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Vector ultrametric spaces and a fixed point theorem for correspondences

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Abstract

In this paper, vector ultrametric spaces are introduced and a fixed point theorem is given for correspondences. Our main result generalizes a known theorem in ordinary ultrametric spaces.

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1. Introduction and preliminaries

An *ultrametric space* (X, d) is a metric space in which the triangle inequality is replaced by

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}, \qquad (x, y, z \in X).$$

A generalization of the notion of ultrametric space via partially ordered sets was given in [12, 13] which led some applications to logic programming [14], computational logic [15], and quantitative domain theory [5].

In this paper we allow ultrametrics to take values in an arbitrary cone of a complete modular space. The main result of this paper is a fixed point theorem for correspondences in vector ultrametric spaces which generalizes the main theorem presented in [11].

We first present some basic notions which will be needed in this paper.

A modular on a real linear space $\mathcal{A}$ is a real valued functional ρ on $\mathcal{A}$ which satisfies the conditions:

1. $\rho(x) = 0$ if and only if $x = 0$,
2. $\rho(x) = \rho(-x)$,

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3. $\rho(\alpha x+\beta y)\leq \rho(x)+\rho(y)$, for all $x,y\in\mathcal{A}$ and $\alpha,\beta\geq 0$, $\alpha+\beta=1$.

Then, the vector subspace $\mathcal{A}_\rho=\{x\in X:\rho(\alpha x)\to 0 \text{ as } \alpha\to 0\}$ of $\mathcal{A}$ is called a modular space.

The modular ρ is called convex (see, e.g., [1, 8] for a more general form of convexity) if Condition (3) is replaced with

$$\rho(ax+by)\leq a\rho(x)+b\rho(y) \text{ for all } x,y\in X \text{ and all } a,b\geq 0 \text{ with } a+b=1.$$

A sequence $(x_n)_{n=1}^\infty$ in $\mathcal{A}_\rho$ is called ρ-convergent (briefly, convergent) to $x\in\mathcal{A}_\rho$ if $\rho(x_n-x)\to 0$ as $n\to\infty$; $(x_n)_{n=1}^\infty$ is said to be a Cauchy sequence if $\rho(x_m-x_n)\to 0$ as $m,n\to\infty$. By a ρ-closed (briefly, closed) set in $\mathcal{A}_\rho$ it is meant that it contains the limit of all its convergent sequences. And, $\mathcal{A}_\rho$ is a complete modular space if every Cauchy sequence in $\mathcal{A}_\rho$ is convergent to a point of $\mathcal{A}_\rho$. The modular ρ is said to satisfy the Δ_2-condition if there exists $k>0$ such that $\rho(2x)\leq k\rho(x)$ for all $x\in\mathcal{A}_\rho$. The reader is referred to [6, 7] for more details in modular spaces. We also suggest the reader see [3, 4, 9, 10].

Definition 1.1. A nonempty subset $\mathcal{P}$ of a complete modular space $\mathcal{A}_\rho$ is called a *cone* if

(i) $\mathcal{P}$ is ρ-closed, and $\mathcal{P}\neq\{0\}$;

(ii) $a,b\in\mathbb{R}$, $a,b\geq 0$, $x,y\in\mathcal{P}\Rightarrow ax+by\in\mathcal{P}$;

(iii) $\mathcal{P}\cap(-\mathcal{P})=\{0\}$.

A partial order $\preceq$ can be induced on $\mathcal{A}_\rho$ by every cone $\mathcal{P}\subset\mathcal{A}$ as $x\preceq y$ whenever $y-x\in\mathcal{P}$. A cone $\mathcal{P}$ is called *normal* (or ρ-normal) if there is a positive real number c (normal constant) such that

$$0\preceq x\preceq y\Rightarrow \rho(x)\leq c\rho(y), \qquad (x,y\in\mathcal{A}_\rho).$$

