Int. J. Nonlinear Anal. Appl. 7 (2016) No. 1, 93-101
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2015.298

Titchmarsh theorem for Jacobi Dini-Lipshitz functions

Mustapha Boujeddaine[a,*], Said Fahlaoui[a], Radouan Daher[b]

[a]*Department of Mathematics and Computer Sciences, Faculty of Sciences, Equipe d'Analyse Harmonique et Probabilités, Université Moulay Ismaïl. BP 11201 , Zitoune, Meknès, Morocco*
[b]*Department of Mathematics, Faculty of Sciences Aïn Chock, University of Hassan II, BP 5366, Maarif, Casablanca, Morocco*

(Communicated by M. Eshaghi Gordji)

Abstract

Our aim in this paper is to prove an analog of Younis's Theorem on the image under the Jacobi transform of a class functions satisfying a generalized Dini-Lipschitz condition in the space $\mathrm{L}^p_{(\alpha,\beta)}(\mathbb{R}^+)$, $(1 < p \le 2)$. It is a version of Titchmarsh's theorem on the description of the image under the Fourier transform of a class of functions satisfying the Dini-Lipschitz condition in L^p.

Keywords: Dini-Lipschitz functions; Jacobi operator; Jacobi transform.
2010 MSC: Primary 26B10; Secondary 26D10.

1. Introduction

In this article, we obtain for Jacobi transform an analog of Younis's theorem ([12, Theorem 5.2]) which is a version of Titchmarsh's theorem ([10, Theorem 84]) on the description of the image under the Fourier transform of a class of functions satisfying the Dini-Lipschitz condition in L^p. This theorem has been generalized in the case of compact groups [11], and was extented in [1] for the Fourier transform in the space $L_2(\mathbb{R}^n)$ using a spherical mean operator. The Younis's theorem has been generalized recently for a class of functions satisfying the Lipschitz condition for the Bessel transform in [3] and also for the Dunkl transform in [4].

A number of years ago, Titchmarsh established in ([10, Theorem 84]) that if $f(x)$ satisfies the Lipschitz condition $\mathrm{Lip}(\alpha, p)$ in the L^p norm $(1 < p \le 2)$ on the real line $\mathbb{R}$, that is

$$\int_{-\infty}^{\infty} |f(x+h) - f(x-h)|^p dx = O\left(h^{\alpha p}\right) \;\; (0 < \alpha \le 1) \quad h \to 0,$$

*Corresponding author
Email addresses: boujeddainemustapha@gmail.com (Mustapha Boujeddaine), saidfahlaoui@gmail.com (Said Fahlaoui), rjdaher024@gmail.com (Radouan Daher)

Received: October 2014 *Revised:* January 2015

then its Fourier transform $\widehat{f}$ belongs to L^β for

$$\frac{p}{p+\alpha p-1} < \beta \leq \frac{p}{p-1}, \quad 0<\alpha\leq 1.$$

He also proved in ([10, Theorem 85]) another reversible form in the L^2, namely:

Theorem 1.1. If $f \in \mathrm{L}^2(\mathbb{R})$, the conditions

$$\int_{-\infty}^{\infty} |f(x+h)-f(x-h)|^2 dx = O\left(h^{2\alpha}\right) \ (0<\alpha\leq 1) \quad h\to 0$$

and

$$\left(\int_{-\infty}^{-r} + \int_{r}^{\infty}\right) (\mathcal{F}(x))^2 dx = O\left(r^{-2\alpha}\right) \ (r\to\infty)$$

are equivalent, where $\mathcal{F}$ stands for the Fourier transform of f in $\mathrm{L}^2(\mathbb{R})$.

