



# Variational inequalities on Hilbert $C^*$ -modules

H. Fathi<sup>a,\*</sup>, S.A.R. Hosseinioun<sup>b</sup>

<sup>a</sup>Department of Mathematics Science, Shahid Beheshti University, Tehran, Iran

<sup>b</sup>Department of Mathematical Sciences, University of Arkansas, Fayetteville, Arkansas 72701, USA

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## Abstract

We introduce variational inequality problems on Hilbert  $C^*$ -modules and we prove several existence results for variational inequalities defined on closed convex sets. Then relation between variational inequalities,  $C^*$ -valued metric projection and fixed point theory on Hilbert  $C^*$ -modules is studied.

*Keywords:* variational inequality; Hilbert  $C^*$ -module; metric projection; fixed point.

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## 1. Introduction and preliminaries

The theory of variational inequalities is an important domain of pure and applied mathematics, introduced in early sixties, by Stampacchia and Hartman [17]. It developed rapidly because of its applications in physics, economics and engineering sciences. A classical variational inequality problem, is to find a vector  $u^* \in K$  such that

$$\langle v - u^*, T(u^*) \rangle \geq 0, \quad \forall v \in K$$

where  $K \subseteq \mathbb{R}^n$  is nonempty, closed and convex set and  $T$  is a mapping from  $\mathbb{R}^n$  into itself. Later, variational inequality expanded to Hilbert and Banach spaces. In real Banach spaces, variational inequality problem is defined similarly, but in this case  $T$  is a mapping from  $K$  to dual of a Banach space. In complex Banach spaces case, it turns to find  $u^* \in K$  such that

$$\operatorname{Re} \langle v - u^*, T(u^*) \rangle \geq 0, \quad \forall v \in K.$$

So far, a large number of existence conditions have been established. The books [7] and [1] provide a suitable introduction to variational inequality and its applications. For another generalizations of variational inequalities see for example [12],[13].

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\*Corresponding author

Email addresses: [Hedayat.fathi@yahoo.com](mailto:Hedayat.fathi@yahoo.com) (H. Fathi), [ahosseinioun@yahoo.com](mailto:ahosseinioun@yahoo.com) (S.A.R. Hosseinioun)

Here we study variational inequalities on Hilbert  $C^*$ -modules. Since in Hilbert  $C^*$ -modules inner product and functionals take their values in a  $C^*$ -algebra (instead of scalars), variational inequalities on Hilbert  $C^*$ -modules is more general and more complicated. Hilbert  $C^*$ -modules contains both Hilbert spaces and  $C^*$ -algebras. So our definition of variational inequalities not only generalize the old one on Hilbert spaces, but also begins a new way to define a special kind of variational inequalities on  $C^*$  algebras.

In section 2 of this paper, we recall some definitions and preliminaries about  $C^*$ -algebras and Hilbert  $C^*$ -modules that we need in the sequel. In section 3 we give some existence theorems for variational inequalities on Hilbert  $C^*$ -modules. Also relation between variational inequalities,  $C^*$ -valued metric projection and fixed point theory is studied.

## 2. preliminaries

A  $C^*$ -algebra  $A$ , is an involutive Banach algebra such that for all  $x \in A$ ,  $\|x^*x\|_A = \|x\|_A^2$ . An element  $x$  in  $C^*$ -algebra  $A$  is called positive if  $x$  is selfadjoint and  $sp(x) \subseteq \mathbb{R}^+$ . We write  $x \geq 0$  if  $x$  is a positive element and denote by  $A^+$  the set of all positive elements of  $A$ . By Theorem 4.2.2 [5],  $A^+$  is a pointed, closed and convex cone i.e.  $A^+$  is closed and

- (i)  $\lambda A^+ \subseteq A^+ \quad (\lambda \in \mathbb{R}^+)$ ,
- (ii)  $A^+ + A^+ \subseteq A^+$ ,
- (iii)  $A^+ \cap (-A^+) = \{0\}$ .

So if we define  $\leq$  on  $A_{sa}$ , the set of self adjoint elements of  $A$ , by:

$$x \leq y \Leftrightarrow y - x \in A^+,$$

then  $A_{sa}$  is a partially ordered set.

For positive element  $a$ , there exists unique positive element  $b$ , denoted by  $a^{\frac{1}{2}}$ , such that  $b^2 = a$ . It is well known that if  $a, b$  are positive elements of  $A$ , then the inequality  $a \leq b$  implies that  $a^{\frac{1}{2}} \leq b^{\frac{1}{2}}$  but the converse holds only in abelian  $C^*$ -algebras.