When the modular ρ of $\mathcal{A}_\rho$ satisfies Δ_2-condition with Δ_2-constant k, it can be replaced with an equivalent modular σ satisfying Δ_2-condition for which the normal constant of $\mathcal{P}$ is 1 with respect to σ. In fact, for such modular ρ it suffices to define

$$\sigma(x)=\inf_{y\preceq x}\rho(y)+\inf_{x\preceq z}\rho(z) \qquad (x\in\mathcal{A}_\rho).$$

Then, σ is a modular on $\mathcal{A}_\rho$ which is equivalent to ρ and satisfies Δ_2-condition. To see this, we just show that $x=0$ if $\rho(x)=0$ and $\rho(\alpha x+\beta y)\leq\rho(x)+\rho(y)$ as $\alpha,\beta\geq 0$, $\alpha+\beta=1$. Let $\varepsilon>0$ be given. There exist $y,z\in\mathcal{A}_\rho$ such that $y\preceq x\preceq z$ and $\max\{\rho(y),\rho(z)\}\leq\varepsilon$. Since $x-y\preceq z-y$, we get

$$\rho(\frac{x}{4})\leq\rho(\frac{x-y}{2})+\rho(\frac{y}{2})\leq c\rho(\frac{z-y}{2})+\rho(\frac{y}{2})\leq c\rho(z)+(c+1)\rho(y),$$

where c is the normal constant. This implies that $x=0$. Now let $x,u\in\mathcal{A}_\rho$. Choose $y_1,y_2,z_1,z_2\in\mathcal{A}_\rho$ such that $y_1\preceq x\preceq z_1$ and $y_2\preceq u\preceq z_2$ with

$$\rho(y_1)+\rho(z_1)\leq\sigma(x)+\varepsilon, \qquad \rho(y_2)+\rho(z_2)\leq\sigma(u)+\varepsilon.$$

Since $\alpha y_1+\beta y_2\preceq\alpha x+\beta u\preceq\alpha z_1+\beta z_2$, we have

$$\sigma(\alpha x+\beta u)\leq\rho(\alpha y_1+\beta y_2)+\rho(\alpha z_1+\beta z_2),$$

and consequently

$$\sigma(\alpha x + \beta u) \leq \sigma(x) + \sigma(u) + 2\varepsilon.$$

To see the normal constant of σ, let $0 \preceq x \preceq u$. Then,

$$\sigma(x) = \inf_{x \preceq z} \rho(z) \leq \inf_{u \preceq z} \rho(z) = \sigma(u),$$

that is the desired constant is 1. Finally, $\sigma(x) \leq 2\rho(x)$, for each $x \in \mathcal{A}_\rho$. On the other hand, if $y \preceq x \preceq z$, we have

$$\rho(\frac{x}{2}) \leq \rho(\frac{x-y}{2}) + \rho(\frac{y}{2}) \leq c\rho(\frac{z-y}{2}) + \rho(\frac{y}{2}) \leq (c+1)(\rho(y) + \rho(z)),$$

therefore,

$$\rho(\frac{x}{2}) \leq (c+1)\sigma(x).$$

Since σ satisfies Δ_2-condition, we get

$$\rho(x) \leq k(c+1)\sigma(x), \qquad (x \in \mathcal{A}_\rho).$$

Hence, by a normal cone we always assume that its normal constant is 1. We also would say that the cone P is *unital* if there exists a vector $e \in \mathcal{P}$ with modular 1 such that

$$x \preceq \rho(x)e \qquad (x \in \mathcal{P}).$$

Throughout this note, we suppose that $\mathcal{P}$ is a cone in complete modular space $\mathcal{A}_\rho$ where its modular is convex and satisfies Δ_2-condition and $\preceq$ is the partial order induced by $\mathcal{P}$.