On the other hand, the Younis's theorem [12] characterized a set of functions in $\mathrm{L}^2(\mathbb{R})$ satisfying the Dini-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms. More precisely, we have:

Theorem 1.2. Let $f \in \mathrm{L}^2(\mathbb{R})$. Then the following conditions are equivalent:

1. $\|f(.+h)-f(.)\|_{\mathrm{L}^2(\mathbb{R})} = O\left(\frac{h^\alpha}{\left(\log\frac{1}{h}\right)^\beta}\right)$ as $h\to 0$, $0<\alpha<1, \beta>0$,
2. $\displaystyle\int_{|\lambda|\geq r} |\mathcal{F}(f)(\lambda)|^2 d\lambda = O\left(r^{-2\alpha}(\log r)^{-2\beta}\right)$ as $r\to+\infty$,

where $\mathcal{F}$ is the Fourier transform of f.

The present article is organized as follows. Section 2, includes some facts on the Jacobi function and basic relations that hold for the Jacobi transform of the first kind. Then we collect a few estimates of this function. We also introduce an appropriate space on which the transform operates and the harmonic analysis associated to the Jacobi tansform. We end this section by presenting some relations related to the transform of the finite differences of the first and higher orders. In Section 3, devoted to the main results, we investigate the validity of Theorem 1.2 for studying some structural properties of functions in the wider Jacobi Dini-Lipschitz class.

2. Preliminaries

In this section, we discuss the basic background material which is necessary for the development of the continuous Jacobi transform. More details about the harmonic analysis associated to the Jacobi transform can be found in [7]. Let

$$(a)_0 = 1 \quad \text{and} \ (a)_k = a(a+1)\cdots(a+k-1).$$

The Gaussian hypergeometric function is defined by

$$F(a,b,c,z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \ |z|<1,$$

where $a, b, z \in \mathbb{C}$ and $c \notin -\mathbb{N}$.

The function $z \mapsto F(a, b, c, z)$ is a unique solution to the differential equation

$$z(1-z)u''(z) + (c - (a+b+1)z)u'(z) - abu(z) = 0$$

which is regular in 0 and equals 1 there.

Let $\alpha \geq -\frac{1}{2}$, $\alpha > \beta \geq -\frac{1}{2}$ and $\rho = \alpha + \beta + 1$. The Jacobi function φ_λ is defined by

$$\varphi_\lambda(t) = \varphi_\lambda^{(\alpha,\beta)}(t) = F\left(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho + i\lambda), \alpha + 1, -\sinh^2 t\right).$$

The Jacobi operator is

$$\mathbf{D} = \mathrm{D}_{\alpha,\beta} = \frac{d^2}{dt^2} + ((2\alpha+1)\coth t + (2\beta+1)\tanh t)\frac{d}{dt}.$$

The Jacobi function φ_λ can be characterized as a unique solution to

$$\mathbf{D}\varphi + (\lambda^2 + \rho^2)\varphi = 0$$

on $\mathbb{R}^+$ satisfying $\varphi_\lambda(0) = 1$, $\varphi_\lambda^{'}(0) = 0$, and such that the function $\lambda \mapsto \varphi_\lambda(t)$ is analytic for each $t \geq 0$.

Lemma 2.1. *The following inequalities hold for a Jacobi function $\varphi_\lambda(t)$, $(\lambda, t \in \mathbb{R}^+)$:*

1. $|\varphi_\lambda(t)| \leq 1$, *and the equality is attained only for $t = 0$,*
2. $|1 - \varphi_\lambda(t)| \leq t^2(\lambda^2 + \rho^2)$,
3. $|1 - \varphi_\lambda(t)| \geq c$, *for $\lambda t \geq 1$, where c is some positive constant which depends only on λ.*

Proof . Similar to Lemmas 3.1 and 3.3 in [9]. □

Consider the space $\mathrm{L}^p_{(\alpha,\beta)}(\mathbb{R}^+) = \mathrm{L}^p(\mathbb{R}^+, \mathrm{A}(t)dt)$ with $1 < p \leq 2$, $\alpha > \beta \geq -\frac{1}{2}$ and

$$\mathrm{A}(t) = \mathrm{A}_{(\alpha,\beta)}(t) = (2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1},$$

with the norm

$$\|f\|_{p,(\alpha,\beta)} = \left(\int_0^\infty |f(x)|^p \mathrm{A}(x)dx\right)^{1/p}.$$

Definition 2.2. For a function $f \in L_{p,(\alpha,\beta)}$, the Jacobi transform is defined by

$$\widehat{f}(\lambda) = \int_0^{+\infty} f(t)\varphi_\lambda(t)\mathrm{A}_{(\alpha,\beta)}(t)dt,$$

where φ_λ is the Jacobi function.

