



# Coupled coincidence point in ordered cone metric spaces with examples in game theory

Alireza Naeimi Sadigh<sup>a,\*</sup>, Samaneh Ghods<sup>b</sup>

<sup>a</sup>Department of Mathematics, Statistics and Computer science, Semnan University, P.O. Box 35195-363, Semnan, Iran

<sup>b</sup>Department of Mathematics, Islamic Azad University, Semnan Branch, Semnan, Iran

(Communicated by A. Ebadian)

---

## Abstract

In this paper, we prove some coupled coincidence point theorems for mappings with the mixed monotone property and obtain the uniqueness of this coincidence point. Then we providing useful examples in Nash equilibrium.

*Keywords:* Coupled fixed point; Coupled coincidence fixed point; Partially ordered sets; Cone metric space; Game theory; Nash equilibrium.

*2010 MSC:* Primary 47H10; Secondary 54H25, 34B15.

---

## 1. Introduction

The concept of cone metric space was introduced by Haung and Zhang [8], replacing the set of real numbers by an ordered Banach space. Recently, many authors have considered fixed point theory in cone metric spaces. Several fixed point theorems are proved for mapping satisfying in some contractions in cone metric spaces [1, 7]. Du [5], showed that the fixed point results in the setting of cone metric spaces in which linear contractive conditions appear, can be reduced to the respective results in the metric setting, but recently, Du [6] and Jankovic, Kadelburg, Radenovic [9], proved when, the cone metric spaces is non-normal, this is impossible. Altun, Damjanovic, Djoric [2] and Shatanawi [12], proved several fixed point and coupled coincidece fixed point theorems on partially ordered cone metric spaces that are not necessarily normal.

First, we recall some definitions and theorems from [8, 4, 2, 10]. Then prove new theorems of coupled coincidence point on ordered cone metric spaces and obtain several interesting corollaries. In our theorems the cone metric space is not necessarily normal.

---

\*Corresponding author

Email addresses: [anaeimi@semnan.ac.ir](mailto:anaeimi@semnan.ac.ir) (Alireza Naeimi Sadigh), [s1ghods@gmail.com](mailto:s1ghods@gmail.com) (Samaneh Ghods)

Let  $E$  be a real Banach space and  $P$  be a subset of  $E$ . By  $\theta$  we denote the zero element of  $E$  and by  $\text{Int}P$  the interior of  $P$ . A subset  $P$  is called a cone if and only if:

- (i)  $P$  is closed, nonempty and  $P \neq \{\theta\}$ ,
- (ii)  $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \implies ax + by \in P$ ,
- (iii)  $x \in P$  and  $-x \in P \implies x = \theta$ .

A cone  $P$  is called solid if it contains its interior points, that is, if  $\text{Int}P \neq \phi$ .

Given a cone  $P \subset E$ , we define a partial ordering  $\preceq$  with respect to  $P$  by  $x \preceq y$  if and only if  $y - x \in P$ . We shall write  $x \prec y$  if  $x \preceq y$  and  $x \neq y$ , while  $x \ll y$  will stand for  $y - x \in \text{Int}P$ .

The cone  $P$  in a real Banach space  $E$  is called normal if there is a number  $K > 0$  such that for all  $x, y \in E$ ,

$$\theta \preceq x \preceq y \text{ implies } \|x\| \preceq K \|y\|.$$

The least positive number  $K$  satisfying the above relation is called the normal constant of  $P$ . It is clear that  $K \geq 1$ .

In the following we always suppose that  $E$  is a Banach space,  $P$  is a cone in  $E$  with  $\text{Int}P \neq \phi$  and  $\preceq$  is a partial ordering with respect to  $P$ .

**Definition 1.1.** ([8]) Let  $X$  be a nonempty set. Suppose the mapping  $d : X \times X \rightarrow E$  satisfies:

- (i)  $\theta \prec d(x, y)$  for all  $x, y \in X$  with  $x \neq y$  and  $d(x, y) = \theta$  if and only if  $x = y$ ,
- (ii)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ,
- (iii)  $d(x, y) \preceq d(x, z) + d(z, y)$  for all  $x, y, z \in X$ .

Then  $d$  is called a cone metric on  $X$  and  $(X, d)$  is called a cone metric space.

**Definition 1.2.** ([8]) Let  $(X, d)$  be a cone metric space. Let  $\{x_n\}$  be a sequence in  $X$  and  $x \in X$ . If for every  $c \in E$  with  $\theta \ll c$  there is an  $N$  such that for all  $n > N$ ,  $d(x_n, x) \ll c$ , then  $\{x_n\}$  is said to be convergent to  $x$  and  $x$  is the limit of  $\{x_n\}$ . We denote this by  $\lim_{n \rightarrow \infty} x_n = x$  or  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . If for every  $c \in E$  with  $\theta \ll c$  there is an  $N$  such that for all  $n, m > N$ ,  $d(x_n, x_m) \ll c$ , then  $\{x_n\}$  is called a Cauchy sequence in  $X$ .  $(X, d)$  is a complete cone metric space if every Cauchy sequence is convergent.

**Lemma 1.3.** ([8]) Let  $(X, d)$  be a cone metric space,  $P$  be a normal cone and  $\{x_n\}$  be a sequence in  $X$ . Then:

- (i)  $\{x_n\}$  is convergent to  $x$  if and only if  $d(x_n, x) \rightarrow \theta$  ( $n \rightarrow \infty$ ),
- (ii)  $\{x_n\}$  is a Cauchy sequence if and only if  $d(x_n, x_m) \rightarrow \theta$  ( $n, m \rightarrow \infty$ ).