Definition 1.2. Let $\mathcal{X}$ be a nonempty set. If the mapping $d : \mathcal{X} \times \mathcal{X} \to \mathcal{A}_\rho$ satisfies the following conditions:

(CUM1) $d(x, y) \succeq 0$ for all $x, y \in \mathcal{X}$ and $d(x, y) = 0$ if and only if $x = y$;

(CUM2) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;

(CUM3) If $d(x, z) \preceq p$ and $d(y, z) \preceq p$, then $d(x, y) \preceq p$, for any $x, y, z \in \mathcal{X}$, and $p \in \mathcal{P}$;

then d is called a *vector ultrametric* on $\mathcal{X}$, and the triple $(\mathcal{X}, d, \mathcal{P})$ is called a *vector ultrametric space.* If $\mathcal{P}$ is unital and normal, then $(\mathcal{X}, d, \mathcal{P})$ is called a unital-normal vector ultrametric space.

For any unital-normal vector ultrametric space $(\mathcal{X}, d, \mathcal{P})$ with a convex modular, since

$$d(x, y) \preceq \rho(d(x, y))e \quad \text{and} \quad d(y, z) \preceq \rho(d(y, z))e,$$

from (CUM3) we have

$$d(x, z) \preceq \max\{\rho(d(x, y)), \rho(d(y, z))\}e,$$

and therefore

$$\rho(d(x, z)) \leq \max\{\rho(d(x, y)), \rho(d(y, z))\}. \tag{1.1}$$

For a unital-normal vector ultrametric space $(\mathcal{X}, d, \mathcal{P})$, if $x \in \mathcal{X}$ and $p \in \mathcal{P} \setminus \{0\}$, the subset

$$B(x; p) := \{y \in \mathcal{X} : \rho(d(x, y)) \leq \rho(p)\},$$

is said to be a ball centered at x with radius p. Every point of a ball is its center and intersecting balls with comparable radii are comparable with respect to inclusion. The unital-normal vector ultrametric space $(\mathcal{X}, d, \mathcal{P})$ is called *spherically complete* if every chain of balls (with respect to inclusion) has a nonempty intersection.

Example 1.3. Consider the full matrix algebra $\mathbb{M}_n$ over complex numbers and choose a nonzero positive definite matrix p of positive cone $\mathcal{P}$ consisting of all positive definite matrices.

1. For any nonempty set $\mathcal{X}$, define the mapping d by
$$d(x,y) = \begin{cases} p & x \neq y \\ 0 & x = y. \end{cases}$$
Then, d is a vector ultrametric on $\mathcal{X}$.
2. Let $(\mathcal{N}, \|\cdot\|)$ be a normed space, (α_n) a sequence of positive real numbers decreasing to zero, and
$$\mathcal{X} := \{x = (x_n)_{n=1}^{\infty} \in \mathcal{N} : \limsup_{n\to\infty} \|x_n\|^{\alpha_n} < \infty\}.$$
Now, the mapping d defined by
$$d(x,y) = \begin{cases} p \limsup_{n\to\infty} \|x_n - y_n\|^{\alpha_n} & x \neq y \\ 0 & x = y, \end{cases}$$
is a vector ultrametric on $\mathcal{X}$.
3. Let $\mathcal{A}$ be a C*-algebra with positive cone $\mathcal{P}$ (consisting of the set of all self-adjoint elements with non-negative spectral values). If $(\mathcal{X}, d)$ is an ultrametric space in the usual sense and $p \in \mathcal{P} \setminus \{0\}$, then the mapping
$$(x,y) \to d(x,y)p \qquad (x, y \in \mathcal{X}),$$
is a vector ultrametric on $\mathcal{X}$.

The next example generalizes the idea given in the previous example.

Example 1.4. Let $\mathcal{A}_\rho$ be a complete modular space with the cone $\mathcal{P}$. For usual ultra metric space $(\mathcal{X}, d)$ and $p \in \mathcal{P} \setminus \{0\}$, the mapping
$$(x,y) \to d(x,y)p \qquad (x, y \in \mathcal{X}),$$
is a vector ultrametric on $\mathcal{X}$.

It is clear that the cones given in Example 1.3 are normal and the cone in 3 of the same example is also unital (see, e.g., [2]).

