations arising in dynamic programme, to illustrate and support our results.

3. Main results

Definition 3.1. Let f and S to be two self mappings of a metric space (X, d), the pair $\{f, S\}$ is said to be weakly subsequentially continuous if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$ and $\lim_{n\to\infty} ASx_n = Az$, $\lim_{n\to\infty} SAx_n = Sz$.

Notice that subsequentially continuous or reciprocally continuous maps are weakly subsequentially continuous, but the converse may be not.

Example 3.2. Let $X = [0, 8]$ and d is the euclidian metric, we define A, S as follows:

$$Ax = \begin{cases} \frac{x+4}{2}, & 0 \le x \le 4 \\ x+1, & 4 \le x \le 8 \end{cases}, \quad Sx = \begin{cases} 8-x, & 0 \le x \le 4 \\ x-2, & 4 \le x \le 8 \end{cases}$$

We consider a sequence $\{x_n\}$ such that for each $n \ge 1$: $x_n = 4 - e^{-n}$, clearly that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 4$, also we have:

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A(4 + e^{-n}) = 5, \quad \lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S(4 - \frac{1}{2}e^{-n}) = 4 = S(4),$$

then $\{A, S\}$ is S-subsequentially continuous.

Theorem 3.3. *Let A, B, S and T be mappings from a metric space (X, d) into itself such that for all $x, y \in X$,*

$$F(d(Ax, By)) \le \psi\Big(F(M(x, y)\Big), \tag{3.1}$$

where $M(x, y) = \max(d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx))$, $F \in \mathcal{F}$ and $\psi \in \Psi$, if the two pairs $\{A, S\}$ and $\{B, T\}$ are weakly subsequentially continuous and compatible of type (E), then A, B, S and T have a common fixed point in X.

Proof . Since $\{A, S\}$ is weakly subsequentially continuous, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ and $\lim_{n\to\infty} ASx_n = Az$, again A and S are compatible of type (E), so $\lim_{n\to\infty} A^2x_n = \lim_{n\to\infty} ASx_n = Sz$ and $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = Az$, which implies that $Az = Sz$.

Also, for B and T are weakly subsequentially continuous, there is a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$ and $\lim_{n\to\infty} BTy_n = Bt$, the pair $\{B, T\}$ is compatible of type (E), then so $\lim_{n\to\infty} B^2y_n = \lim_{n\to\infty} BTy_n) = Tt$, and $\lim_{n\to\infty} T^2y_n = \lim_{n\to\infty} TBy_n) = Bt$, which implies that $Bt = Tt$.

We claim $Az = Bt$, if not by using (3.1) we get:

$$F(d(Az, Bt)) \le \psi(F(\max(d(Sz, Tt), d(Az, Sz), d(Bt, Tt), d(Az, Tt), d(Bt, Sz)))).$$

$$\le \psi(F(d(Az, Bt), 0, 0, d(Az, Bt), d(Az, Bt))).$$

$$\le \psi(F(d(Az, Bt))) < F(d(Az, Bt))$$

which is a contradiction, then $Az = Sz = Bt = Tt$.
Now we shall prove $z = t$, if not by using (3.1) we get:

$$F(d(Ax_n, By_n)) \le \psi(F(\max(d(Sx_n, Ty_n), d(Ax_n, Sx_n), d(By_n, Ty_n), d(Ax_n, Ty_n), d(By_n, Sx_n)))).$$

Letting $n \to \infty$, we obtain

$$F(d(z,t)) \leq \psi(F(\max(d(z,t),0,0,d(z,t),d(z,t)))) < F((d(z,Az)),$$

which is a contradiction, then $z = t$ and so z is a common fixed point for A, B, S and T.

For the uniqueness, suppose there exists another common fixed point w forA, B, S and T, by using (3.1) we get:

$$F(d(z,w)) = F(d(Az,Bw)) \leq \psi(F(\max(d(Sz,Tw),d(Az,Sz),d(Bw,Tw),d(Az,Tw),d(Bw,Sz)))).$$

$$\leq \psi(F(d(z,w),0,0,d(z,w),d(z,w))).$$

$$\leq \psi(F(d(z,w))) < F(d(z,w))$$

which is a contradiction, then z is unique. □

Theorem 3.3 improves and generalizes Theorem 2 in [4]. If we combine Theorem 3.3 with Example 2.7, we obtain the following corollary:

Corollary 3.4. For the self mappings A, B, S and T of a metric space (X,d), such for all $x, y \in X$:

$$d(Ax,By) \leq \psi(\max(d(Sx,Ty),d(Ax,Sx),d(Ty,By),d(Ax,Ty),d(Ty,Ax))),$$

if the pair $\{A,S\}$ is weakly subsequentially continuous and compatible of type (E), as well as $\{B,T\}$, then A, B, S and T have a unique fixed point in X.