**Theorem 1.4.** ([2]) Let  $(X, \sqsubseteq)$  be a partially ordered set and suppose that there exists a cone metric  $d$  in  $X$  such that the cone metric space  $(X, d)$  is complete. Let  $f : X \rightarrow X$  be a continuous and nondecreasing mapping w.r.t  $\sqsubseteq$ . Suppose that the following two assertions hold:

- (i) There exist  $\alpha, \beta, \gamma \geq 0$  with  $\alpha + 2\beta + 2\gamma < 1$  such that

$$d(fx, fy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, fy)] + \gamma[d(x, fy) + d(y, fx)],$$

for all  $x, y \in X$  with  $y \sqsubseteq x$ ,

- (ii) There exists  $x_0 \in X$  such that  $x_0 \sqsubseteq fx_0$ .

Then,  $f$  has a fixed point  $x^* \in X$ .

**Theorem 1.5.** ([2]) Let  $(X, \sqsubseteq)$  be a partially ordered set and suppose that there exists a metric  $d$  in  $X$  such that the cone metric space  $(X, d)$  is complete. Let  $f : X \rightarrow X$  be a nondecreasing mapping w.r.t  $\sqsubseteq$ . Suppose that the following three assertions hold:

(i) There exist  $\alpha, \beta, \gamma \geq 0$  with  $\alpha + 2\beta + 2\gamma < 1$  such that

$$d(fx, fy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, fy)] + \gamma[d(x, fy) + d(y, fx)],$$

for all  $x, y \in X$  with  $y \sqsubseteq x$ ,

(ii) There exists  $x_0 \in X$  such that  $x_0 \sqsubseteq fx_0$ ,

(iii) If an increasing sequence  $\{x_n\}$  converges to  $x$  in  $X$ , then  $x_n \sqsubseteq x$  for all  $n$ .

Then,  $f$  has a fixed point  $x^* \in X$ .

**Definition 1.6.** ([4]) Let  $(X, \sqsubseteq)$  be a partially ordered set and  $F : X \times X \rightarrow X$ . We say that  $F$  has the mixed monotone property if  $F(x, y)$  is monotone nondecreasing in  $x$  and is monotone nonincreasing in  $y$ , that is, for any  $x, y \in X$ ,

$$x_1, x_2 \in X, x_1 \sqsubseteq x_2 \implies F(x_1, y) \sqsubseteq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \sqsubseteq y_2 \implies F(x, y_1) \supseteq F(x, y_2).$$

**Definition 1.7.** ([4]) We call an element  $(x, y) \in X \times X$  a coupled fixed point of the mapping  $F$  if

$$F(x, y) = x, F(y, x) = y.$$

**Definition 1.8.** ([10]) Let  $(X, \sqsubseteq)$  be a partially ordered set and  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$ . We say  $F$  has the mixed g-monotone property if  $F$  is monotone-g-non-decreasing in its first argument and is monotone-g-non-increasing in its second argument, that is, for any  $x, y \in X$ ,

$$x_1, x_2 \in X, g(x_1) \sqsubseteq g(x_2) \text{ implies } F(x_1, y) \sqsubseteq F(x_2, y)$$

and

$$y_1, y_2 \in X, g(y_1) \sqsubseteq g(y_2) \text{ implies } F(x, y_1) \supseteq F(x, y_2).$$

**Definition 1.9.** ([10]) An element  $(x, y) \in X \times X$  is called a coupled coincidence point of the mappings  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$ , if

$$F(x, y) = g(x), F(y, x) = g(y).$$

**Definition 1.10.** ([10]) Let  $X$ , be a non-empty set and  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$ . We say  $F$  and  $g$  are commutative if

$$g(F(x, y)) = F(g(x), g(y)),$$

for all  $x, y \in X$ .

Now, we can mention the cases of game theory that is needed.

**Definition 1.11.** ([13]) A strategy profile  $(\sigma_1, \dots, \sigma_I)$  is a Nash equilibrium of  $G$  if for every  $i$ , and every  $s_i \in S_i$ ,

$$u_i(\sigma_i, \sigma_i) \geq u_i(s_i, \sigma_i).$$

**Proposition 1.12.** ([13]) Nash equilibria exist in finite games.

Now, we prove our main theorems.

## 2. The main results

Let  $(X, \sqsubseteq)$  be a partially ordered set and  $d$  be a cone metric on  $X$ . We endow the product space  $X \times X$  with the following partial order:

for  $(x, y), (u, v) \in X \times X; (u, v) \sqsubseteq (x, y) \iff y \sqsubseteq v, x \sqsupseteq u$ .

**Theorem 2.1.** *Let  $(X, \sqsubseteq)$  be a partially ordered set and suppose there exists a cone metric  $d$  in  $X$  such that the cone metric space  $(X, d)$  is complete. Suppose  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  are such that  $F(X \times X) \subseteq g(X)$  and  $F$  has the mixed  $g$ -monotone property on  $X$  w.r.t  $\sqsubseteq$ . Assume there is a function  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  with  $\varphi(t) \leq t$  and  $\lim_{r \rightarrow t^+} \varphi(r) < t$  for each  $t > 0$ . Suppose the following four assertions hold:*

- (i)  $g$  is continuous and commutes with  $F$ ,
- (ii) There exists  $\alpha \geq 0$  with  $\alpha < \frac{1}{8}$  such that

$$d(F(x, y), F(u, v)) \preceq \varphi(\alpha[d(g(x), g(u)) + d(g(x), F(x, y)) + d(g(u), F(u, v)) + d(g(x), F(u, v)) + d(g(u), F(x, y))])$$

for all  $x, y, u, v \in X$  with  $g(x) \sqsubseteq g(u), g(y) \sqsupseteq g(v)$ ,

- (iii) There exist  $x_0, y_0 \in X$  such that  $g(x_0) \sqsubseteq F(x_0, y_0)$  and  $g(y_0) \sqsupseteq F(y_0, x_0)$ ,

(iv)  $F$  is continuous.

