Some new results
using integration of arbitrary order

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Abstract
In this paper, we present recent results in integral inequality theory. Our results are based on the fractional integration in the sense of Riemann-Liouville

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1. Introduction

The integral inequalities involving functions of independent variables play a fundamental role in the theory of differential equations. Motivated by certain applications, many such new inequalities have been discovered in the past few years (see \cite{2, 5, 13, 14, 15}). Moreover, the fractional type inequalities are of great importance. We refer the reader to \cite{1, 16} for some applications. Let us now turn our attention to some results that have inspired our work. We consider the quantity

\begin{equation}
R_{a,b}(p, q, f, g) := \int_a^b p f^2(x) \, dx \int_a^b q g^2(x) \, dx + \int_a^b q f^2(x) \, dx \int_a^b p g^2(x) \, dx - 2 \left( \int_a^b p |f g| (x) \, dx \right) \left( \int_a^b q |f g| (x) \, dx \right) - 2 \left( \int_a^b q |f g| (x) \, dx \right) \left( \int_a^b p |f g| (x) \, dx \right),
\end{equation}

where \( f \) and \( g \) are two continuous functions on \([a, b]\) and \( p \) and \( q \) are two positive and continuous functions on \([a, b]\).

In the case, when \( p = q \), S.S. Dragomir \cite{10} proved the inequality:

\begin{equation}
0 < R_{1,\Omega}(p, f, g) := R_{\Omega}(p, p, f, g) \leq \frac{(M - m)^2}{2mM} \left( \int_\Omega |f g| (x) \, d\mu(x) \right),
\end{equation}

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providing \( f \) and \( g \) are Lebesgue \( \mu \)-measurable, \( pf^2, pg^2 \) are Lebesgue \( \mu \)-integrable on \( \Omega \) and 
\[ 0 < m \leq \frac{|f(x)|}{g(x)} \leq M \leq \infty, \text{ for } \mu \text{ a.e. } x \in \Omega. \]
For other results related to the Cauchy-Schwarz difference \((1)\), in the case \( p = q \), a number of valued extensions can be found in \([3, 6, 7, 8, 9, 12, 18]\) and the references cited therein.

The main aim of this paper is to establish some new fractional integral inequalities of Cauchy-Schwarz type by giving an upper and a lower bound for the quantity \((1.1)\). Some new fractional results related to Cassel’s inequality \([4], [17], [19]\) are also generated. For our results, some classical inequalities can be deduced as some special cases. Our results have some relationships with \([3, 10]\).

### 2. Description of the fractional calculus

We introduce some definitions and properties which will be used in this paper:

**Definition 2.1.** A real valued function \( f \) is said to be in the space \( C^\mu([0, \infty[), \mu \in \mathbb{R} \) if there exists a real number \( r > \mu \), such that 
\[ f(t) = t^r f_1(t), \text{ where } f_1 \in C([0, \infty]). \]

**Definition 2.2.** A function \( f \) is said to be in the space \( C^n\mu([0, \infty[), n \in \mathbb{N}, \) if \( f^{(n)} \in C^\mu([0, \infty[). \)

**Definition 2.3.** The Riemann-Liouville fractional integral operator of order \( \alpha \geq 0 \), for a function \( f \in C^\mu([0, \infty[), \mu \geq -1 \), is defined as
\[
J_\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, t > 0
\]
\[
J_0^\alpha f(t) = f(t).
\]

For the convenience of establishing the results, we give the following property:
\[
J_\alpha J_\beta f(t) = J_{\alpha+\beta} f(t).
\]

For the expression \((2.1)\), when \( f(t) = t^\beta \) we get another expression that will be used later:
\[
J_\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}.
\]

For more details, see \([11, 16]\).