⋄ For $f \in \mathrm{L}^1_{(\alpha,\beta)}(\mathbb{R}^+)$, if $\widehat{f} \in \mathrm{L}^1(\mathbb{R}^+, \frac{1}{2\pi}d\mu(\lambda))$, the inversion formula is given as [6]:

$$f(t) = \frac{1}{2\pi}\int_0^{+\infty} \widehat{f}(\lambda)\varphi_\lambda(t)d\mu(\lambda),$$

where $d\mu(\lambda) = |c(\lambda)|^{-2}d\lambda$ and $c(\lambda)$ is the c-function defined by

$$c(\lambda) = \frac{2^{\rho}\Gamma\left(\frac{1}{2}(1+i\lambda)\right)\Gamma(i\lambda)}{\Gamma\left(\frac{1}{2}(i\lambda+\alpha+\beta+1)\right)\Gamma\left(\frac{1}{2}(i\lambda+\alpha-\beta+1)\right)},$$

and $\rho = \alpha+\beta+1$.

⋄ One may show that the Jacobi transform extends to an isometry on $\mathrm{L}_{2,(\alpha,\beta)}(\mathbb{R}^+)$ (see [8]):

$$\|f\|_{\mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)} = \|\widehat{f}\|_{\mathrm{L}^2(\mathbb{R}^+,\frac{1}{2\pi}d\mu(\lambda))}. \qquad (Parseval's\ identity) \tag{2.1}$$

If $f \in \mathrm{L}^p_{(\alpha,\beta)}(\mathbb{R}^+)$, and under suitable conditions, it also satisfies the Hausdorff-Young inequality on $\mathrm{L}^p_{(\alpha,\beta)}(\mathbb{R}^+)$, $1 < p \leq 2$:

$$\|\widehat{f}\|_{q,(\alpha,\beta)} \leq C.\|f\|_{p,(\alpha,\beta)}, \tag{2.2}$$

where C is a positive constant and $\frac{1}{p}+\frac{1}{q}=1$.

We note that if $\alpha=\beta=-\frac{1}{2}$, the Jacobi transform coincides with the classical Fourier transform.

For $f \in \mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$, we have

$$\widehat{(\mathrm{D}f)}(\lambda) = -(\lambda^2+\rho^2)\widehat{f}(\lambda) \tag{2.3}$$

⋄ The Generalized translation operator for a function f on $\mathbb{R}^+$ was defined in [5] as:

$$\tau_h f(x) = \int_0^\infty f(z)K(x,h,z)\mathrm{A}(z)dz,$$

where K is an explicitly known kernel function such that

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_0^\infty \varphi_\lambda(z)K(x,y,z)\mathrm{A}(z)dz$$

with the kernel

$$\begin{cases} K(x,y,z) = \frac{2^{-2\rho}\Gamma(\alpha+1)(\cosh x\cosh y\cosh z)^{-\alpha-\beta-1}}{\Gamma(1/2)\Gamma(\alpha+\frac{1}{2})(\sinh x\sinh y\sinh z)^{2\alpha}}(1-\mathrm{B}^2)^{\alpha-\frac{1}{2}} \times F\left(\alpha+\beta,\alpha-\beta,\alpha+\frac{1}{2},\frac{1}{2}(1-\mathrm{B})\right) \\ \qquad\qquad \text{for } |x-y|<z<x+y \\ K(x,y,z)=0 \quad \text{elsewhere} \end{cases}$$

and

$$\mathrm{B} = \frac{\cosh^2 x+\cosh^2 y+\cosh^2 z-1}{2\cosh x\cosh y\cosh z}.$$

In [8], for $f \in \mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$, we have

$$\widehat{(\tau_h f)}(\lambda) = \varphi_\lambda(h)\widehat{f}(\lambda). \tag{2.4}$$