Then, there exist  $x^*, y^* \in X$  such that  $g(x^*) = F(x^*, y^*)$  and  $g(y^*) = F(y^*, x^*)$ .

**Proof .** Since  $F(X \times X) \subseteq g(X)$  and  $x_0, y_0 \in X$ , there exist  $x_1, y_1 \in X$  such that  $F(x_0, y_0) = g(x_1)$  and  $F(y_0, x_0) = g(y_1)$ . Again there exist  $x_2, y_2 \in X$  such that  $F(x_1, y_1) = g(x_2)$  and  $F(y_1, x_1) = g(y_2)$ . Continuing this process we can construct sequences  $\{x_n\}$  and  $\{y_n\}$  in  $X$  such that

$$F(x_n, y_n) = g(x_{n+1}) \text{ and } F(y_n, x_n) = g(y_{n+1}), \forall n \geq 0. \tag{2.1}$$

We show that

$$g(x_n) \sqsubseteq g(x_{n+1}), \forall n \geq 0 \tag{2.2}$$

and

$$g(y_n) \sqsupseteq g(y_{n+1}), \forall n \geq 0. \tag{2.3}$$

We shall use mathematical induction. Let  $n = 0$ . Since  $g(x_0) \sqsubseteq F(x_0, y_0)$  and  $g(y_0) \sqsupseteq F(y_0, x_0)$ , and as  $F(x_0, y_0) = g(x_1)$  and  $F(y_0, x_0) = g(y_1)$ , we have  $g(x_0) \sqsubseteq g(x_1)$  and  $g(y_0) \sqsupseteq g(y_1)$ . Thus (2.2) and (2.3) hold for  $n = 0$ .

Suppose now that (2.2) and (2.3), hold for some  $n \geq 0$ . Since  $g(x_n) \sqsubseteq g(x_{n+1})$  and  $g(y_{n+1}) \sqsupseteq g(y_n)$  and as  $F$  has the mixed  $g$ -monotone property, from (2.1) we obtain

$$g(x_{n+1}) = F(x_n, y_n) \sqsubseteq F(x_{n+1}, y_n) \text{ and } g(y_{n+1}) = F(y_n, x_n) \sqsupseteq F(y_{n+1}, x_n). \tag{2.4}$$

Also

$$g(x_{n+2}) = F(x_{n+1}, y_{n+1}) \sqsupseteq F(x_{n+1}, y_n) \text{ and } g(y_{n+2}) = F(y_{n+1}, x_{n+1}) \sqsubseteq F(y_{n+1}, x_n). \tag{2.5}$$

Now, from (2.4) and (2.5) we get

$$g(x_{n+1}) \sqsubseteq g(x_{n+2}) \text{ and } g(y_{n+1}) \sqsupseteq g(y_{n+2}).$$

Thus by mathematical induction we conclude that (2.2) and (2.3) hold for all  $n \geq 0$ . Therefore,

$$g(x_0) \sqsubseteq g(x_1) \sqsubseteq \cdots \sqsubseteq g(x_n) \sqsubseteq g(x_{n+1}) \sqsubseteq \cdots \tag{2.6}$$

and

$$g(y_0) \supseteq g(y_1) \supseteq \cdots \supseteq g(y_n) \supseteq g(y_{n+1}) \supseteq \cdots \tag{2.7}$$

We let

$$\delta_n = d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})). \tag{2.8}$$

Since  $g(x_{n-1}) \sqsubseteq g(x_n)$  and  $g(y_{n-1}) \supseteq g(y_n)$ , from (ii) and (2.1) and the triangle inequality we have

$$\begin{aligned} d(g(x_n), g(x_{n+1})) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\preceq \varphi(\alpha[d(g(x_{n-1}), g(x_n)) + d(g(x_{n-1}), F(x_{n-1}, y_{n-1})) + d(g(x_n), F(x_n, y_n)) \\ &\quad + d(g(x_{n-1}), F(x_n, y_n)) + d(g(x_n), F(x_{n-1}, y_{n-1}))]) \\ &= \varphi(2\alpha d(g(x_{n-1}), g(x_n)) + \alpha d(g(x_n), g(x_{n+1})) + \alpha d(g(x_{n-1}), g(x_{n+1}))) \\ &\leq 2\alpha d(g(x_{n-1}), g(x_n)) + \alpha d(g(x_n), g(x_{n+1})) + \alpha d(g(x_{n-1}), g(x_{n+1})) \\ &\preceq 2\alpha d(g(x_{n-1}), g(x_n)) + \alpha d(g(x_n), g(x_{n+1})) + \alpha d(g(x_{n-1}), g(x_n)) + \alpha d(g(x_n), g(x_{n+1})). \end{aligned}$$

So,

$$(1 - 2\alpha)d(g(x_n), g(x_{n+1})) \preceq 3\alpha d(g(x_{n-1}), g(x_n))$$

i.e,

$$d(g(x_n), g(x_{n+1})) \preceq \left(\frac{3\alpha}{1 - 2\alpha}\right)d(g(x_{n-1}), g(x_n)), \quad \forall n \geq 1.$$

Using this relation repeatedly, we get

$$d(g(x_n), g(x_{n+1})) \preceq k^n d(g(x_0), g(x_1)),$$

where  $k = \frac{3\alpha}{1 - 2\alpha} < \frac{1}{2} < 1$ .