### 3. Main results

Our first result is the following theorem:

**Theorem 3.1.** Suppose that \( f \) and \( g \) are two continuous functions on \([0, \infty[\) and \( p \) and \( q \) are two positive continuous function on \([0, \infty[\), such that \( p|f|^2, p|g|^2, q|f|^2, q|g|^2, pf^2, pg^2, qf^2 \) and \( qg^2 \) are integrable functions on \([0, \infty[\). If there exist \( m \) and \( M \) two positive real numbers, such that
\[
0 < m \leq |f(\tau)g(\tau)| \leq M; \tau \in [0, t], t > 0,
\]
\[(3.1)\]
then we have

\[ m^2 \left( J^\alpha (q/g) (t)(J^\alpha (p|f|)(t) + J^\alpha (p|g|)(t) - 2J^\alpha p(t)\right) \]

\[ \leq J^\alpha p f^2 (t) J^\alpha q g^2 (t) + J^\alpha q f^2 (t) J^\alpha p g^2 (t) - 2 J^\alpha (p|f g|)(t) \]

(3.2)

\[ \leq M^2 \left( J^\alpha (p|\frac{f}{g}|)(t) J^\alpha (q|\frac{g}{f}|)(t) + J^\alpha (q|\frac{f}{g}|)(t) J^\alpha (p|\frac{g}{f}|)(t) - 2 J^\alpha p(t) J^\alpha q(t) \right) , \]

for any \( \alpha > 0, t > 0 \).

**Proof**.

In the identity

\[ \frac{v^2 + u^2}{2} - uv = \frac{1}{2} uv \left( \sqrt{\frac{u}{v}} - \sqrt{\frac{v}{u}} \right)^2 ; u > 0, v > 0, \]

we take \( u = |f(\tau)g(\rho)| \) and \( v = |f(\rho)g(\tau)|, \tau, \rho \in [0, t], t > 0 \). Then we can write

\[ \frac{f^2(\tau)g^2(\rho) + f^2(\rho)g^2(\tau)}{2} - |f(\tau)g(\rho)||f(\tau)g(\rho)| \]

(3.3)

\[ = \frac{1}{2} |f(\tau)g(\tau)||f(\rho)g(\rho)| \left( \sqrt{\frac{|f(\tau)||g(\rho)|}{f(\tau)|g(\rho)|}} - \sqrt{\frac{|f(\rho)||g(\tau)|}{f(\tau)|g(\tau)|}} \right)^2. \]

On the other hand, we have

\[ \left( \sqrt{|\frac{f(\tau)}{g(\tau)}||g(\rho)||f(\rho)|} - \sqrt{|\frac{f(\rho)}{g(\rho)}||g(\tau)||f(\tau)|} \right)^2 = |\frac{f(\tau)}{g(\tau)}||g(\rho)||f(\rho)| + |\frac{f(\rho)}{g(\rho)}||g(\tau)||f(\tau)| - 2. \]

(3.4)

Using (3.4) and the condition (3.1) we can write

\[ \frac{m^2}{2} \left( |\frac{f(\tau)}{g(\tau)}||g(\rho)||f(\rho)| + |\frac{f(\rho)}{g(\rho)}||g(\tau)||f(\tau)| - 2 \right) \]

(3.5)

\[ \leq \frac{f^2(\tau)g^2(\rho) + f^2(\rho)g^2(\tau)}{2} - |f(\tau)g(\tau)||f(\rho)g(\rho)| \]

\[ \leq \frac{M^2}{2} \left( |\frac{f(\tau)}{g(\tau)}||g(\rho)||f(\rho)| + |\frac{f(\rho)}{g(\rho)}||g(\tau)||f(\tau)| - 2 \right). \]

Hence we get,

\[ \frac{m^2}{2} \left( \frac{|g(\rho)|}{f(\rho)}||J^\alpha (p|\frac{f}{g}|)(t) + \frac{|f(\rho)|}{g(\rho)}||J^\alpha (p|\frac{g}{f}|)(t) - 2J^\alpha p(t) \right) \]

\[ \leq \frac{g^2(\rho)J^\alpha p f^2(t) + f^2(\rho)J^\alpha p g^2(t)}{2} - |f(\rho)g(\rho)||J^\alpha (p|f g|)(t) \]