For any nonunital  $C^*$ -algebra  $A$  we set  $\tilde{A}$  the unitization of  $A$ . The map

$$A \rightarrow \tilde{A}, \quad a \longmapsto (a, 0)$$

is an embedding, and we can identify  $A$  as an ideal of  $\tilde{A}$ . For more details about  $C^*$ -algebras we refer to [5] and [10].

Let  $A$  be a  $C^*$ -algebra. A pre-Hilbert  $A$ -module is a linear space  $E$  which is a right  $A$ -module together with an  $A$ -valued mapping  $\langle \cdot, \cdot \rangle : E \times E \rightarrow A$  with following properties:

- (i)  $\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle \quad (x, y, z \in E, \lambda \in \mathbb{C})$ ,
- (ii)  $\langle x, ya \rangle = \langle x, y \rangle a \quad (x, y \in E, a \in A)$ ,
- (iii)  $\langle y, x \rangle = \langle x, y \rangle^* \quad (x, y \in E)$ ,
- (iv)  $\langle x, x \rangle \geq 0, \quad \langle x, x \rangle = 0$  then  $x = 0$ .

The map  $\langle \cdot, \cdot \rangle$  is called the  $A$ -valued inner product on  $E$ . A pre-Hilbert  $A$ -module  $(E, \langle \cdot, \cdot \rangle)$  is called Hilbert  $A$ -module if it is complete with respect to the norm  $\| \cdot \| = \| \langle \cdot, \cdot \rangle \|_A^{\frac{1}{2}}$ . We always suppose that the linear structure of  $A$  and  $E$  are compatible.

If we define  $|x| = \langle x, x \rangle^{\frac{1}{2}}$ , then  $| \cdot |$  is called  $A$ -valued "norm". This is not actually a norm, since for example it need not satisfies  $|x + y| \leq |x| + |y|$  (see [8]). More percisly triangle inequality is

satisfied for  $A$ -valued norms if and only if  $\langle E, E \rangle$  is commutative, where  $\langle E, E \rangle = \text{clspan}\{\langle x, y \rangle | x, y \in E\}$  (see [6]).  $A$ -valued norm is very important because of its applications also it may motivate us to study the geometry in case the triangle inequality does not hold. [11] is an example for this point of view.

If  $I$  is a closed right ideal of  $C^*$ -algebra  $A$ , then  $I$  is a Hilbert  $A$ -module if we define

$$\langle a, b \rangle = a^*b \quad (a, b \in I).$$

In particular any  $C^*$ -algebra is a Hilbert module over itself. On the other hand any Hilbert module over the field of complex numbers  $\mathbb{C}$  is a Hilbert space. Thus Hilbert  $C^*$ -modules generalize both  $C^*$ -algebras and Hilbert spaces.

There are some similarities between Hilbert  $C^*$ -modules and Hilbert spaces, but there is a fundamental way in which Hilbert  $C^*$ -modules differ from Hilbert spaces. To see this it is well known that in a Hilbert space any closed and convex set has best approximation property, but this approximation property is not valid in Hilbert  $C^*$ -modules.

More information about Hilbert  $C^*$ -modules can be found in [8] and [18].

### 3. Variational inequalities on Hilbert $C^*$ -modules

Here we give the natural generalization of variational inequality on Hilbert  $C^*$ -modules. In this paper  $A$  denotes a  $C^*$ -algebra and  $E$  is a Hilbert  $A$ -module unless otherwise indicated.

**Definition 3.1.** Let  $K$  be an arbitrary nonempty subset of  $E$  and  $T : K \rightarrow E$  be a mapping. ( $C^*$ -valued) variational inequality correspond to  $T$  and  $K$ , denoted by  $VI(T, K)$ , is to find  $x_0 \in K$  such that:

$$\text{Re} \langle x - x_0, T(x_0) \rangle \geq 0 \quad (x \in K).$$

Note that in general  $A$  is neither unital nor commutative. Before proving existence theorems we recall the following classical notions and prove a lemma due to Minty [9].