i.e,

$$d(g(x_n), g(x_{n+1})) \preceq k^n d(g(x_0), F(x_0, y_0)). \tag{2.9}$$

Similarly,

$$\begin{aligned} d(g(y_n), g(y_{n+1})) &= d(F(y_n, x_n), F(y_{n-1}, x_{n-1})) \\ &\preceq \varphi(\alpha[d(g(y_{n-1}), g(y_n)) + d(g(y_{n-1}), F(y_{n-1}, x_{n-1})) + d(g(y_n), F(y_n, x_n)) \\ &\quad + d(g(y_{n-1}), F(y_n, x_n)) + d(g(y_n), F(y_{n-1}, x_{n-1}))]) \\ &= \varphi(2\alpha d(g(y_{n-1}), g(y_n)) + \alpha d(g(y_n), g(y_{n+1})) + \alpha d(g(y_{n-1}), g(y_{n+1}))) \\ &\leq 2\alpha d(g(y_{n-1}), g(y_n)) + \alpha d(g(y_n), g(y_{n+1})) + \alpha d(g(y_{n-1}), g(y_{n+1})) \\ &\preceq 2\alpha d(g(y_{n-1}), g(y_n)) + \alpha d(g(y_n), g(y_{n+1})) + \alpha d(g(y_{n-1}), g(y_n)) + \alpha d(g(y_n), g(y_{n+1})). \end{aligned}$$

So,

$$(1 - 2\alpha)d(g(y_n), g(y_{n+1})) \preceq 3\alpha d(g(y_{n-1}), g(y_n))$$

i.e,

$$d(g(y_n), g(y_{n+1})) \preceq \left(\frac{3\alpha}{1 - 2\alpha}\right)d(g(y_{n-1}), g(y_n)), \quad \forall n \geq 1.$$

Using this relation repeatedly, we obtain

$$d(g(y_n), g(y_{n+1})) \preceq k^n d(g(y_0), g(y_1)),$$

where  $k = \frac{3\alpha}{1-2\alpha} < \frac{1}{2} < 1$ .

i.e,

$$d(g(y_n), g(y_{n+1})) \preceq k^n d(g(y_0), F(y_0, x_0)). \tag{2.10}$$

Let  $m > n$ ; from (2.9) we have

$$\begin{aligned} d(g(x_m), g(x_n)) &\preceq d(g(x_m), g(x_{m-1})) + d(g(x_{m-1}), g(x_{m-2})) + \dots + d(g(x_{n+1}), g(x_n)) \\ &\preceq (k^{m-1} + k^{m-2} + \dots + k^n) d(g(x_0), F(x_0, y_0)) = \left(\frac{k^n - k^m}{1 - k}\right) d(g(x_0), F(x_0, y_0)) \\ &\preceq \frac{k^n}{1 - k} d(g(x_0), F(x_0, y_0)). \end{aligned}$$

Therefore,

$$d(g(x_m), g(x_n)) \preceq \frac{k^n}{1 - k} d(g(x_0), F(x_0, y_0)). \tag{2.11}$$

Now, we show that  $\{g(x_n)\}_{n \geq 1}$  is a Cauchy sequence in  $(X, d)$ . Let  $\theta \ll c$  be arbitrary. Since  $c \in \text{Int}P$ , there is a neighborhood of  $\theta$  :

$$N_\delta(\theta) = \{y \in E : \|y\| < \delta\}, \delta > 0,$$

such that  $c + N_\delta(\theta) \subseteq \text{Int}P$ . Choose a natural number  $N_1$  such that

$$\left\| -\frac{k^{N_1}}{1 - k} d(g(x_0), F(x_0, y_0)) \right\| < \delta.$$

Then  $-\frac{k^n}{1 - k} d(g(x_0), F(x_0, y_0)) \in N_\delta(\theta)$  for all  $n \geq N_1$ . Hence  $c - \frac{k^n}{1 - k} d(g(x_0), F(x_0, y_0)) \in c + N_\delta(\theta) \subseteq \text{Int}P$ . Thus we have

$$\frac{k^n}{1 - k} d(g(x_0), F(x_0, y_0)) \ll c, \quad \forall n \geq N_1.$$

Therefore, from (2.11) we get

$$d(g(x_m), g(x_n)) \preceq \frac{k^n}{1 - k} d(g(x_0), F(x_0, y_0)) \ll c, \quad \forall m > n \geq N_1.$$

So,

$$d(g(x_m), g(x_n)) \ll c, \quad \forall m > n \geq N_1.$$

Hence we conclude that  $\{g(x_n)\}_{n \geq 1}$  is a Cauchy sequence in  $(X, d)$ . Similarly, we can verify that  $\{g(y_n)\}_{n \geq 1}$  is also a Cauchy sequence in  $(X, d)$ . Since  $(X, d)$  is a complete cone metric space, there exist  $x^*, y^* \in X$  such that  $g(x_n) \rightarrow x^*$  as  $n \rightarrow \infty$  and  $g(y_m) \rightarrow y^*$  as  $m \rightarrow \infty$ . Now, from the continuity of  $g$ ,

$$\lim_{n \rightarrow \infty} g(g(x_n)) = g(\lim_{n \rightarrow \infty} g(x_n)) = g(x^*), \quad \lim_{n \rightarrow \infty} g(g(y_n)) = g(\lim_{n \rightarrow \infty} g(y_n)) = g(y^*). \tag{2.12}$$

From (2.1) and the commutativity of  $F$  and  $g$ ,

$$g(g(x_{n+1})) = g(F(x_n, y_n)) = F(g(x_n), g(y_n)), \tag{2.13}$$

$$g(g(y_{n+1})) = g(F(y_n, x_n)) = F(g(y_n), g(x_n)). \tag{2.14}$$

From (2.12), (2.13), (2.14), the continuity of  $F$  and  $g$  and the commutativity of  $F$  and  $g$  we have

$$\begin{aligned} g(x^*) &= \lim_{n \rightarrow \infty} g(g(x_{n+1})) = \lim_{n \rightarrow \infty} (F(g(x_n), g(y_n))) \\ &= F(\lim_{n \rightarrow \infty} g(x_n), \lim_{n \rightarrow \infty} g(y_n)) = F(x^*, y^*), \end{aligned}$$

and

$$\begin{aligned} g(y^*) &= \lim_{n \rightarrow \infty} g(g(y_{n+1})) = \lim_{n \rightarrow \infty} (F(g(y_n), g(x_n))) \\ &= F(\lim_{n \rightarrow \infty} g(y_n), \lim_{n \rightarrow \infty} g(x_n)) = F(y^*, x^*). \end{aligned}$$