⋄ The finite differences of the first and higher orders are defined as follows:

$$\Delta_h f(x) = \tau_h f(x) - f(x) = (\tau_h - I)f(x), \qquad f \in \mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+),$$

$$\Delta_h^k f(x) = \Delta_h\left(\Delta_h^{k-1}f(x)\right) = \tau_h\left(\tau_h^{k-1}f(x)\right) = \sum_{j=0}^{k} C_j^k(-1)^{k-j}(\tau_h)^j f(x), \tag{2.5}$$

where $\tau_h^0 f(x) = f(x)$, $\tau_h^j f(x) = \tau_h\left(\tau_h^{j-1} f(x)\right)$, $(j = 1,2,3,\dots; k = 1,2,3,\dots)$, and I is a unit operator in $\mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$. Therefore,

$$\widehat{\Delta_h^k f}(\lambda) = (\varphi_\lambda(h) - 1)^k \,.\widehat{f}(\lambda), \quad h \geq 0. \tag{2.6}$$

Let $W^k_{p,(\alpha,\beta)}$, $(1 < p \leq 2)$ be the Sobolev space constructed by the Jacobi operator D, that is:

$$W^k_{p,(\alpha,\beta)} = \{f \in \mathrm{L}^p_{(\alpha,\beta)}(\mathbb{R}^+) : \mathbf{D}^j f \in \mathrm{L}^p_{(\alpha,\beta)}(\mathbb{R}^+),\ j = 1,2,\dots,k\}.$$

Lemma 2.3. *Let $f \in W^k_{2,(\alpha,\beta)}$. Then*

$$\|\Delta_h^k \mathbf{D}^r f(.)\|^2_{L_{2,(\alpha,\beta)}} = \int_0^\infty (\lambda^2+\rho^2)^{2r} |1-\varphi_\lambda(h)|^{2k} |\widehat{f}(\lambda)|^2 \mathrm{d}\mu(\lambda) \tag{2.7}$$

where $r = 0,1,\dots,k$.

Proof . Analog to ([2, Lemma 2.1]). □

3. Main Results

In this section, we give the principal result of this paper. For this objective, we first need to define the k-Jacobi Dini-Lipschitz class.

Definition 3.1. Let $\delta \in (0,1)$. We say that a function $f \in W^k_{2,(\alpha,\beta)}$ belongs to the Jacobi Dini-Lipshitz class $J - DLip[2;(\delta,\gamma),k,r]$ if $f(x)$ belongs to $\mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$ and

$$\|\Delta_h^k \mathbf{D}^r f(.)\|_{L_{2,(\alpha,\beta)}} = O\left(\frac{h^\delta}{(\log\frac{1}{h})^\gamma}\right) \ \textit{as } h \to 0,\ \delta,\gamma > 0,$$

where $r = 0,1,\dots,k$.

Theorem 3.2. *Let $f \in W^k_{2,(\alpha,\beta)}$. The following two conditions are equivalent:*

$$f(x) \in J - DLip[2;(\delta,\gamma),k,r],\ \delta,\gamma > 0, \tag{3.1}$$

$$\int_s^\infty (\lambda^2+\rho^2)^{2r} |\widehat{f}(\lambda)|^2 \mathrm{d}\mu(\lambda) = O\left(\frac{s^{-2\delta}}{(\log s)^{2\gamma}}\right) \ \textit{as } s \to +\infty. \tag{3.2}$$

Proof . From now on, the letter c indicates a positive constant that is not necessarily the same in each occurrence.

(3.1) ⇒ (3.2): Let $f \in J - DLip[2;(\delta,\gamma),k,r]$. From Lemma 2.3, we have

$$\|\Delta_h^k \mathbf{D}^r f(.)\|^2 = \int_0^\infty (\lambda^2+\rho^2)^{2r} |1-\varphi_\lambda(h)|^{2k} |\widehat{f}(\lambda)|^2 \mathrm{d}\mu(\lambda).$$