**Definition 3.2.** Let  $K \subseteq E$  be a closed convex set and  $T : K \rightarrow E$  be a mapping.

(i) We say  $T$  is monotone if

$$\text{Re} \langle x - y, T(x) - T(y) \rangle \geq 0 \quad (x, y \in K).$$

(ii) We say  $T$  is strictly monotone if  $x \neq y$  implies that

$$\text{Re} \langle x - y, T(x) - T(y) \rangle > 0 \quad (x, y \in K).$$

(iii) We say  $T$  is pseudo monotone if  $\text{Re} \langle x - y, T(y) \rangle \geq 0$  implies

$$\text{Re} \langle x - y, T(x) \rangle \geq 0 \quad (x, y \in K).$$

**Lemma 3.3.** Let  $K \subseteq E$  be a closed convex set and  $T : K \rightarrow E$  be a continuous and pseudo monotone. Then an element  $x_0 \in K$  is a solution of  $VI(T, K)$  if and only if

$$\text{Re} \langle x - x_0, T(x) \rangle \geq 0 \quad (x \in K). \tag{3.1}$$

**Proof .** Suppose that  $x_0 \in K$  is solution of  $VI(T, K)$ . Then for any  $x \in K$ , we have  $Re\langle x - x_0, T(x_0) \rangle \geq 0$  and the pseudo monotonicity implies that  $Re \langle x - x_0, T(x) \rangle \geq 0$ .

Conversely, suppose that an element  $x_0 \in K$  satisfies (3.1). In this case, if  $x \in K$ , we define  $x_t$  by

$$x_t = (1 - t)x_0 + tx, \quad t \in (0, 1).$$

Insert  $x_t$  in the (3.1), we have

$$Re \langle x_t - x_0, T(x_t) \rangle \geq 0,$$

which implies

$$Re \langle t(x_t - x_0), T(x_t) \rangle \geq 0$$

and finally

$$Re \langle x - x_0, T(x_t) \rangle \geq 0.$$

Let  $t \rightarrow 0$ . Using the continuity of  $T$  we have

$$Re \langle x - x_0, T(x_0) \rangle \geq 0,$$

i.e.  $x_0$  is a solution of  $VI(T, K)$ .  $\square$

**Remark 3.4.** The solution of  $VI(T, K)$  need not be unique but when  $T$  is strictly monotone the uniqueness property holds. In fact if  $x, x' \in K$  be solutions of  $VI(T, K)$  then

$$Re\langle y - x, T(x) \rangle \geq 0 \quad (y \in K),$$

$$Re\langle y - x', T(x') \rangle \geq 0 \quad (y \in K).$$

So, setting  $y = x'$  in first inequality and  $y = x$  in second, we have

$$Re\langle x - x', T(x) - T(x') \rangle \leq 0.$$

Now strictly monotonicity implies that  $x = x'$ .

By definition of variational inequality,  $x_0 \in K$  is solution of  $VI(T, K)$  if and only if

$$Re \langle x, T(x_0) \rangle \geq Re \langle x_0, T(x_0) \rangle \quad (x \in K)$$

or equivalently

$$\min_{x \in K} Re \langle x, T(x_0) \rangle = Re \langle x_0, T(x_0) \rangle.$$

Since every compact subset of  $\mathbb{R}$  attains its minimum, existence of solution for ordinary  $VI(T, K)$  is easy, but in  $C^*$  case we have only a partial order on  $A_{sa}$ . So if we can find a  $C^*$ -algebra  $A$  such that every compact subset of  $A_{sa}$  attains its minimum or  $A_{sa}$  being totally order, then we can extend most of the theorems in ordinary variational case to Hilbert  $C^*$ -module on these  $C^*$ -algebras without any extra assumption. In next proposition we show that the only  $C^*$ -algebra with one of the above properties is  $\mathbb{C}$ , the set of complex numbers.

**Proposition 3.5.** *The following statements are equivalent:*

- (i)  $A = \mathbb{C}$ .
- (ii)  $A_{sa}$  is totally ordered set.
- (iii)  $A_{sa} = A^+ \cup (-A^+)$ .

(iv) Any compact subset of  $A_{sa}$  attains its minimum.

**Proof .**  $i \Rightarrow ii$  is obvious.

$ii \Rightarrow iii$  if  $x \in A_{sa}$  then from (ii)  $x \geq 0$  or  $x \leq 0$ , because  $0 \in A_{sa}$ . Thus  $A_{sa} = A^+ \cup (-A^+)$ .

$iii \Rightarrow i$  Let  $x, y \in A_{sa}$ . Then  $x - y \in A_{sa}$ . Now from (iii),  $x \geq y$  or  $y \geq x$ . Thus  $A_{sa}$  is totally order.(in special case it has lattice structure.) But Sherman [16] proved that if  $A_{sa}$  has lattice structure then  $A$  should be commutative. So  $A = C_0(X)$  for some locally compact Hausdorff  $X$ . Now if  $a$  and  $b$  be two disjoint element of  $X$  by Orysohn's lemma there exists  $f, g \in C_0(X)$  such that

$$f(a) = 1, f(b) = 0, g(a) = 0, g(b) = 1.$$

We know that order in  $C_0(X)_{sa}$  is as usual

$$f \leq g \iff f(x) \leq g(x) \quad (x \in X).$$

Hence  $f$  and  $g$  are two element of  $A_{sa}$  that cant be compare. Thus  $X$  has only one element and hence  $A = C_0(X) = \mathbb{C}$ .