Thus we proved that  $g(x^*) = F(x^*, y^*)$  and  $g(y^*) = F(y^*, x^*)$ .  $\square$

If we use the conditions (iv), (v) instead of the condition (iv) in Theorem 2.1 we have the following result:

**Theorem 2.2.** *Let  $(X, \sqsubseteq)$  be a partially ordered set and suppose there exists a cone metric  $d$  in  $X$  such that the cone metric space  $(X, d)$  is complete. Suppose  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  are such that  $F(X \times X) \subseteq g(X)$  and  $F$  has the mixed  $g$ -monotone property on  $X$  w.r.t  $\sqsubseteq$ . Assume there is a function  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  with  $\varphi(t) \leq t$  and  $\lim_{r \rightarrow t^+} \varphi(r) < t$  for each  $t > 0$ . Suppose that the following five assertions hold:*

- (i)  $g$  is continuous and commutes with  $F$ ,
- (ii) There exists  $\alpha \geq 0$  with  $\alpha < \frac{1}{8}$  such that

$$\begin{aligned} d(F(x, y), F(u, v)) &\preceq \varphi(\alpha[d(g(x), g(u)) + d(g(x), F(x, y)) \\ &\quad + d(g(u), F(u, v)) + d(g(x), F(u, v)) + d(g(u), F(x, y))] \end{aligned}$$

for all  $x, y, u, v \in X$  for which  $g(x) \sqsubseteq g(u), g(y) \supseteq g(v)$ ,

(iii) there exist  $x_0, y_0 \in X$  such that  $g(x_0) \sqsubseteq F(x_0, y_0)$  and  $g(y_0) \supseteq F(y_0, x_0)$ ,

(iv) if a non-decreasing sequence  $\{x_n\} \rightarrow x$ , then  $x_n \sqsubseteq x$  for all  $n$ ,

(v) if a non-increasing sequence  $\{y_n\} \rightarrow y$ , then  $y_n \supseteq y$  for all  $n$ .

Then, there exist  $x^*, y^* \in X$  such that  $g(x^*) = F(x^*, y^*)$  and  $g(y^*) = F(y^*, x^*)$ .

**Proof .** Following the proof of the Theorem 2.1 we only have to show  $g(x^*) = F(x^*, y^*)$  and  $g(y^*) = F(y^*, x^*)$ . Let  $\theta \ll c$ . Since  $\{g(x_n)\}_{n \geq 1} \rightarrow x^*$  and  $\{g(y_n)\}_{n \geq 1} \rightarrow y^*$ , there exist  $n_1 \in \mathbb{N}, n_2 \in \mathbb{N}$  such that for all  $n \geq n_1$  and  $m \geq n_2$ , we have:

$$d(g(x_n), x^*) \ll \frac{c}{3}, d(g(y_m), y^*) \ll \frac{c}{3}.$$

Taking  $n \in \mathbb{N}, n \geq \text{Max}\{n_1, n_2\}$  and using  $g(x_n) \sqsubseteq x^*$ , we get:

$$\begin{aligned} d(F(x^*, y^*), g(x^*)) &\preceq d(F(x^*, y^*), g(g(x_{n+1}))) + d(g(g(x_{n+1})), g(x^*)) \\ &= d(F(x^*, y^*), F(g(x_n), g(y_n)) + d(g(g(x_{n+1})), g(x^*)) \\ &\preceq \varphi(\alpha[d(g(g(x_n)), g(x^*)) + d(F(x^*, y^*), g(x^*)) + d(g(g(x_n)), F(g(x_n), g(y_n)) \\ &\quad + d(g(x^*), F(g(x_n), g(y_n)) + d(g(g(x_n)), F(x^*, y^*))]) + d(g(g(x_{n+1})), g(x^*)) \\ &\preceq \alpha[d(g(g(x_n)), g(x^*)) + d(F(x^*, y^*), g(x^*)) + d(g(g(x_n)), F(g(x_n), g(y_n)) \\ &\quad + d(g(x^*), F(g(x_n), g(y_n)) + d(g(g(x_n)), F(x^*, y^*))]) + d(g(g(x_{n+1})), g(x^*)). \end{aligned}$$

Taking  $n \rightarrow \infty$ , we have

$$d(F(x^*, y^*), g(x^*)) \leq 2\alpha d(g(x^*), F(x^*, y^*)).$$

By the condition (ii), we have

$$d(F(x^*, y^*), g(x^*)) \leq \frac{1}{4}d(F(x^*, y^*), g(x^*)).$$

Hence,

$$\frac{3}{4}d(F(x^*, y^*), g(x^*)) \leq \theta.$$

Therefore  $-d(F(x^*, y^*), g(x^*)) \in P$  and so, as  $d(F(x^*, y^*), g(x^*)) \in P$ , we have

$$d(F(x^*, y^*), g(x^*)) = \theta.$$

Hence  $F(x^*, y^*) = g(x^*)$ . Similarly, we can show that  $F(y^*, x^*) = g(y^*)$ .