If we combine Example 2.8 with Theorem 3.3, we get the following corollary:

Corollary 3.5. For the self mappings A, B, S and T of a metric space (X,d) into itself which satisfy for all $x, y \in X$:

$$\int_0^{d(Ax,By)} \varphi(t)dt \leq \psi(\int_0^{M(x,y)} \varphi(t)dt),$$

where $M(x,y) = \max(d(Sx,Ty),d(Ax,Sx),d(By,Ty),d(Ax,Ty),d(By,Sx))$, suppose that the pair $\{A,S\}$ is weakly subsequentially continuous and compatible of type (E), as well as $\{B,T\}$, then A, B, S and T have a unique fixed point.

If we combine Example 2.9 with Theorem 3.3, we get the following corollary:

Corollary 3.6. Let A, B, S and T be self mappings of metric space (X,d) into itself, such for all $x, y \in X$:

$$\int_0^{\int_0^{d(Ax,By)}\phi(s)ds} \varphi(t)dt \leq \psi(\int_0^{\int_0^{M(x,y)}\phi(s)ds} \varphi(t)dt),$$

where $M(x,y) = \max(d(Sx,Ty),d(Ax,Sx),d(By,Ty),d(Ax,Ty),d(By,Sx))$, if the pair $\{A,S\}$ is weakly subsequentially continuous and compatible of type (E), as well as $\{B,T\}$, then A, B, S and T have a unique fixed point.

Example 3.7. Let $X = [0,2]$ and d is the euclidian metric, we define f, S by

$$Ax = \begin{cases} \frac{x+1}{2}, & 0 \leq x \leq 1 \\ \frac{3}{4}, & 1 < x \leq 2 \end{cases} \qquad Bx = \begin{cases} 1, & 0 \leq x \leq 1 \\ \frac{1}{2}, & 1 < x \leq 2 \end{cases}$$

$$Sx = \begin{cases} 2-x, & 0 \le x \le 1 \\ 0, & 1 < x \le 2 \end{cases} \qquad Tx = \begin{cases} x, & 0 \le x \le 1 \\ \frac{1}{4}, & 1 < x \le 2 \end{cases}$$

We consider a sequence $\{x_n\}$, which defined for each $n \ge 1$ by: $x_n = 1 - \frac{1}{n}$,
clearly that $\lim_{n\to\infty} Ax_n = 1$ and $\lim_{n\to\infty} Sx_n = 1$, also we have: $\lim_{n\to\infty} ASx_n = 1 = A(1)$, then $\{A, S\}$ is weakly subsequentially continuous. On other hand, we have $\lim_{n\to\infty} ASx_n = S1 = 1$ and $\lim_{n\to\infty} SAx_n = A(1) = 1$, which implies that $\{A, S\}$ is compatible of type (E), as well as the pair $\{B, T\}$, we have $\lim_{n\to\infty} BTx_n = \lim_{n\to\infty} B(x_n) = 1 = B(1)$, this yields $\{B, T\}$ is weakly subsequentially continuous, again we have:

$$\lim_{n\to\infty} BTx_n = \lim_{n\to\infty} B(x_n) = 1 = T(1)$$

and

$$\lim_{n\to\infty} TBx_n = 1 = B(1),$$

and so the pair $\{B, T\}$ is compatible of type (E).
We choose $\psi(t) = \frac{2}{3}t$, clearly $\psi \in \Psi$ and $F(t) = 1$

1. For $x, y \in [0, 1]$, we have
$$d(Ax, By) = \frac{1}{2}|x-1| \le |x-1| = \frac{2}{3}d(Ax, Sx)$$
2. For $x \in [0, 1]$ and $y \in]1, 2]$, we have
$$d(Ax, By) = \frac{x}{2} \le \frac{1}{6}|2x+1| = \frac{2}{3}d(Ax, Ty)$$
3. For $x \in]1, 2], y \in [0, 1]$ we have
$$d(Ax, By) = \frac{1}{4} \le \frac{2}{3} = \frac{2}{3}d(By, Sx)$$
4. For $x, y \in]1, 2]$, we have
$$d(Ax, By) = \frac{1}{4} \le \frac{1}{2} = \frac{2}{3}d(Ax, Sx)$$

Consequently all hypotheses of Corollary 3.4 satisfied, therefore 1 is the unique common fixed for A, B, S and T.