$i \Rightarrow iv$  is obvious.

$iv \Rightarrow ii$  Let  $x, y \in A_{sa}$ . Then by (iv)  $\{x, y\}$  attains its minimum. So  $A_{sa}$  is totally ordered.  $\square$

There is two approach to the existence theorems for variational inequalities. Some of them add some restriction on underlying set  $K$  and the others on mapping  $T$ . We consider both of them, first study the cases that underlying set  $K$  has some extra assumption.

Solutions of variational inequality on a set that is closed under scalar multiplication is characterized by its orthogonal complement.

**Theorem 3.6.** Let  $K \subseteq E$  be nonempty and closed under scalar multiplication and  $T : K \rightarrow E$  be a mapping. Then  $x_0$  is solution to  $VI(T, K)$  if and only if  $T(x_0) \in K^\perp$ .

**Proof .** If  $x_0$  be a solution of  $VI(T, K)$  and  $x \in K$  then  $Re\langle x - x_0, T(x_0) \rangle \geq 0$ . Now  $0, -x \in K$  implies that  $Re\langle x, T(x_0) \rangle = 0$ . On the other hand  $Re\langle x, T(x_0) \rangle = Im\langle -ix, T(x_0) \rangle$ . Thus  $\langle x, T(x_0) \rangle = 0$ . Hence  $T(x_0) \in K^\perp$ . Conversely if  $T(x_0) \in K^\perp$  then clearly  $Re\langle x - x_0, T(x_0) \rangle = 0$  for every  $x \in K$  and the proof is complete.  $\square$

Now we define special subsets of  $E$  which are useful for our purpose.

**Definition 3.7.** We say  $K \subseteq E$  has minimum in  $A$  if for any  $x \in E$ ,  $Re\langle x, K \rangle$  attains its minimum in  $A_{sa}$ . i.e. for any  $x \in E$  there exists  $k \in K$  such that  $Re\langle x, k \rangle = \min_{y \in K} Re\langle x, y \rangle$ .

The next proposition consists some ways to build new sets with minimum in  $A$ , from the old. Proof of this proposition is straightforward, just proof of (iv) is based on the fact that  $a, b, c \in A$ ,  $a \leq b$  and  $c \in A^+$  commutes with both  $a, b$  then  $ac \leq bc$ .

**Proposition 3.8.** If  $K \in E$  has minimum in  $A$  then

- (i)  $co(K)$  has minimum in  $A$ .
- (ii)  $\bar{K}$  has minimum in  $A$ .
- (iii) If  $C \subseteq \mathbb{R}$  attains its minimum then  $CK$  attains its minimum in  $A$ .
- (iv) If  $A$  is commutative,  $\varphi(K) \subseteq A^+$  ( $\varphi \in E'$ ) and  $C \subseteq A^+$  attains its minimum then  $KC$  has minimum in  $A$ .

**Remark 3.9.** (i) If  $x \in E$  and  $K \subseteq \mathbb{R}$  be compact then  $\bar{co}(xK)$  has minimum in  $A$ .  
 (ii) We Know by Krein-Milman theorem that for a compact convex set  $K$ , we have  $K = \overline{Co}(E(K))$ , where  $E(K)$  is the set of extreme points of  $K$ . So above proposition implies that  $K$  has minimum in  $A$  if  $E(K)$  is so.

The fixed point theory plays important role in variational inequalities. Most of existence theorems for variational inequalities is based on a fixed point theorem. In fact connection of variational inequalities with fixed point theory is an important factor in its development. Next theorem is Fundamental existence theorem for compact convex subsets. Corresponding theorem for locally convex space can be found in [4]. Since in Hilbert  $C^*$ -modules, inner product takes its values in a -not necessary unital or commutative  $C^*$ -algebra - and its order is partial, it needs many adaptations so we give the exact proof. This proof is based on Fan-Kakutani fixed point theorem which asserts that for compact convex subset  $K$  of locally convex space  $X$  and upper semicontinuous multivalued mapping  $F : K \rightarrow 2^K$  whose values are nonempty closed convex subset of  $K$ , there exists  $x_0 \in K$  with  $x_0 \in F(x_0)$ .