□ Now we shall prove the existence and uniqueness theorem for a coupled common fixed point. Note that if  $(X, \sqsubseteq)$  is a partially ordered sets, then we endow the product  $S \times S$  with the following partial order:

$$\forall (x, y), (u, v) \in X \times X, (x, y) \sqsubseteq (u, v) \iff x \sqsubseteq u, y \sqsupseteq v.$$

**Theorem 2.3.** *In addition to the hypotheses of Theorems 2.1, 2.2 suppose that for every  $(x, y), (u, v) \in X \times X$  there exists a  $(z_1, z_2) \in X \times X$  such that  $(F(z_1, z_2), F(z_2, z_1))$  is comparable to  $(F(x, y), F(y, x))$  and  $(F(u, v), F(v, u))$ . Then  $F$  and  $g$  have a unique coupled common fixed point, that is, there exists a unique  $(x^*, y^*) \in X \times X$ , such that*

$$x^* = g(x^*) = F(x^*, y^*), y^* = g(y^*) = F(y^*, x^*).$$

**Proof.** From Theorems 2.1 and 2.2 the set of coupled coincidences is non-empty. We shall show that if  $(x^*, y^*)$  and  $(x, y)$  are coupled coincidence points, then  $g(x) = g(x^*)$  and  $g(y) = g(y^*)$ . By assumption there is  $(z_1, z_2) \in X \times X$  such that  $(F(z_1, z_2), F(z_2, z_1))$  is comparable with  $(F(x^*, y^*), F(y^*, x^*))$  and  $(F(x, y), F(y, x))$ . Put  $u_0 = z_1, v_0 = z_2$ , then as in the proof of the Theorem 2.1, we can inductively define sequences  $\{g(u_n)\}$  and  $\{g(v_n)\}$  such that

$$g(u_{n+1}) = F(u_n, v_n), g(v_{n+1}) = F(v_n, u_n).$$

Further, set  $x_0 = x^*, y_0 = y^*, x_0 = x, y_0 = y$  and, on the same way define the sequences  $\{g(x_n^*)\}, \{g(y_n^*)\}$  and  $\{g(x_n)\}, \{g(y_n)\}$ . Then it is easy to show that

$$g(x_n^*) = F(x^*, y^*), g(y_n^*) = F(y^*, x^*), g(x_n) = F(x, y), g(y_n) = F(y, x) \forall n \in \mathbb{N}.$$

Since  $(F(x^*, y^*), F(y^*, x^*)) = (g(x_1^*), g(y_1^*)) = (g(x^*), g(y^*))$  and  $(F(z_1, z_2), F(z_2, z_1)) = (g(u_1), g(v_1))$  are comparable, it follows that  $g(x^*) \sqsubseteq g(u_1)$  and  $g(y^*) \sqsupseteq g(v_1)$ . It is easy to show that  $(g(x^*), g(y^*))$  and  $(g(u_n), g(v_n))$  are comparable, that is,  $g(x^*) \sqsubseteq g(u_n)$  and  $g(y^*) \sqsupseteq g(v_n)$  for all  $n \geq 1$ . Thus, for each  $n \geq 1$ , we have

$$\begin{aligned} d(g(x^*), g(u_{n+1})) &= d(F(x^*, y^*), F(u_n, v_n)) \\ &\leq \varphi(\alpha[d(g(x^*), g(u_n)) + d(F(x^*, y^*), g(x^*)) + d(g(u_n), F(u_n, v_n)) \\ &\quad + d(g(x^*), F(u_n, v_n)) + d(g(u_n), F(x^*, y^*))]) \\ &\leq \alpha[d(g(x^*), g(u_n)) + d(F(x^*, y^*), g(x^*)) + d(g(u_n), F(u_n, v_n)) \\ &\quad + d(g(x^*), F(u_n, v_n)) + d(g(u_n), F(x^*, y^*))]. \end{aligned}$$



Hence,

$$\lim_{n \rightarrow \infty} d(g(x^*), g(u_{n+1})) \leq \lim_{n \rightarrow \infty} 3\alpha d(g(x^*), g(u_n)) \leq \lim_{n \rightarrow \infty} \alpha d(g(x^*), g(u_n)).$$

Continuing this process, we get

$$\lim_{n \rightarrow \infty} d(g(x^*), g(u_{n+1})) \leq \lim_{n \rightarrow \infty} \alpha^n d(g(x^*), g(u_n)) = 0.$$

Thus

$$\lim_{n \rightarrow \infty} d(g(x^*), g(u_{n+1})) = 0. \tag{2.15}$$

Similarly,

$$\lim_{n \rightarrow \infty} d(g(y^*), g(v_{n+1})) = 0. \tag{2.16}$$

Also, we can prove that

$$\lim_{n \rightarrow \infty} d(g(x), g(u_{n+1})) = 0, \lim_{n \rightarrow \infty} d(g(y), g(v_{n+1})) = 0. \tag{2.17}$$

By the triangle inequality, (2.15), (2.16) and (2.17)

$$d(g(x^*), g(x)) \leq d(g(x^*), g(u_{n+1})) + d(g(x), g(u_{n+1})) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

$$d(g(y^*), g(y)) \leq d(g(y^*), g(v_{n+1})) + d(g(y), g(v_{n+1})) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus,

$$g(x) = g(x^*), g(y) = g(y^*). \tag{2.18}$$

Since  $g(x^*) = F(x^*, y^*)$  and  $g(y^*) = F(y^*, x^*)$ , by the commutativity of  $F$  and  $g$  we have

$$g(g(x^*)) = g(F(x^*, y^*)) = F(g(x^*), g(y^*)) \text{ and } g(g(y^*)) = g(F(y^*, x^*)) = F(g(y^*), g(x^*)). \tag{2.19}$$

Denote  $g(x^*) = X^*, g(y^*) = Y^*$ . Then, from (2.19)

$$g(X^*) = F(X^*, Y^*), g(Y^*) = F(Y^*, X^*). \tag{2.20}$$

Thus,  $(X^*, Y^*)$  is a coupled coincidence point. Thus from (2.18) with  $x = X^*$  and  $y = Y^*$  it follows that  $g(X^*) = g(x^*)$  and  $g(Y^*) = g(y^*)$ , that is,

$$X^* = g(X^*), Y^* = g(Y^*). \tag{2.21}$$

From (2.20) and (2.21),

$$X^* = g(X^*) = F(X^*, Y^*), Y^* = g(Y^*) = F(Y^*, X^*).$$

Therefore  $(X^*, Y^*)$  is a coupled common fixed point of  $F$  and  $g$ .