If $\lambda \in [\frac{1}{h}, \frac{2}{h}]$ then $|\lambda h| \geq 1$, and by Lemma 2.1, there exists a constant $c > 0$ for which

$$1 \leq \frac{1}{c^{2k}} |1-\varphi_\lambda(h)|^{2k}.$$

Therefore,

$$\begin{aligned}\int_{1/h}^{2/h}(\lambda^2+\rho^2)^{2r}|\widehat{f}(\lambda)|^2\mathrm{d}\mu(\lambda) &\leq \frac{1}{c^{2k}}\int_{1/h}^{2/h}(\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda(h)|^{2k}|\widehat{f}(\lambda)|^2\mathrm{d}\mu(\lambda)\\ &\leq \frac{1}{c^{2k}}\int_{0}^{\infty}(\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda(h)|^{2k}|\widehat{f}(\lambda)|^2\mathrm{d}\mu(\lambda)\\ &= O\left(\frac{h^{2\delta}}{(\log\frac{1}{h})^{2\gamma}}\right) \;\; as\; h\to 0\end{aligned}$$

Putting $s=h^{-1}$, we can write this inequality in the following form:

$$\int_{s}^{2s}(\lambda^2+\rho^2)^{2r}|\widehat{f}(\lambda)|^2\mathrm{d}\mu(\lambda)\leq c.\left(\frac{s^{-2\delta}}{(\log s)^{2\gamma}}\right),$$

where $c>0$ is some positive constant. As a consequence,

$$\begin{aligned}\int_{s}^{\infty}(\lambda^2+\rho^2)^{2r}|\widehat{f}(\lambda)|^2\mathrm{d}\mu(\lambda) &= \left[\int_{s}^{2s}+\int_{2s}^{4s}+\int_{4s}^{8s}+\dots\right](\lambda^2+\rho^2)^{2r}|\widehat{f}(\lambda)|^2\mathrm{d}\mu(\lambda)\\ &\leq c.\big(\frac{s^{-2\delta}}{(\log s)^{2\gamma}}+\frac{(2s)^{-2\delta}}{(\log 2s)^{2\gamma}}+\frac{(4s)^{-2\delta}}{(\log 4s)^{2\gamma}}+\dots\big)\\ &\leq c.\frac{s^{-2\delta}}{(\log s)^{2\gamma}}\left[1+2^{-2\delta}+(2^{-2\delta})^2+(2^{-2\delta})^3+\dots\right]\\ &\leq c.K.\frac{s^{-2\delta}}{(\log s)^{2\gamma}},\end{aligned}$$

where $K=(1-2^{-2\delta})^{-1}$. It follows that

$$\int_{s}^{\infty}(\lambda^2+\rho^2)^{2r}|\widehat{f}(\lambda)|^2\mathrm{d}\mu(\lambda)=O\left(\frac{s^{-2\delta}}{(\log s)^{2\gamma}}\right)\;\; as\; s\to+\infty.$$

(3.2) $\Rightarrow$ (3.1): Now, suppose that

$$\int_{s}^{\infty}(\lambda^2+\rho^2)^{2r}|\widehat{f}(\lambda)|^2\mathrm{d}\mu(\lambda)=O\left(\frac{s^{-2\delta}}{(\log s)^{2\gamma}}\right)\;\; as\; s\to+\infty.$$

According to Lemma 2.3, we write $\|\Delta_h^k\mathbf{D}^r f(.)\|^2_{L_{2,(\alpha,\beta)}}=I_1+I_2$, where

$$I_1=\int_0^{\frac{1}{h}}(\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda(h)|^{2k}|\widehat{f}(\lambda)|^2\mathrm{d}\mu(\lambda)$$

and

$$I_2=\int_{\frac{1}{h}}^{\infty}(\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda(h)|^{2k}|\widehat{f}(\lambda)|^2\mathrm{d}\mu(\lambda).$$

Let us estimate the summands I_1 and I_2 from above. To estimate I_1, we use both the first two estimates of φ_λ in Lemma 2.1. Therefore