Theorem 3.8. *Let A, B, S and T be mappings from a metric space (X, d) into itself, such that for all $x, y \in X$,*

$$F(d(Ax, By)) \le \psi\Big(F(\max(d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{d(Ax, Ty) + d(By, Sx)}{2}\Big), \tag{3.2}$$

where $F \in \mathcal{F}$ and $\psi \in \Psi$, if the two pairs $\{A, S\}$ and $\{B, T\}$ are weakly subsequentially continuous and compatible of type (E), then A, B, S and T have a unique common fixed point in X.

This Theorem improves Theorem 2.11 of Zhang, and Theorem 3.1 in [4].

Theorem 3.9. Let A, B, S and T be mappings from a metric space (X, d) into itself, such that for all $x, y \in X$,

$$F(d(Ax, By)) \le \psi\Big(F(M(x, y)), \tag{3.3}$$

where $M(x, y) = \max(d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx))$, $F \in \mathcal{F}$ and $\psi \in \Psi$, if one of the following hypotheses satisfied:

1. the pair $\{A, S\}$ is subsequentially continuous and A-compatible, again $\{B, T\}$ is subsequentially continuous and B-compatible of type (E),or
2. $\{A, S\}, \{B, T\}$ are A-subsequentially continuous (B-subsequentially continuous resp) and A-compatible (Bcompatible resp) of type (E),or
3. $\{A, S\}, \{B, T\}$ are S-subsequentially continuous (T-subsequentially continuous resp) and S-compatible (Tcompatible resp) of type (E),or
4. $\{A, S\}, \{B, T\}$ are subsequentially continuous and S-compatible (Tcompatible resp) of type (E),

then A, B, S and T have a common fixed point in X.

4. Application

In this section, as an application for our results, we establish existence of the solution, to the following systems of functional equations, arising in dynamic programming:

$$\begin{cases} F_i(x) = \sup\limits_{x \in W}\{u(x, y) + H_i(x, y, F_i(T(x, y)))\}, & i = 1, 2 \\ G_i(x) = \sup_{x \in W}\{u(x, y) + K_i(x, y, G_i(T(x, y)))\}, & i = 1, 2 \end{cases} \tag{4.1}$$

Let X, Y be two Banach space, $W \subset X, D \subset Y$ are state and decision space respectively, we denote $B(W)$ set of all bounded functions defined on W, endowed with the following metric

$$h, k \in B(W), d(h, k) = \sup_{x \in S} |h(x) - k(x)|$$

Theorem 4.1. *If the following hypotheses are satisfied:*

(C_1) : *H_i and K_i are bounded.*

(C_2) : *For every $x, y \in S$ and $h, k \in B(W)$ there exists a function $\psi \in \Psi$ such that:*

$$\begin{aligned} &|H_1(x, y, h) - K_1(x, y, k)| \\ &\quad \le \psi(\max(d(A_2h, B_2k), d(A_1h, A_2h), d(B_1k, B_2k), d(A_2h, B_1k), d(B_2k, A_1h))), \end{aligned}$$

(C_3) : *there exists two sequences $\{h_n\}$ in S and $h, k \in B(W)$ such that* $\lim\limits_{n\to\infty} \sup |A_1h_n - h| = \lim\limits_{n\to\infty} \sup |A_2h_n - h| = 0$ *and* $\lim\limits_{n\to\infty} \sup |A_1A_2h_n - A_1h| = 0$.

(C_4) : *for any sequence $\{p_n\}$ in S and $p \in B(W)$ such that* $\lim\limits_{n\to\infty} \sup |p_n - p| =$ *and* $\lim\limits_{n\to\infty} \sup |A_1^2p_n - A_2p| = \lim\limits_{n\to\infty} \sup |A_1A_2p_n - p_2k| = 0$.