**Theorem 3.10.** *Let  $K$  be a compact convex subset of  $E$ , which has minimum in  $A$ . If  $T : K \rightarrow E$  is continuous, then  $VI(T, K)$  has a solution.*

**Proof .** Let  $F : K \rightarrow 2^K$  defined by

$$F(x) := \{z \in K : Re \langle z, T(x) \rangle = \min_{y \in K} Re \langle y, T(x) \rangle\}.$$

The values of  $F$  are nonempty (by our hypothesis) closed convex subset of  $K$  and fixed points of  $F$  are exactly solutions of  $VI(T, K)$ . Thus if we show that  $F$  is upper semicontinuous, Fan-Kakutani Theorem completes the proof. Let  $O$  be an open set in  $K$  and choose  $y_0 \in K$  such that  $F(y_0) \subseteq O$  we have to find neighborhood  $N$  of  $y_0$  such that for all  $y \in N$ ,  $F(y) \subseteq O$ . There exists an  $\epsilon_0 > 0$  such that  $x \in K \setminus O$  implies

$$Re \langle x, T(y_0) \rangle \not\leq \epsilon_0 I + \min_{y \in K} Re \langle y, T(y_0) \rangle. \tag{3.2}$$

Or equivalently

$$\epsilon_0 I + \min_{y \in K} Re \langle y, T(y_0) \rangle - Re \langle x, T(y_0) \rangle \notin (\tilde{A})^+,$$

where  $I$  is unit of  $\tilde{A}$ . For otherwise there would be a sequence  $\{x_n\} \subseteq K \setminus O$  such that

$$Re \langle x_n, T(y_0) \rangle \leq \frac{1}{n} I + \min_{y \in K} Re \langle y, T(y_0) \rangle.$$

Compactness of  $K \setminus O$  implies that  $\{x_n\}$  has cluster point and any cluster point of this sequence would be in  $K \setminus O$ . If  $x$  is the cluster point of  $\{x_n\}$  then

$$\min_{y \in K} Re \langle y, T(y_0) \rangle - Re \langle x, T(y_0) \rangle \in (\tilde{A})^+.$$

But  $(\tilde{A})^+ \cap A = A^+$ . Thus

$$\min_{y \in K} Re \langle y, T(y_0) \rangle - Re \langle x, T(y_0) \rangle \in A^+.$$

Hence  $T(y_0)$  attain its minimum in  $x$ . So  $x$  should be in  $F(y_0)$  and this contradicts  $F(y_0) \subseteq O$ .  
 Let  $\epsilon := \frac{\epsilon_0}{3}$  and define

$$K_\epsilon := \{w \in E : \epsilon I \not\leq \operatorname{Re} \langle x, w - T(y_0) \rangle \not\leq -\epsilon I \quad \forall x \in K\}.$$

Then  $K_\epsilon$  is an open subset of  $E$  and Continuity of  $T$  implies that  $N = T^{-1}(K_\epsilon)$  is a neighborhood of  $y_0$ . Now let  $y \in N$  and  $w = T(y)$  then  $w \in K_\epsilon$ . For all  $z \in K \setminus O$  by (3.2) we have

$$\operatorname{Re} \langle z, w \rangle \not\leq \operatorname{Re} \langle z, T(y_0) \rangle - \frac{\epsilon_0}{3} I \not\leq \frac{2\epsilon_0}{3} I + \min_{y \in K} \operatorname{Re} \langle y, w \rangle \not\leq \frac{\epsilon_0}{3} I + \min_{y \in K} \operatorname{Re} \langle y, w \rangle.$$

Thus  $z \notin F(y)$ . In other word  $z \in K \setminus O$  implies  $z$  is not in  $F(y)$  and hence for all  $y \in N$ ,  $F(y) \subseteq O$  and this proves the theorem.  $\square$

If  $E$  is reflexive and  $T$  is pseudo monotone, compactness condition in Theorem 3.10 can be reduced to closed and bounded sets. Proof of this theorem is similar to the corresponding theorem in Hilbert spaces, just it need some adaptation which is similar to above theorems so we omit it. The proof is based on above theorem, Lemma 3.3, Banach-Alaoglu theorem and finite intersection property.

**Theorem 3.11.** (Theorem 1.4 in [7]) Let  $K$  be a closed bounded and convex subset of  $E$  that any finite dimension subset of  $K$  has minimum in  $A$ . Let  $T : K \rightarrow E$  be pseudo monotone and continuous. If  $E$  is reflexive (as a Banach space) then  $VI(T, K)$  has a solution.

The following result is a necessary and sufficient condition for solvability of  $VI(T, K)$ .

**Corollary 3.12.** Let  $T$  be as previous theorem and  $K$  be a closed convex subset of  $E$  such that any finite dimension subset of  $K$  has minimum in  $A$ . A necessary and sufficient condition to exist a solution to the  $VI(T, K)$  is that for a positive real number  $R$  there exists a solution  $x_R$  of the variational inequality  $VI(T, K_R)$  where  $(K_R = K \cap \{v : \|v\| \leq R\})$ , satisfies the inequality  $\|x_R\| < R$ .