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{h}} (\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda(h)|^{2k}|\widehat{f}(\lambda)|^2 \mathrm{d}\mu(\lambda) \\ &= \int_0^{\frac{1}{h}} (\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda(h)||1-\varphi_\lambda(h)|^{2k-1}|\widehat{f}(\lambda)|^2 \mathrm{d}\mu(\lambda) \\ &\leq 2^{2k-1}\int_0^{\frac{1}{h}} (\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda(h)||\widehat{f}(\lambda)|^2 \mathrm{d}\mu(\lambda) \\ &\leq 2^{2k-1}\int_0^{\frac{1}{h}} h^2(\lambda^2+\rho^2)^{2r+1}|\widehat{f}(\lambda)|^2 \mathrm{d}\mu(\lambda). \end{aligned}$$

Now, we apply integration by parts for a function $\Phi(s) = \displaystyle\int_s^\infty (\lambda^2+\rho^2)^{2r}|\widehat{f}(\lambda)|^2 \mathrm{d}\mu(\lambda)$ to get

$$\begin{aligned} I_1 &\leq 2^{k-1}h^2.\int_0^{\frac{1}{h}} (\lambda^2+\rho^2)^{2r+1}|\widehat{f}(\lambda)|^2 \mathrm{d}\mu(\lambda) \leq -2^{k-1}h^2.\int_0^{\frac{1}{h}} (s^2+\rho^2)\Phi'(s)\mathrm{d}s \\ &\leq -2^{k-1}h^2.\int_0^{\frac{1}{h}} s^2\Phi'(s)\mathrm{d}s \leq -2^{k-1}\Phi(\frac{1}{h}) + 2^k h^2\int_0^{\frac{1}{h}} s\Phi(s)\mathrm{d}s \leq 2^k h^2\int_0^{\frac{1}{h}} s\Phi(s)\mathrm{d}s \\ &\leq c.h^2\int_0^{\frac{1}{h}} \frac{s^{1-2\delta}}{(\log s)^{2\gamma}}\mathrm{d}s \quad \text{since} \quad \Phi(s) = \int_s^\infty (\lambda^2+\rho^2)^{2r}|\widehat{f}(\lambda)|^2 \mathrm{d}\mu(\lambda) = O\Big(\frac{s^{-2\delta}}{(\log s)^{2\gamma}}\Big) \\ &\leq c.\frac{h^{2\delta}}{(\log\frac{1}{h})^{2\gamma}}, \end{aligned}$$

where c is a positive constant.

On the other hand, it follows from the first inequality of Lemma 2.1 that

$$I_2 \leq 4^k \int_{\frac{1}{h}}^\infty (\lambda^2+\rho^2)^{2r}|\widehat{f}(\lambda)|^2 \mathrm{d}\mu(\lambda)$$

so that

$$I_2 = O\left(\frac{h^{2\delta}}{(\log\frac{1}{h})^{2\gamma}}\right).$$

Consequently,

$$\|\Delta_h^k \mathbf{D}^r f(.)\|_{L_{2,(\alpha,\beta)}} = O\Big(\frac{h^{2\delta}}{(\log\frac{1}{h})^{2\gamma}}\Big)$$

and this ends the proof of the theorem. □

In the rest of this work, we will give the version of Theorem 3.2 in the space $f \in \mathrm{L}^p_{(\alpha,\beta)}(\mathbb{R}^+)$. The Plancherel formula (2.1) will likewise be replaced by the Hausdorff-Young inequality (2.2) to derive the following non-reversible result.

Definition 3.3. We say that a function f belongs to the k-Jacobi Dini-Lipshitz class $J-DLip_\psi(p,\gamma,k)$, $\gamma>0$ if $f(x)$ belongs to $W^k_{p,(\alpha,\beta)}$ and

$$\|\Delta_h^k \mathbf{D}^r f(.)\|_{p,(\alpha,\beta)} = O\left(\frac{\psi(h)}{(\log\frac{1}{h})^{\gamma}}\right) \ \ as\ h\to 0,\ \gamma>0,$$

where

1. $\psi(t)$ is a continuous increasing function on $[0,\infty)$,
2. $\psi(a)\psi(b) \leq \psi(ab)$ for all $a, b \in [0,\infty)$.