(3.6)

\[ \leq \frac{M^2}{2} \left( \frac{|g(\rho)|}{f(\rho)}||J^\alpha (p|\frac{f}{g}|)(t) + \frac{|f(\rho)|}{g(\rho)}||J^\alpha (p|\frac{g}{f}|)(t) - 2J^\alpha p(t) \right). \]

Multiplying both sides of (3.6) by \( (q(\rho)^{\frac{n-1}{\alpha}})q(\rho) \), then integrating the resulting inequalities with respect to \( \rho \) over \( [0, t] \), we obtain
two positive real numbers, such that
\[ p \cdot q \leq \int_0^\infty \rho(t) \, dt \]

Applying Theorem 3.1 for Remark 3.2.

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The previous result can be generalized to the following:

**Theorem 3.3.** Suppose that \( f \) and \( g \) are two continuous functions on \([0, t]\) and let \( p \) and \( q \) be two positive continuous functions on \([0, t]\), such that \( p|f|^2, q|g|^2 \) and \( pg^2 \) are integrable functions on \([0, t]\). If there exist \( m \) and \( M \) two positive real numbers, such that

\[
0 < m \leq |f(\tau)g(\tau)| \leq M; \tau \in [0, t], t > 0, \tag{3.8}
\]

then the inequalities

\[
m^2 \left( J^\alpha(p|f|^2(t)J^\beta(q|g|^2))(t) + J^\beta(p|f|^2(t)J^\alpha(q|g|^2))(t) - 2J^\alpha p(t)J^\beta q(t) \right)
\]

\[
\leq J^\alpha p f^2(t)J^\beta q g^2(t) + J^\beta q f^2(t)J^\alpha p g^2(t) - 2J^\alpha p f(t)J^\beta q(t) \tag{3.9}
\]

\[
\leq M^2 \left( J^\alpha(p|f|^2(t)J^\beta(q|g|^2))(t) + J^\beta(p|f|^2(t)J^\alpha(q|g|^2))(t) - 2J^\alpha p(t)J^\beta q(t) \right)
\]

are valid for any \( \alpha > 0, \beta > t > 0. \)

**Proof.** Multiplying both sides of (3.6) by \( \frac{(t-\rho)^\beta-1}{\Gamma(\beta)} q(\rho) \), then integrating the resulting inequalities with respect to \( \rho \) over \([0, t] \), we obtain:

\[
m^2 \left( J^\alpha(p|f|^2(t)J^\beta(q|g|^2))(t) + J^\beta(p|f|^2(t)J^\alpha(q|g|^2))(t) - 2J^\alpha p(t)J^\beta q(t) \right)
\]

\[
\leq \frac{J^\alpha p f^2(t)J^\beta q g^2(t) + J^\beta q f^2(t)J^\alpha p g^2(t) - 2J^\alpha p f(t)J^\beta q(t)}{2} \tag{3.10}
\]

\[
\leq M^2 \left( J^\alpha(p|f|^2(t)J^\beta(q|g|^2))(t) + J^\beta(p|f|^2(t)J^\alpha(q|g|^2))(t) - 2J^\alpha p(t)J^\beta q(t) \right).
\]

The proof of Theorem 3.3 is thus achieved. \( \Box \)

**Remark 3.4.** It is clear that Theorem 3.1 would follow as a special case of of Theorem 3.3 when \( \alpha = \beta. \)

Now, we shall propose a new generalization of Cassel’s inequality. We have:
Theorem 3.5. Let \( f, g \) be two continuous functions on \([0, \infty[\) and let \( p \) and \( q \) be two positive continuous functions on \([0, \infty[\), such that \( pf^2, qf^2, pg^2 \) and \( qg^2 \) are integrable on \([0, \infty[\). If there exist \( m \) and \( M \) two positive real numbers, such that

\[
0 < m \leq \left| \frac{f(\tau)}{g(\tau)} \right| \leq M; \tau \in [0, t], t > 0,
\]

then we have

\[
J^\alpha pf^2(t)J^\alpha qg^2(t) - J^\alpha(p|fg|)(t)J^\alpha(q|fg|)(t)
\]

\[
\leq \frac{(M-m)^2}{4mM} J^\alpha(p|fg|)(t)J^\alpha(q|fg|)(t),
\]

for any \( \alpha > 0, t > 0 \).