Example 1.5. Consider the Euclidean space $\mathbb{R}^2$ with the lexicographical order $\preceq$ (i.e., $(a,b) \preceq (a',b')$ if $a < a'$ or $[a = a'$ and $b \leq b'])$. Then, it is clear that $\mathcal{P} = \{x \in \mathbb{R}^2 : x \succeq 0\}$ is not normal. For any nonempty set $\mathcal{X}$ equipped with the mapping
$$d(x,y) = \begin{cases} u & x \neq y \\ 0 & x = y, \end{cases}$$
where $u \in \mathcal{P}$ is a fixed element, we obtain a non-normal and unital vector ultrametric space. In fact, $(a,b) \preceq \|(a,b)\|(1,1)$, for every $(a,b) \in \mathbb{R}^2$.

2. Main theorem

We recall that a correspondence φ on a set Ω, denoted by $\varphi : \Omega \twoheadrightarrow \Omega$, assigns to each w in Ω a (nonempty) subset $\varphi(w)$ of Ω. For any subset C of Ω and correspondence $\varphi : C \twoheadrightarrow \Omega$, an element $w \in C$ is said to be a fixed point if $w \in \varphi(w)$.

By a convergent sequence $(x_n)_{n=1}^{\infty}$ in vector ultrametric space $(\mathcal{X}, d, \mathcal{P})$, we mean that there exists an element $x \in \mathcal{X}$ such that $\rho(d(x_n, x)) \to 0$ as $n \to \infty$. It is not difficult to see that for any unital-normal vector ultrametric space $(\mathcal{X}, d, \mathcal{P})$, the vector ultrametric d is jointly continuous, i.e, if $x_n \to x$ and $y_n \to y$, then $d(x_n, y_n) \to d(x, y)$.

We also say that a subset G of $(\mathcal{X}, d, \mathcal{P})$ is compact if every sequence in G has a convergent subsequence in G. In the following by $\varphi : \mathcal{X} \twoheadrightarrow c(\mathcal{X})$ we mean that φ is a correspondence with compact values.

Theorem 2.1. *Let $(\mathcal{X}, d, \mathcal{P})$ be a spherically complete unital-normal vector ultrametric space and $\varphi : \mathcal{X} \twoheadrightarrow c(\mathcal{X})$. If for every $x, y \in \mathcal{X}$, $x \neq y$, and $p \in \varphi(x)$ there exists $q \in \varphi(y)$ such that*

$$\rho(d(p, q)) < \max\{\rho(d(x, p)), \rho(d(x, y)), \rho(d(y, q))\}, \tag{2.1}$$

then there exists $g \in \mathcal{X}$ such that $g \in \varphi(g)$.

Proof . Let

$$\Gamma = \{B_{(a,p)} \mid a \in \mathcal{X},\ p \in \varphi(a)\},$$

where $B_{(a,p)} = B(a; d(a, p))$. Consider the partial order $\sqsubseteq$ on Γ defined by

$$B_{(a,p)} \sqsubseteq B_{(b,q)} \quad \text{iff} \quad B_{(b,q)} \subseteq B_{(a,p)},$$

where $a, b \in \mathcal{X}$, $p \in \varphi(a)$, and $q \in \varphi(b)$. If Γ' is any chain in Γ, then the spherically completeness of $\mathcal{X}$ implies that the intersection Ω of elements of Γ' is nonempty. Choose $c \in \Omega$ and $B_{(a,p)} \in \Gamma'$. If $x \in B_{(c,q)}$, where $q \in \varphi(c)$ and satisfies (2.1) then

$$\rho(d(x, c)) \leq \rho(d(c, q)) \leq \max\{\rho(d(c, a)), \rho(d(a, p)), \rho(d(p, q))\},$$

and since $\rho(d(c, a)) \leq \rho(d(a, p))$ (because of $c \in B_{(a,p)}$), we get