Theorem 3.4. *Let f belong to $J - DLip_\psi(p,\gamma,k)$. Then*

$$\int_s^\infty (\lambda^2+\rho^2)^{qr}|\widehat{f}(\lambda)|^q \mathrm{d}\mu(\lambda) = O(\psi(s^{-q})(\log s)^{-q\gamma}) \text{ as } s \to +\infty,$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof . Let $f \in J - DLip_\psi(p,\gamma,k)$. Then we have

$$\|\Delta_h^k \mathbf{D}^r f(.)\|_{p,(\alpha,\beta)} = O\left(\frac{\psi(h)}{(\log\frac{1}{h})^\gamma}\right) \text{ as } h \to 0,\ \gamma > 0.$$

If $\lambda \in [\frac{1}{h}, \frac{2}{h}]$, then $|\lambda h| \geq 1$, and from the third inequality of Lemma 2.1, we obtain

$$1 \leq \frac{1}{c^{q.k}}|1-\varphi_\lambda(h)|^{q.k}.$$

Using the Hausddorff-Young inequality (2.2), we deduce

$$\begin{aligned}
\int_{1/h}^{2/h}(\lambda^2+\rho^2)^{r.q}|\widehat{f}(\lambda)|^q \mathrm{d}\mu(\lambda) &\leq \frac{1}{c^{q.k}}\int_{1/h}^{2/h}|1-\varphi_\lambda(h)|^{q.k}(\lambda^2+\rho^2)^{r.q}|\widehat{f}(\lambda)|^q \mathrm{d}\mu(\lambda)\\
&\leq \frac{1}{c^{q.k}}\int_0^\infty |1-\varphi_\lambda(h)|^{q.k}(\lambda^2+\rho^2)^{r.q}|\widehat{f}(\lambda)|^q \mathrm{d}\mu(\lambda)\\
&\leq c\|\widehat{\Delta_h^k \mathbf{D}^r f}(.)\|_{q,(\alpha,\beta)}^q \leq c\|\Delta_h^k \mathbf{D}^r f(.)\|_{p,(\alpha,\beta)}^q\\
&= O\left(\frac{[\psi(h)]^q}{(\log\frac{1}{h})^{q\gamma}}\right) \text{ as } h \to 0,\ \gamma > 0,
\end{aligned}$$

where c is a positive constant. We get

$$\int_s^{2s}(\lambda^2+\rho^2)^{r.q}|\widehat{f}(\lambda)|^q \mathrm{d}\mu(\lambda) = O\left(\frac{[\psi(s^{-1})]^q}{(\log s)^{q\gamma}}\right).$$

Then, there exists a positive constant C such that

$$\int_s^\infty (\lambda^2+\rho^2)^{r.q}|\widehat{f}(\lambda)|^q \mathrm{d}\mu(\lambda) \leq C.\frac{[\psi(s^{-1})]^q}{(\log s)^{q\gamma}}$$

and so

$$\begin{aligned}
\int_s^\infty (\lambda^2+\rho^2)^{r.q}|\widehat{f}(\lambda)|^q \mathrm{d}\mu(\lambda) &= \left[\int_s^{2s}+\int_{2s}^{4s}+\int_{4s}^{8s}+\cdots\right](\lambda^2+\rho^2)^{r.q}|\widehat{f}(\lambda)|^q \mathrm{d}\mu(\lambda)\\
&\leq C\frac{[\psi(s^{-1})]^q}{(\log s)^{q\gamma}} + C\frac{[\psi((2s)^{-1})]^q}{(\log 2s)^{q\gamma}} + C\frac{[\psi((4s)^{-1})]^q}{(\log 4s)^{q\gamma}} + \cdots\\
&\leq C\frac{[\psi(s^{-1})]^q}{(\log s)^{q\gamma}} + C\frac{[\psi(s^{-1})]^q\,[\psi(2^{-1})]^q}{(\log s)^{q\gamma}} + C\frac{[\psi(s^{-1})]^q[\psi(4^{-1})]^q}{(\log s)^{q\gamma}} + \cdots\\
&\leq C\frac{[\psi(s^{-1})]^q}{(\log s)^{q\gamma}}\left[1+(\psi(2^{-1}))^q+(\psi(2^{-1}))^{2q}+(\psi(2^{-1}))^{3q}+\cdots\right].
\end{aligned}$$