Proof. From the condition \( \left| \frac{f(\tau)}{g(\tau)} \right| \leq M; \tau \in [0, t], t > 0 \), we have

\[
f^2(\tau) \leq M|f(\tau)g(\tau)|; \tau \in [0, t], t > 0.
\]

Therefore,

\[
\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} p(\tau)f^2(\tau)d\tau \leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} p(\tau)|f(\tau)g(\tau)|d\tau.
\]

Consequently,

\[
J^\alpha pf^2(t) \leq MJ^\alpha(p|fg|)(t).
\]

Now, using the condition \( m \leq \left| \frac{f(\tau)}{g(\tau)} \right|; \tau \in [0, t], t > 0 \), we can write

\[
mJ^\alpha qg^2(t) \leq J^\alpha(q|fg|)(t).
\]

Multiplying (3.15) and (3.16) we obtain

\[
J^\alpha pf^2(t)J^\alpha qg^2(t) \leq \frac{M}{m} J^\alpha(p|fg|)(t)J^\alpha(q|fg|)(t).
\]

Consequently, we get

\[
J^\alpha pf^2(t)J^\alpha qg^2(t) - J^\alpha(p|fg|)(t)J^\alpha(q|fg|)(t)
\]

\[
\leq \frac{M-m}{m} J^\alpha(p|fg|)(t)J^\alpha(q|fg|)(t),
\]

which implies (3.12) Theorem 3.5 is thus proved. \( \square \)

Remark 3.6. If we take \( \alpha = 1, p = q \), then we obtain Cassel’s inequality \([10],[19]\) on \([0, t]\).

Also, with the same assumptions as before, we get the following generalization of Theorem 3.5.
In Theorem 3.5, we take \( pF^2, qF^2, pg^2 \) and \( qg^2 \) are integrable on \([0, \infty[\). If there exist \( m, M \) positive real numbers, such that

\[
0 < m \leq \left| \frac{f(\tau)}{g(\tau)} \right| \leq M, \tau \in [0, t], t > 0,
\]  

then, for any \( \alpha > 0, \beta > 0, t > 0 \), the inequality

\[
J^\alpha pf^2(t) J^\beta qg^2(t) - J^\alpha (p|f|g)(t) J^\beta (q|f|g)(t) \leq \frac{(M - m)^2}{4mM} J^\alpha (p|f|g)(t) J^\beta (q|f|g)(t)
\]  

is valid.

**Proof.** From the condition \( m \leq \left| \frac{f(\tau)}{g(\tau)} \right|; \tau \in [0, t], t > 0 \), we can write

\[
mJ^\beta qg^2(t) \leq J^\beta (q|f|g)(t).
\]  

Thanks to (3.16) and (3.21) we obtain

\[
J^\alpha pf^2(t) J^\beta qg^2(t) \leq \frac{M}{m} J^\alpha (p|f|g)(t) J^\beta (q|f|g)(t).
\]  

Therefore,

\[
J^\alpha pf^2(t) J^\beta qg^2(t) - J^\alpha (p|f|g)(t) J^\beta (q|f|g)(t) \leq \frac{M - m}{m} J^\alpha (p|f|g)(t) J^\beta (q|f|g)(t).
\]

Hence, we deduce the desired inequality (3.20). \( \square \)

We give also the following corollaries:

**Corollary 3.8.** Let \( F, G \) be two continuous functions on \([0, \infty[\) and let \( p \) and \( q \) be two positive continuous functions on \([0, \infty[\). If there exist \( n, N, M \) positive real numbers, such that \( |F(\tau)G(\tau)| \leq M \) and

\[
0 < n \leq \left| \frac{F(\tau)}{G(\tau)} \right| \leq N, \tau \in [0, t], t > 0,
\]  

then, for any \( \alpha > 0, t > 0 \), the inequality

\[
J^\alpha pF^2(t) J^\alpha qG^2(t) + J^\alpha qF^2(t) J^\alpha pG^2(t) - 2J^\alpha (p|FG|)(t) J^\alpha (q|FG|)(t)
\]  

\[
\leq \frac{M^2(N - n)^2}{2nN} J^\alpha pJ^\alpha q(t)
\]  

is valid.

**Proof.** In Theorem 3.5, we take \( f := \sqrt{\frac{F}{G}}, g := \sqrt{\frac{G}{F}} \). We constat that \( n \leq \frac{f(\tau)}{g(\tau)} \leq N; \tau \in [0, t], t > 0 \), and then

\[
J^\alpha (p\frac{F}{G})(t) J^\alpha (q\frac{G}{F})(t) - J^\alpha p(t) J^\alpha q(t)
\]  

\[
\leq \frac{(N - n)^2}{4nN} J^\alpha p(t) J^\alpha q(t).
\]
We have also
\[
J^\alpha(q \frac{F}{G})(t)J^\alpha(p \frac{G}{F})(t) - J^\alpha(p(t))J^\alpha(q(t)) 
\leq \frac{(N-n)^2}{4nN} J^\alpha(p(t))J^\alpha(q(t)).
\] (3.27)
Combining (3.26) and (3.27), we obtain
\[
J^\alpha(p \frac{F}{G})(t)J^\alpha(q \frac{G}{F})(t) + J^\alpha(q \frac{F}{G})(t)J^\alpha(p \frac{G}{F})(t) - 2J^\alpha(p(t))J^\alpha(q(t)) 
\leq \frac{(N-n)^2}{2nN} J^\alpha(p(t))J^\alpha(q(t)).
\] (3.28)

Since \(|F(\tau)G(\tau)| \leq M; \tau \in [0, t], t > 0\), then thanks to the second inequality of (3.2) (Theorem 3.1), we claim that
\[
J^\alpha(pF^2(t)J^\alpha(qG^2(t) + J^\alpha(qF^2(t)J^\alpha(pG^2(t) - 2J^\alpha(p|FG|)(t)J^\alpha(q|FG|)(t) 
\leq M^2 \left( J^\alpha(p \frac{F}{G})(t)J^\alpha(q \frac{G}{F})(t) + J^\alpha(q \frac{F}{G})(t)J^\alpha(p \frac{G}{F})(t) - 2J^\alpha(p(t))J^\alpha(q(t)) \right).\]
(3.29)
Using (3.28) and (3.29), we obtain the desired inequality (3.25). □

**Remark 3.9.** If we take \(p = q, \alpha = 1, d\mu(\tau) = d\tau\), then we obtain Corollary 3.8 on \(\Omega = [0, t]\).

**Corollary 3.10.** Let \(F, G, p\) and \(q\) satisfy the conditions of Corollary 3.8. Then, for any \(\alpha > 0, \beta > 0, t > 0\), we have
\[
J^\alpha(pF^2(t)J^\alpha(qG^2(t) + J^\alpha(qF^2(t)J^\alpha(pG^2(t) - 2J^\alpha(p|FG|)(t)J^\alpha(q|FG|)(t) 
\leq \frac{M^2(N-n)^2}{2nN} J^\alpha(pJ^\beta q(t).\]
(3.30)
**Proof.** We apply Theorem 3.5 and Theorem ??.

**Remark 3.11.** If we take \(\alpha = \beta\), then we obtain Corollary 3.8.

**References**