In nonreflexive case we have the following theorem:

**Theorem 3.13.** Let  $K$  be a closed convex set in  $E$  that any compact subset of  $K$  attains its minimum in  $A$ . Let  $T : K \rightarrow E$  be a continuous mapping such that there exists a nonempty compact and convex subset  $D$  in  $K$  such that, for every  $x \in K \setminus D$  there exists  $z \in D$  such that  $\langle x - z, T(x) \rangle > 0$ . Then  $VI(T, K)$  has a solution.

**Proof .** For every  $u \in K$  Let

$$D_u = \{x \in D | \operatorname{Re} \langle u - x, T(x) \rangle \geq 0\}.$$

Since  $T$  is continuous,  $D_u$  is closed in  $D$  and any element of  $\bigcap_{u \in K} D_u$  is a solution of  $VI(T, K)$ . Using the finite intersection property of compact sets it is enough to prove that for arbitrary  $u^1, \dots, u^m \in K$ ,  $\bigcap_{i=1}^m D_{u^i} \neq \emptyset$ . So suppose that  $u^1, \dots, u^m \in K$  be arbitrary. Suppose that  $\hat{D}$  is close convex cone of  $D \cup \{u^1, \dots, u^m\}$ . Since  $\hat{D}$  is a non-empty compact and convex subset in  $K$ , it follows from 3.10 that  $VI(T, \hat{D})$  has a solution, say  $\hat{x} \in \hat{D}$ . In particular

$$\operatorname{Re} \langle u^i - \hat{x}, T(\hat{x}) \rangle \geq 0, \quad (i = 1, \dots, m).$$

It remains to be shown that  $\hat{x} \in D$ . But, if  $\hat{x} \notin D$ , then it follows from assumption there exists  $z \in D$  such that  $Re\langle z - \hat{x}, T(\hat{x}) \rangle \leq 0$  which is a contradiction. This contradiction proves theorem.  $\square$

Rest of this paper studies relation between variational inequalities and  $C^*$ -metric projection. As mentioned before in any Hilbert space  $H$ , if  $K$  is a closed convex subset of  $H$ , then there exists a unique element  $P_K(x)$  of  $K$  such that

$$\| x - P_K(x) \| = \inf_{y \in K} \| x - y \| .$$

$P_K(x)$  is projection of  $x$  on  $K$  (or best approximation to  $x$  from  $K$ ). The mapping  $P_K : H \rightarrow K$  is called metric projection onto  $K$ . In Hilbert  $C^*$ -modules neither existence nor uniqueness of projection holds.( For example see [2] for subspaces of Hilbert  $C(X)$ -module  $C(X)$ ). But if we work with  $A$ -valued norm  $|\cdot|$ , then uniqueness condition is satisfied.

**Proposition 3.14.** *Let  $K \subseteq E$  be a closed convex set. Let*

$$P_K^A(x) = \{y_0 \in K : | x - y_0 |^2 = \inf_{y \in K} | x - y |^2\}.$$

*then  $P_K^A(x)$  has at most one element.*

**Proof .** Recall that the triangle inequality does not hold in  $A$ -valued norms, but parallelogram law satisfies. Let  $P_K^A(x)$  be nonempty and  $y_1, y_2 \in P_K^A(x)$ . We set  $| x - y_1 | = d$ . Then

$$\begin{aligned} 0 \leq | y_1 - y_2 |^2 &= | (y_1 - x) + (x - y_2) |^2 \\ &= 2 | y_1 - x |^2 + 2 | x - y_2 |^2 - 4 | x - \frac{1}{2}(y_1 + y_2) |^2 \\ &\leq 2d^2 + 2d^2 - 4d^2 = 0 \end{aligned}$$

Thus  $y_1 = y_2$ .  $\square$

Next proposition shows that  $C^*$ -metric projection  $P_K^A(x)$  is characterized by a variational inequality. In fact if  $(E, \langle \cdot, \cdot \rangle)$  is a Hilbert  $A$ -module,  $x \in E$ ,  $K \subseteq E$  is closed convex and  $T : y \mapsto y - x$ , ( $y \in K$ ), then  $p = P_K^A(x)$  if and only if  $p$  is the solution of  $VI(T, K)$ .

**Proposition 3.15.** *Let  $K$  be a closed convex subset of  $E$  and  $x \in E$ . The following statements are equivalent:*

- (i)  $p \in K$  and  $Re \langle x - p, p - y \rangle \geq 0$  ( $y \in K$ ),
- (ii)  $p = P_K^A(x)$ .