Since $\psi(2^{-1}) < \psi(1) < 1$, and under the hypotheses satisfied by the function ψ in definition 3.3, we have

$$\int_s^{\infty} (\lambda^2+\rho^2)^{q.k} |\widehat{f}(\lambda)|^q \mathrm{d}\mu(\lambda) \le C_{\psi,q} \frac{[\psi(s^{-1})]^q}{(\log s)^{q\gamma}},$$

where $C_{\psi,q} = C[1-(\psi(1/2))^q]^{-1}$. Finally, we find

$$\int_s^{\infty} (\lambda^2+\rho^2)^{q.k} |\widehat{f}(\lambda)|^q \mathrm{d}\mu(\lambda) = O\left(\frac{\psi(s^{-q})}{(\log s)^{q\gamma}}\right) \ as \ s \to +\infty$$

which completes the proof. □

We conclude this work by the following immediate consequence.

Corollary 3.5. Let $\psi(t) = t^{\delta}$ with $0 < \delta$. If a function $f \in W^k_{p,(\alpha,\beta)} \cap J - DLip[p;(\delta,\gamma),k,r]$ with $\delta, \gamma > 0$, then

$$\int_s^{\infty} (\lambda^2+\rho^2)^{qr} |\widehat{f}(\lambda)|^q \mathrm{d}\mu(\lambda) = O((s^{-\delta q})(\log s)^{-q\gamma}) \ as \ s \to +\infty,$$

where $1 < p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

References

[1] A. Abouelaz, R. Daher and M. El Hamma, *Fourier transform of Dini-Lipschitz functions in the space*, Romanian J. math. Comput. Sci. 3 (2013) 41–47.

[2] A. Abouelaz, R. Daher and M. El Hamma, *Generalization of Titchmarch's theorem for the Jacobi transform.* Facta Univ., Ser. Math. Info. 28 (2013) 43–51.

[3] R. Daher and M. El Hamma, *Bessel Transform of -Bessel Lipschitz Functions.* Journal of Mathematics, vol.2013, Article ID 418546, 3 pages , (2013).

[4] R. Daher, M. Boujeddaine and M. El Hamma, *Dunkl transform of (β,γ)-Dunkl Lipschitz functions.* Proc. Japan Acad. Ser. A Math. Sci. 90 (2014) 135–137.

[5] M. Flensted-Jensen and T. Koornwinder, *The convolution structure for Jacobi function expansions*, Ark. Math. 11 (1973) 245–262.

[6] T.H. Koornwinder, *Jacobi functions and analysis on noncompact semisimple Lie groups, Special functions, group theoretical aspects and applications*, Math. Appl., Dordrecht, (1984), pp. 1–85.

[7] T.H. Koornwinder, *Special orthogonal polynomial systems mapped onto each other by the Fourier-Jacobi transform, Orthogonal polynomials and applications*, (Bar-le-Duc, 1984) Lecture Notes in Math., vol. 1171, Springer, Berlin, 1985, 174–183.

[8] M.A. Mourou and K. Trimeche, *Calderons formula associated with a differential operator on $(0,\infty)$ and inversion of the Abel transform.* J. Fourier Anal. Appl. 4 (1998) 229–245.

[9] S.S. Platonov, *Approximation of functions in the L^2 Metric on nonCompact Rank 1 Symmetric Spaces*, Algebra i Analyz 11 (1999) 244–270.

[10] E.C. Titchmarsh, *Introduction to the theory of Fourier Integrals*, Clarendon Press, Oxford (1937) 115–118.

[11] M.S. Younis, *Fourier Transforms of Lipschitz Functions on compact Groups*, Ph.D. Thesis, McMaster Univ., Hamilton, Ontario, Canada, 1974.

[12] M.S. Younis, *Fourier transforms of Dini-Lipschitz Functions*, Int. J. Math. Math. Sci. 9 (1986) 301–312.