To prove the uniqueness, assume that  $(X_1^*, Y_1^*)$  is another coupled fixed point. Then by (2.18), we have

$$X_1^* = g(X_1^*) = g(X^*) = X^*, Y_1^* = g(Y_1^*) = g(Y^*) = Y^*.$$

□

**Corollary 2.4.** *Let  $(X, \sqsubseteq)$  be a partially ordered set and suppose there exists a cone metric  $d$  in  $X$  such that the cone metric space  $(X, d)$  is complete. Suppose  $F : X \times X \rightarrow X$  is a continuous mapping having the mixed monotone property on  $X$  w.r.t  $\sqsubseteq$ . Assume there is a function*

$\varphi : [0, +\infty) \rightarrow [0, +\infty)$  with  $\varphi(t) \leq t$  and  $\lim_{r \rightarrow t^+} \varphi(r) < t$  for each  $t > 0$ . Suppose that the following two assertions hold:

(i) There exists  $\alpha \geq 0$  with  $\alpha < \frac{1}{8}$  such that

$$d(F(x, y), F(u, v)) \preceq \varphi(\alpha[d(x, u) + d(x, F(x, y)) + d(u, F(u, v)) + d(x, F(u, v)) + d(u, F(x, y))])$$

for all  $x, y, u, v \in X$  for which  $x \sqsubseteq u, y \sqsupseteq v$ , (i.e for all  $(u, v) \sqsubseteq (x, y)$ .)

(ii) There exist  $x_0, y_0 \in X$  such that  $x_0 \sqsubseteq F(x_0, y_0)$  and  $y_0 \sqsupseteq F(y_0, x_0)$ . Then, there exist  $x^*, y^* \in X$  such that  $x^* = F(x^*, y^*)$  and  $y^* = F(y^*, x^*)$ . Furthermore, if  $x_0, y_0$  are comparable, then  $x^* = y^*$ , that is  $x^* = F(x^*, x^*)$ .

**Proof .** Following the proof of the Theorem 2.1 with  $g = I$ , (the identity mapping), we only have to show that  $x = F(x^*, x^*)$ . Let us suppose that  $x_0 \sqsubseteq y_0$ . We shall show that

$$x_n \sqsubseteq y_n, \forall n \geq 0, \tag{2.22}$$

where  $x_n = F(x_{n-1}, y_{n-1}), y_n = F(y_{n-1}, x_{n-1}); n \in \mathbb{N}$ . Suppose that (2.22) holds for some fixed  $n \geq 0$ . Then, by the mixed monotone property of  $F$ ,

$$x_{n+1} = F(x_n, y_n) \sqsubseteq F(y_n, x_n) = y_{n+1}.$$

Thus, (2.22) holds. From (2.22) and (i) we have

$$\begin{aligned} d(F(x_n, y_n), F(y_n, x_n)) &\preceq \varphi(\alpha[d(x_n, y_n) + d(x_n, F(x_n, y_n)) + d(y_n, F(y_n, x_n)) \\ &\quad + d(x_n, F(y_n, x_n)) + d(y_n, F(x_n, y_n))] \\ &= \varphi(\alpha[d(x_n, y_n) + d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(x_n, y_{n+1}) + d(y_n, x_{n+1})]) \\ &\preceq \alpha[d(x_n, y_n) + d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(x_n, y_{n+1}) + d(y_n, x_{n+1})]. \end{aligned}$$

Now, by the triangle inequality,

$$\begin{aligned} d(x^*, y^*) &\preceq d(x^*, x_{n+1}) + d(x_{n+1}, y_{n+1}) + d(y_{n+1}, y^*) \\ &= d(F(x_n, y_n), F(y_n, x_n)) + d(x^*, x_{n+1}) + d(y_{n+1}, y^*) \\ &\preceq \alpha[d(x_n, y_n) + d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(x_n, y_{n+1}) + d(y_n, x_{n+1})] \\ &\quad + d(x^*, x_{n+1}) + d(y_{n+1}, y^*). \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} x_n = x^*$  and  $\lim_{n \rightarrow \infty} y_n = y^*$ , we get by taking the limit as  $n \rightarrow \infty$ ,

$$d(x^*, y^*) \preceq 3\alpha d(x^*, y^*) \preceq \frac{3}{8}d(x^*, y^*).$$

i.e,

$$\frac{5}{8}d(x^*, y^*) \preceq \theta.$$

Therefore  $-d(x^*, y^*) \in P$  and so, as  $d(x^*, y^*) \in P$ , we have  $d(x^*, y^*) = \theta$ . Hence  $x^* = y^*$ , that is  $x^* = F(x^*, y^*)$ .  $\square$

**Remark 2.5.** If we use the conditions (iv), (v) from the Theorem 2.2 instead of the continuity of  $F$  in Corollary 2.4 then we obtain the results of Corollary 2.4.