(C_5) : *there exists a sequence $\{k_n\}$ in W and $k \in B(W)$ such that* $\lim\limits_{n\to\infty} \sup |B_1k_n - k| = \lim\limits_{n\to\infty} \sup |B_2h_n - k| = 0$ *and* $\lim\limits_{n\to\infty} \sup |B_1B_2h_n - B_1h| = 0$. $\lim\limits_{n\to\infty} \sup |A_1A_2h_n - A_1h| = 0$.

(C_6) : *for any sequence $\{q_n\}$ in W and $q \in B(W)$ such that* $\lim\limits_{n\to\infty} \sup |q_n - q| = 0$ *and* $\lim\limits_{n\to\infty} \sup |B_1^2q_n - B_2p| = \lim\limits_{n\to\infty} \sup |B_1B_2q_n - B_2q| = 0$.

where the mappings A_i and B_i defined by:

$$A_ih = \sup_{x \in W}\{u(x, y) + H_i(x, y, h(T(x, y)))\}, \quad B_ik = \sup_{x \in W}\{u(x, y) + K_i(x, y, k(T(x, y)))\}$$

then the systems (4.1) *have a unique solution in $B(W)$.*

Proof . The systems (4.1) have a solution, if and only if the self mappings $A_i, B_i, i = 1, 2$ have a common fixed point. In the first, remark that the condition (C_1) implies that for $i = 1, 2$, the four mappings A_i, B_i are from $B(W)$ into itself. For the contractive condition, the condition (C_2) give that, for all $h, k \in B(W)$ and $\varepsilon > 0$, there exists $y, z \in D$, such that

$$A_1h < u(x, y) + H_1(x, y, h(T(x, y))) + \varepsilon, \tag{4.2}$$

$$B_1h < u(x, z) + K_1(x, z, h(T(x, z))) + \varepsilon, \tag{4.3}$$

and since

$$A_1h \geq u(x, z) + H_1(x, z, h(T(x, z))), \tag{4.4}$$

$$B_1h \geq u(x, y) + K_1(x, y, h(T(x, y))), \tag{4.5}$$

then from (4.2) and (4.5), we get

$$A_1h - B_1k \leq H_1(x, y, h(T(x, y))) - K_1(x, y, k(T(x, y))) + \varepsilon$$

$$\leq \psi(\max(d(A_2h, B_2k), d(A_1h, A_2h), d(B_1k, B_2k), d(A_2h, B_1k), d(B_2k, A_1h))) + \varepsilon, \tag{4.6}$$

on the other hand and from (4.3) and (4.4), we get

$$A_1h - B_1k > K_1(x, y, h(T(x, y))) - h_1(x, y, k(T(x, y))) - \varepsilon$$

$$\geq -\psi(\max(d(A_2h, B_2k), d(A_1h, A_2h), d(B_1k, B_2k), d(A_2h, B_1k), d(B_2k, A_1h))) - \varepsilon, \tag{4.7}$$

consequently, (4.6) and (4.7) imply that

$$d(A_1h, B_1k) = \sup |A_1h - B_1k| \leq |H_1(x, y, h(T(x, y))) - K_1(x, y, k(T(x, y)))| + \varepsilon$$

$$\leq \psi(\max(d(A_2h, B_2k), d(A_1h, A_2h), d(B_1k, B_2k), d(A_2h, B_1k), d(B_2k, A_1h))) + \varepsilon,$$

since the last inequality is true, for any arbitrary $\varepsilon > 0$, we can write

$$d(A_1h, B_1k) \leq \psi(\max(d(A_2h, B_2k), d(A_1h, A_2h), d(B_1k, B_2k), d(A_2h, B_1k), d(B_2k, A_1h))), \tag{4.8}$$

the conditions (C_3) and (C_4) imply that $\{A_1, A_2\}$ is A_1-subsequentially continuous and A_1-compatible of type (E), as well as the pair $\{B_1, B_2\}$ is B_1-subsequentially continuous and B_1-compatible of type (E) from (C_5) and (C_6).

Consequently all the conditions of Corollary 3.4 are satisfied, A_1, A_2, B_1, B_2 have a common fixed point in $B(W)$, and this point is a common solution of system of functional equations (4.1). $\square$

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