$$\rho(d(x, c)) \leq \max\{\rho(d(a, p)), \rho(d(p, q))\}. \tag{2.2}$$

We claim that $\rho(d(x, c)) \leq \rho(d(a, p))$. If $\rho(d(p, q)) \leq \rho(d(a, p))$, then the inequality is clear. If, otherwise $\rho(d(p, q)) > \rho(d(a, p))$, then from (2.2) we obtain

$$\rho(d(x, c)) \leq \rho(d(p, q)).$$

From (2.1) it follows that

$$\rho(d(x, c)) < \max\{\rho(d(a, p)), \rho(d(a, c)), \rho(d(c, q))\},$$

and hence

$$\rho(d(x, c)) < \max\{\rho(d(a, p)), \rho(d(c, q))\}.$$

Now, if $\rho(d(a, p)) < \rho(d(c, q))$, then

$$\rho(d(c, q)) \leq \max\{\rho(d(c, a), \rho(d(a, p)), \rho(d(p, q))\},$$

that is,

$$\rho(d(c,q)) \leq \rho(d(p,q)),$$

and so from (2.1) we get the contradiction $\rho(d(p,q)) < \rho(d(p,q))$. Therefore

$$\rho(d(x,c)) \leq \rho(d(a,p)),$$

and because $B_{(a,p)} = B(c; d(a,p))$, it implies that

$$\rho(d(x,a)) \leq \rho(d(a,p)).$$

That is, $x \in B_{(a,p)}$, and consequently $B_{(c,q)} \subseteq B_{(a,p)}$. Now,

$$\inf_{q\in\varphi(c)} \rho(d(c,q)) = \rho(d(c,\tilde{q})),$$

for some $\tilde{q} \in \varphi(c)$ (because of (1.1) and Δ_2-condition). If $\rho(d(c,\tilde{q})) = 0$, then $c \in \varphi(c)$. Otherwise, $B_{(c,\tilde{q})}$ is an upper bound for the chain Γ'. Therefore, by Zorn's lemma Γ admits a maximal element $B_{(g,w)}$, where $g \in \mathcal{X}$ and $w \in \varphi(g)$. We show that $g \in \varphi(g)$. Suppose on the contrary that $g \notin \varphi(g)$. Then, by (2.1), setting $x = g$ and $y = p = w \in \varphi(g)$, there exists $s \in \varphi(w)$ such that

$$\rho(d(s,w)) < \max\{\rho(d(g,w)), \rho(d(w,s))\}$$

and therefore

$$\rho(d(s,w)) < \rho(d(g,w)). \tag{2.3}$$

On the other hand, from the maximality of $B_{(g,w)}$ and that $w \in B_{(g,w)}$, we have

$$B_{(g,w)} \subseteq B_{(w,s)} = B(g; d(w,s)),$$

and so

$$\rho(d(w,g) \leq \rho(d(w,s)),$$

which contradicts (2.3). □

The following corollaries obtain immediately from preceding theorem. We suppose that $(\mathcal{X}, d, \mathcal{P})$, γ, and φ are as given in the previous theorem.

Corollary 2.2. If for every $x, y \in \mathcal{X}$, $x \neq y$, and $p \in \varphi(x)$ there exists $q \in \varphi(y)$ such that

$$\rho(d(p,q)) < \rho(d(x,y)),$$

then there exists $g \in \mathcal{X}$ such that $g \in \varphi(g)$.

Corollary 2.3. If for every $x, y \in \mathcal{X}$, $x \neq y$, and $p \in \varphi(x)$ there exists $q \in \varphi(y)$ such that

$$\rho(d(p,q)) < \max\{\rho(d(x,p)), \rho(d(x,y)), \rho(d(y,q))\},$$

then φ has a fixed point.

Corollary 2.4. If for every $x, y \in \mathcal{X}$, $x \neq y$, and $p \in \varphi(x)$ there exists $q \in \varphi(y)$ such that

$$\rho(d(p,q)) < \rho(d(x,y)),$$

then φ has a fixed point.

As seen, the last corollary generalizes Theorem 1 in [11].

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