*Moreover if  $K$  is closed submodule then  $p = P_K^A(x)$  if and only if  $x - p \in K^\perp$ .*

**Proof .** (i)  $\Rightarrow$  (ii) Let  $Re \langle x - p, p - y \rangle \geq 0$  for all  $y \in K$ . Then

$$\begin{aligned} | x - p |^2 - | x - y |^2 &= | x - p |^2 - | (x - p) + (p - y) |^2 \\ &= - | p - y |^2 - 2Re \langle x - p, p - y \rangle \leq 0. \end{aligned}$$

Thus  $| x - p |^2 \leq | x - y |^2$ . So  $p = P_K^A(x)$ .

(ii)  $\Rightarrow$  (i) For any  $(t \in (0, 1])$

$$\begin{aligned} 0 \geq | x - p |^2 - | x - (ty - (1 - t)p) |^2 \\ = | x - p |^2 - | x - p - t(y - p) |^2 - 2tRe \langle x - p, p - y \rangle - t^2 | y - p |^2 . \end{aligned}$$



Thus

$$2\operatorname{Re} \langle x - p, p - y \rangle + t \|p\|^2 \geq 0.$$

Now if  $t \rightarrow 0$  then

$$\operatorname{Re} \langle x - p, p - y \rangle \geq 0.$$

For the second part of proof, let  $x \in E$  and  $y \in K$ . By first part of proof it is clear that left hand side implies right hand. Conversely if  $p = P_K^A(x)$  then  $\operatorname{Re} \langle x - p, p - y \rangle \geq 0$ . On the other hand  $2p - y \in K$  so  $\operatorname{Re} \langle x - p, p - y \rangle = -\operatorname{Re} \langle x - p, p - (2p - y) \rangle \leq 0$ . Thus  $\operatorname{Re} \langle x - p, p - y \rangle = 0$ .

$$\operatorname{Re} \langle x - p, y \rangle = \operatorname{Re} \langle x - p, y - p \rangle + \operatorname{Re} \langle x - p, p \rangle = 0.$$

Also we have  $\operatorname{Im} \langle x - p, y \rangle = -\operatorname{Re} \langle x - p, iy \rangle = 0$ . which completes the proof.  $\square$

Next two propositions present equivalent statements for solvability of variational inequalities.

**Proposition 3.16.** *Let  $K \subseteq E$  be closed convex and  $T : K \rightarrow E$  be a mapping. Then  $x_0$  is a solution of  $VI(T, K)$  if and only if  $x_0$  is a fixed point of the map  $P_K^A(I - \rho T) : K \rightarrow K$ . That is  $x_0 = P_K^A(x_0 - \rho T(x_0))$ , where  $\rho > 0$  is constant.*

**Proof .** Let  $x \in K$  and  $x_0$  be a solution of  $VI(T, K)$ . Thus  $\operatorname{Re} \langle x - x_0, \rho T(x_0) \rangle \geq 0$  or equivalently  $\operatorname{Re} \langle x - x_0, x_0 + \rho T(x_0) - x_0 \rangle \geq 0$ . Now above lemma implies that  $x_0 = P_K^A(x_0 - \rho T(x_0))$ .

Conversely, if  $x_0 = P_K^A(x_0 - \rho T(x_0))$  then  $\operatorname{Re} \langle x_0 - x, x_0 + \rho T(x_0) - x_0 \rangle \geq 0$ . Hence  $\operatorname{Re} \langle x_0 - x, T(x_0) \rangle \geq 0$ . That is  $x_0$  is a solution of  $VI(T, K)$ .  $\square$

Let  $K$  be a closed convex subset of  $E$ . The operator  $T_K : E \rightarrow E$  defined by  $T_K(z) = T(P_K^A(z)) + z - P_K^A(z)$  ( $z \in E$ ) is called the normal operator associated with  $T$  and  $K$  (see [14]).

**Proposition 3.17.** *An element  $x_0 \in E$  is a solution to the equation  $T_K(x_0) = 0$  if and only if  $p = P_K^A(x_0)$  is a solution to the  $VI(T, K)$ .*

**Proof .** If  $T_K(x_0) = 0$  then we have  $T(p) + x_0 - p = 0$ . On the other hand by Proposition 3.15,

$$\operatorname{Re} \langle x_0 - p, p - x \rangle \geq 0 \quad (x \in K).$$

So we have  $\operatorname{Re} \langle x - p, T(p) \rangle \geq 0$ .