**Corollary 2.6.** *Let  $(X, \sqsubseteq)$  be a partially ordered set and suppose there exists a cone metric  $d$  in  $X$  such that the cone metric space  $(X, d)$  is complete. Suppose  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  are such that  $F(X \times X) \subseteq g(X)$  and  $F$  has the mixed  $g$ -monotone property on  $X$  w.r.t  $\sqsubseteq$ . Suppose that the following four assertions hold:*

- (i)  $g$  is continuous and commutes with  $F$ ,
- (ii) There exists  $\alpha \geq 0$  with  $\alpha < \frac{1}{8}$  such that

$$d(F(x, y), F(u, v)) \leq \alpha[d(g(x), g(u)) + d(g(x), F(x, y)) + d(g(u), F(u, v)) + d(g(x), F(u, v)) + d(g(u), F(x, y))]$$

for all  $x, y, u, v \in X$  with  $g(x) \sqsubseteq g(u), g(y) \sqsupseteq g(v)$ ,

- (iii) There exist  $x_0, y_0 \in X$  such that  $g(x_0) \sqsubseteq F(x_0, y_0)$  and  $g(y_0) \sqsupseteq F(y_0, x_0)$ ,
- (iv)  $F$  is continuous.

Then, there exist  $x^*, y^* \in X$  such that  $g(x^*) = F(x^*, y^*)$  and  $g(y^*) = F(y^*, x^*)$ . Furthermore, if the conditions of the Theorem 2.3 is satisfied, then  $F$  and  $g$  have a unique coupled common fixed point, that is, there exists a unique  $(x^*, y^*) \in X \times X$ , such that

$$x^* = g(x^*) = F(x^*, y^*), y^* = g(y^*) = F(y^*, x^*).$$

**Proof .** Taking  $\varphi(t) = t$  in Theorems 2.1 and 2.3, we obtain Corollary 2.6.  $\square$

**Remark 2.7.** If we use the conditions (iv), (v) from Theorem 2.2 instead of the condition (iv) in Corollary 2.6 then we have the conclusious of Corollary 2.6.

**Corollary 2.8.** *Let  $(X, \sqsubseteq)$  be a partially ordered set and suppose there exists a cone metric  $d$  in  $X$  such that the cone metric space  $(X, d)$  is complete. Suppose  $F : X \times X \rightarrow X$  be a continuous mapping having the mixed monotone property on  $X$  w.r.t  $\sqsubseteq$ . Suppose that the following two assertions hold:*

- (i) There exists  $\alpha \geq 0$  with  $\alpha < \frac{1}{8}$  such that

$$d(F(x, y), F(u, v)) \leq \alpha[d(x, u) + d(x, F(x, y)) + d(u, F(u, v)) + d(x, F(u, v)) + d(u, F(x, y))]$$

for all  $x, y, u, v \in X$  for which  $x \sqsubseteq u, y \sqsupseteq v$ ,

- (ii) There exist  $x_0, y_0 \in X$  such that  $x_0 \sqsubseteq F(x_0, y_0)$  and  $y_0 \sqsupseteq F(y_0, x_0)$ .

Then, there exist  $x^*, y^* \in X$  such that  $x^* = F(x^*, y^*)$  and  $y^* = F(y^*, x^*)$ . Furthermore, if  $x_0, y_0$  are comparable, then  $x^* = y^*$ , that is  $x^* = F(x^*, x^*)$ .

**Proof .** Taking  $\varphi(t) = t$  in Corollary 2.4, the result follows.  $\square$

**Remark 2.9.** If we use the conditions (iv), (v) from Theorem 2.2 instead of the continuity of  $F$  in Corollary 2.8 then we obtain the results of Corollary 2.8.

Now, we give two examples in game theory.

**Example 2.10.** This game has three Nash equilibria  $(U, L), (D, R)$  and  $(\frac{1}{2}U + \frac{1}{2}D, \frac{1}{2}L + \frac{1}{2}R)$  with payoffs  $(5, 1), (1, 5)$  and  $(\frac{5}{2}, \frac{5}{2})$ .

	L	R
U	5,1	0,0
D	4,4	1,5

In fact, these Nash equilibrias are three fixed points of function and the number of fixed point are not unique.

**Example 2.11.** Bernheim in [3] shows that this game is a unique Nash equilibrium  $(F, B)$ .

	A	B	C	D
E	0,7	2,5	7,0	0,1
F	5,2	3,3	5,2	0,1
G	7,0	2,5	0,7	0,1
H	0,0	0,-2	0,0	10,-1

In fact this Nash equilibrium is the unique fixed point of function.

## References

- [1] M. Alimohammady, J. Balooee, S. Radojevic, V. Rakocevic and M. Roohi, *Conditions of regularity in cone metric spaces*, Appl. Math. Comput. 217 (2011) 6359–6363.
- [2] I. Altun, B. Damjanovic and D. Djoric, *Fixed point and common fixed point theorems on ordered cone metric spaces*, Appl. Math. Lett. 23 (2010) 310–316.
- [3] D. Bernheim, *Rationalizable strategic behavior*, Economet-rica 52 (1984) 1007–1028.
- [4] T.G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. 65 (2006) 1379–1393.
- [5] W.S. Du, *A note on cone metric fixed point theory and its equivalence*, Nonlinear Anal. 72 (2010) 2259–2261.
- [6] W.S. Du, *New cone fixed point theorems for nonlinear multivalued maps with their applications*, Appl. Math. Lett. 24 (2011) 172–178.
- [7] A.P. Farajzadeh, A. Amini-Harandi and D. Baleanu, *Fixed point theory for generalized contractions in cone metric spaces*, Common. Nonlinear Sci. Numer. Simul. Order, (In press).
- [8] L.G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mapping*, J. Math. Anal. Appl. 332 (2007) 1468–1476.
- [9] S. Jankovic, Z. Kadelburg and S. Radenovic, *On cone metric spaces: A survey*, Nonlinear Anal. 74 (2011) 2591–2601.
- [10] V. Lakshmikantham and Lj.B. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. 70 (2009) 4341–4349.
- [11] J.J. Nieto and R.R. Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order (in press).
- [12] W. Shatanawi, *Partially ordered cone metric spaces and coupled fixed point results*, Comput. Math. Appl. 60 (2010) 2508–2515.
- [13] D. Tirole and J. Tirole, *Game Theory*, MIT Press, 1991.