Conversely, if  $x \in K, x_0 = p - T(p)$  and  $\operatorname{Re} \langle x - p, T(p) \rangle \geq 0$ , we have  $T(p) = p - x_0$ . So  $\operatorname{Re} \langle x - p, p - x_0 \rangle \geq 0$ . Which implies that  $p = P_K^A(x_0)$ . Therefore we have  $T_K(x_0) = 0$ . and this completes the proof.  $\square$

Next proposition shows that  $P_K^A$  is strongly monotone and nonexpansive.

**Proposition 3.18.** *If  $K$  is closed and convex subset of Hilbert  $C^*$ -module  $E$  then  $P_K^A$  satisfies the following properties:*

- (i)  $\operatorname{Re} \langle P_K^A(x) - P_K^A(x'), x - x' \rangle \geq \|P_K^A(x) - P_K^A(x')\|^2 \quad (x, x' \in E)$ .
- (ii)  $\|P_K^A(x) - P_K^A(x')\| \leq \|x - x'\| \quad (x, x' \in E)$ .

**Proof .** Let  $x, x' \in E$  and  $p = P_K^A(x), p' = P_K^A(x')$ . Then by Proposition 3.15

$$\operatorname{Re} \langle p, y - p \rangle \geq \operatorname{Re} \langle x, y - p \rangle \quad (y \in K),$$

$$\operatorname{Re} \langle p', y - p' \rangle \geq \operatorname{Re} \langle x', y - p' \rangle \quad (y \in K).$$

Now if we set  $y = p'$  in first and  $y = p$  in second inequality and adding two inequalities we have

$$\operatorname{Re}\langle p - p', p - p' \rangle \leq \operatorname{Re}\langle x' - x, p' - p \rangle.$$

So

$$\begin{aligned} |p - p'|^2 &= \langle p - p', p - p' \rangle = \operatorname{Re}\langle p - p', p - p' \rangle \\ &\leq \operatorname{Re}\langle x' - x, p - p' \rangle \leq |\langle x' - x, p - p' \rangle| \\ &\leq \|x' - x\| \|y - y'\|. \end{aligned}$$

Thus

$$\|p - p'\|^2 \leq \|x' - x\| \|p - p'\|.$$

Which implies that  $\|p - p'\| \leq \|x' - x\|$ .  $\square$

**Remark 3.19.** By proof of above proposition if  $\langle E, E \rangle$  is commutative then

$$|P_K^A(x) - P_K^A(x')| \leq |x - x'| \quad (x, x' \in E).$$

We closed this paper with a theorem which is based on Above remark and Banach fixed point theorem.

**Theorem 3.20.** Let  $\langle E, E \rangle$  be commutative,  $K$  be a nonempty closed convex subset of  $E$  which  $P_K^A(x)$  is nonempty for all  $x \in E$ , and  $T : K \rightarrow E$  an operator satisfying

$$\begin{aligned} \operatorname{Re}\langle T(u) - T(v), u - v \rangle &\geq m|u - v|^2 \quad (u, v \in K), \\ |T(u) - T(v)| &\leq M|u - v| \quad (u, v \in K) \end{aligned}$$

Where  $m$  and  $M$  are positive constants. Then there exists a unique solution for  $VI(T, K)$ .

**Proof .** We show that if the number  $\rho$  is chosen properly  $U = P_K^A(I - \rho T)$  is a contraction mapping then Proposition 3.16 will proves the theorem

$$\begin{aligned} |U(u) - U(v)|^2 &\leq |(I - \rho T)(u) - (I - \rho T)(v)|^2 \\ &= \langle u - \rho T(u) - v + \rho T(v), u - \rho T(u) - v + \rho T(v) \rangle \\ &= |u - v|^2 - 2\rho \operatorname{Re}\langle T(u) - T(v), u - v \rangle + \rho^2 |T(u) - T(v)|^2 \\ &\leq (1 - 2\rho m + \rho^2 M^2) |u - v|^2. \end{aligned}$$

Thus

$$\|U(u) - U(v)\|^2 \leq |1 - 2\rho m + \rho^2 M^2| \|u - v\|^2.$$

Now if we suppose that  $0 < \rho < \frac{2m}{M^2}$  then  $U$  will be a contraction. Since we can always choose  $\rho$  so,  $U$  can always be constructed so. Hence there exists a unique solution to the  $VI(T, K)$ .  $\square$

More ever, proof of above theorem shows that, the solution of  $VI(T, K)$  can be obtained as the limit of the sequence generated by the classical iterative process

$$u^{n+1} = U(u^{n+1}) = P_K^A(u^n - \rho T(u^n))$$

whenever  $0 < \rho < \frac{2m}{M^2}$ .

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